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NOTE ON A POINCARÉ MAP

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ABSTRACT. We construct a C^{r-1}-vector field, the flow of which gnerates a C^r-perturbation of a given C^r-Poincaré map.

Recently M. Medveď [1] has constructed a C^{r-1}-vector field, the flow of which generates a C^r-perturbation of a given C^r-Poincaré map. He has used a surjective mapping theorem, which is a corollary of the Nash–Moser implicit function theorem. We give in this paper a simple proof of this theorem using only the implicit function theorem. We study also a similar problem which solves the question whether a local mapping can be imbedded into a local flow.

The first part

We shall prove our main result (see Theorem 1.1) of this paper. Let X be a compact C^r-manifold, $\infty \geqq r \geqq 2$. Let us denote the set of all C^r-vector fields on X by $\Gamma^r(X)$. We assume that $v \in \Gamma^r(X)$ has a periodic orbit which passes through a point x_0. By the small flow box lemma [2] there is an open neighbouhood U of x_0 and a C^r-diffeomorphism $G\colon U \to \mathbf{R}^n$ such that $x_0 \in U$, $G(x_0) = 0 \in \mathrm{R}^n$ and the vector field defined by v on $G(U)$ has the form

$$x' = \frac{\mathrm{d}}{\mathrm{d}t} x = 1$$

$$y' = 0$$

where$x \in \mathbf{R}$, $|x| \leqq 2$, $y \in \mathbf{R}^{n-1}$, $|y| \leqq 2$.

We can suppose that for the sets $V_1 = G^{-1}(\{0\} \times V)$ and $V_2 = G^{-1}(\{1\} \times V)$ there is a quasi-Poincaré map, where V is some open neighbourhood of $0 \in \mathbf{R}^{n-1}$. It means that there exists a mapping $\bar{P}\colon V_2 \to V_1$ such that

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$\bar{P}(\bar{x}) = \Phi(t_0, \bar{x})$ for each $\bar{x} \in V_2$ and some $t_0 \in \mathbf{R}$ with the property $\Phi(t, \bar{x}) \notin$ $\notin V_1$ for $0 \leqq t < t_0$, where $\Phi(t, \bar{x})$ is the flow for v. By $P: V \to V$ we denote the representation of $\bar{P}$ in these new coordinates (x, y). Since v has the above simple structure on $G(U)$, we note that the mapping P is a Poincaré map [1] of v of our periodic orbit for the cross section V_2 in the coordinates $(1, y)$. Now we take a mapping $h: \mathbf{R} \to \mathbf{R}$ such that

1. $h \in C^\infty$, $h(0) = 0$, $h(1) = 1$
2. there exists $d > 0$ such that $h'(t) = 0$ for $t < d$ or $t > 1 - d$.

It is clear that such a mapping exists.

Let W_1, W_2, W_3, W_4 be open neighbourhoods of $0 \in \mathbf{R}^{n-1}$ such that

$$W_4 \subset \bar{W}_4 \subset W_3 \subset \bar{W}_3 \subset W_2 \subset \bar{W}_2 \subset W_1 \subset \bar{W}_1 \subset V,\ P(W_4) \subset W_3,\ P^{-1}(W_1) \subset V.$$

Let us consider a C^r-mapping $f: V \to V$. If f is sufficiently C^1-small (i.e. $|f|$, $|Df(.)| \ll 1$ on V), then for each $t \in \mathbf{R}$ the mapping $Q_t(z) = z + h(t) \cdot f(P^{-1}(z))$, $Q_t: W_1 \to V$ is a diffeomorphism with the property $Q_t(W_1) \subset W_2$ and $Q_t(P(W_4)) \subset W_3$.

