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TOPOLOGICAL DIFFERENCE POSETS

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ABSTRACT. Difference posets (D-posets) are partially ordered sets with a partial difference operation. Special cases of D-posets are orthomodular posets or systems of fuzzy sets. In this paper, we define a topological D-poset as a D-poset with a topology guaranteeing the continuity of the difference operation, and a topological lattice D-poset as a lattice D-poset with a topology guaranteeing the continuity of the difference operation and lattice operations. If these topologies are uniform and the operations are uniformly continuous, we speak of uniform D-posets and uniform lattice D-posets. In the paper, several examples of uniform D-posets are exhibited. The main result is the theorem asserting that the topological completion of a uniform Hausdorff lattice D-poset in which all monotone nets are Cauchy is also a uniform Hausdorff lattice D-poset, which is a complete lattice. This is the generalization of a known result for orthomodular lattices ([14]).

1. Introduction

In recent decades, many extensions of Kolmogoroff axiomatics were introduced. After Boolean algebras, there followed quantum logics, orthomodular lattices and fuzzy sets. Several years ago, orthoalgebras were defined (see [1]), and the most recent notion is that of D-posets (see [6], [7]), which include all the previously mentioned structures.

DEFINITION 1.1. A difference poset (briefly D-poset) is a quadruple $(D, \leq, \ominus, 1)$, where $D