Let $\bar{g}$ be a C^∞-function $\bar{g}: V \to \langle 0, 1 \rangle$ such that

$$\bar{g}(\bar{x}) = 0 \text{ for } \bar{x} \notin W_2$$

$$\bar{g}(\bar{x}) = 1 \text{ for } \bar{x} \in W_3.$$

Then we define the mapping $g: \mathbf{R} \times V \to V$

$$\begin{aligned} g(t, \bar{x}) &= h'(t) \cdot \bar{g}(\bar{x}) \cdot f(P^{-1}Q_t^{-1}(\bar{x})) && \text{for } \bar{x} \in W_2 \\ \bar{g}(t, \bar{x}) &= 0 && \text{for } \bar{x} \notin W_2. \end{aligned}$$

Using the mapping g we define the vector field v_1 as follows:

$$\begin{aligned} &v_1 = (1, g(x, y)) \text{ in the coordinates } (x, y) \in G(U) \\ &v_1 = v \text{ on } X \setminus U. \end{aligned}$$

Since $g(x, y) = 0$ on the boudary of $G(U)$, v_1 is well defined. Note that there is the loss of the derivative of v_1 due to the transformation G, i.e. $v_1 \in \Gamma^{r-1}(X)$. We know that if $\bar{x} \in W_4$, then $Q_t(P(\bar{x})) \in W_3$ and, therefore, the function $y(t) =$ $= P(\bar{x}) + h(t) \cdot f(\bar{x})$, $x(t) = t$ satisfies the equation

$$x' = 1,\ y' = g(x, y)$$

with the initial condition $x(0) = 0$, $y(0) = P(x)$.
Indeed, we obtain

$$y'(t) = h'(t) \cdot f(\bar{x}) = g(t, Q_t(P(\bar{x})) = g(t, y(t)).$$

On the other hand

$$y(1) = P(\bar{x}) + h(1) \cdot f(\bar{x}) = P(\bar{x}) + f(\bar{x}),$$

for each $\bar{x} \in W_4$. Since $v_1 = v$ on $X \setminus U$ and $v_1 = (1, g(x, y))$ in the coordinates $(x, y) \in G(U)$, we see that the Poincaré map of v_1 of our periodic orbit for the cross section $G^{-1}(\{1\} \times W_4) \subset V_2$ in the coordinates $(1, y)$ has the form $\bar{x} \to P(\bar{x}) + f(\bar{x})$.
Thus we obtain the following

Theorem 1.1. *Let X be a C^r-manifold ($\infty \geqq r \geqq 2$) and let $v \in \Gamma^r(X)$ be a C^r-vector field on X with a periodic orbit θ and $x \in \theta$. Then we can define the Poincaré map P: $V \to V$ where V passes through x transversally to θ [1]. If $W \subset \bar{W} \subset V$ is a small neighbourhood of x in V, then there exists a C^1-neighbourhood U of the mapping P in $C^r(V, V)$ and a C^0-mapping S: $U \to \Gamma^{r-1}(X)$ such that if $P_1 \in U$, then $P_{S(P_1)}/W = P_1$, where $P_{S(P_1)}$ is the Poincaré map of $S(P_1)$ for the cross section V.*

The second part

Let H be a Hilbert space. We consider the set M^r, $\infty \geqq r \geqq 1$,

$$M^r \subset \{(f, V), f \in C^r(V, H), V \text{ is a connected open neighbourhood of } 0 \in H, f(0) = 0\}$$

such that $(f, V) \in M^r$ if and only if there exists an open set V_1, $V \subset V_1$ and a C^r-mapping g: $I \times V_1 \to H$ for $I = (-2, 2)$ with the following property

$$\begin{aligned} &\text{If } y' = 1,\ z' = g(y, z),\ y \in I,\ z \in V_1 \\ &\quad y(0) = 0,\ z(0) = x \in V, \end{aligned} \tag{2}$$

then the solution $z(., x)$ of the equation (2) satisfies

$$z(1, x) = f(x).$$

Hence M^r is the set of all mappings f: $V \to H, f \in C^r$, which can be imbedded into local flows, where V has the above properties. We shall give assumptions which guarantee that a mapping f: $V \to H, f \in C^r$ and $f(0) = 0$ belongs to M^r for some neighbourhood $V \subset H$ of 0. By the above small flow box lemma we can consider for a fixed $(f, v) \in M^r$ that the equation (2) has the formy $y' = 1$, $z' = 0$ on $I \times V$. Thus using similar arguments as in the previous section we obtain the following

Theorem 2.1. *If $(f, V) \in M^r$, then $(f + q, W) \in M^{r-1}$ for a small neighbourhood W of 0, $W \subset \bar{W} \subset V$ and a C^1-small mapping $q \in C^r(V, H)$, $q(0) = 0$ (i.e. $|q|$, $|Dq(.)| \ll 1$ on V).*
Furhter, by (2) we have

$$\begin{aligned} &D_x z'(t, 0) = D_z g(t, z(t, 0)) D_x z(t, 0) \\ &D_x z(0, x) = \text{Idenity}. \end{aligned} \tag{3}$$

It is well known [3, p. 280] that $D_x z(t, 0)$ is invertible for each $t \in (-2, 2)$ and moreover, $D_x z(0, 0) = \text{Identity}$, $D_x z(1, 0) = Df(0)$.

Reversely, let us consider a C^∞-mapping $B: (-2, 2) \to \mathscr{L}(H, H)$ such that $B(0) = \text{Identity}$ and $b(t) \in \mathscr{L}^I(H, H) = \{A \in \mathscr{L}(H, H), A^{-1} \in \mathscr{L}(H, H)\}$ for each $t \in (-2, 2)$. Then the equation $z' = A(t)z$ with the initial condition $z(0) = z_0 \in H$ has the solution $z(t) = B(t)z_0$, $t \in (-2, 2)$, provided that $A(t) = B'(t) \cdot B^{-1}(t)$. Hence $(B(1), h) \in M^\infty$, and by Theorem 2.1 we obtain

Theorem 2.2. *Let $f: V \to H$ be a C^r-diffeomorphism ($\infty \geqq r \geqq 2$), with properties $0 \in V \subset H$, $f(0) = 0$ and V has the above properties. Then $(f, W) \in M^r$ for some $W \subset V$ if and only if* $Df(0) \in \text{Comp Identity}$, *where* Comp Identity *is the arcwise connected component of* Identity *in* $\mathscr{L}^I(H, H)$.

Let us assume that the mapping $g \in C^\infty$ (see (2)) is defined on $\mathbf{R} \times V_1$ and 1-periodic in y. Then by (3) we obtain that $z(t + 1) = z(t, 0)$ and

$$D_z g(t + 1, z(t + 1, 0)) = D_z g(t, z(t, 0)).$$

Hence $D_x z(t, 0)$ satisfies the equation (3) with the 1-periodic map $D_z g(t, z(t, 0))$. Reversely, if a competent mapping (see (2)) of f, $(f, V) \in M^\infty$ is defined on $R \times V_1$ and 1-periodic in y, then for a C^1-small $q \in C^\infty(V, H)$, $q(0) = 0$ the competent mapping (see (2)) of $f + q$ constructed in the proof of Theorem 2.1 is also 1-periodic in y. It follows from the fact that we used the implicit function theorem in the proof of Theorem 2.1. Using similar arguments as in the proof of Theorem 2.2 we obtain

Theorem 2.3. *Let $(f, V) \in M^\infty$. If a competent mapping* (see (2)) *of f is defined on $\mathbf{R} \times V_1$ and 1-periodic in y, then* $Df(0) = z(1)$, *where $z(.)$ is a fundamental solution of some linear differential equation $z'(t) = A(t)z(t)$, $A \in C^\infty$, $A(t + 1) = A(t)$. Reversely, if* $Df(0)$ *has this property, then $(f, W) \in M^\infty$ for some $W \subset V$ and a competent mapping* (see (2)) *of f is defined for each $y \in \mathbf{R}$ and 1-periodic in y.*

Let us assume that $0 < \dim H < \infty$. It is well known that

$$\text{Comp Identity} = \{A \in \mathscr{L}(H, H), \det A > 0\}.$$

Hence by Theorem 2.2 we obtain

Theorem 2.4. *Let $f: V \to \mathbf{R}^n$ be a C^r-mapping ($\infty \geqq r \geqq 2$), where V is a neighbourhood of* 0 *and* $f(0) = 0$. *Then f can be imbedded into a local flow, i.e. $(f, W) \in M^r$ for some $W \subset V$ if and only if* $\det Df(0) > 0$.

REFERENCES

[1] MEDVEĎ, M.: Construction of realizations of perturbations of Poincaré maps. Math. Slovaca, 36, 1986, 179—190.
[2] IRWIN, M. C.: Smooth Dynamical Systems. Academic Press 1980.
[3] ŠILOV, G. J.: Matematická analýza. Alfa, Bratislava 1974.

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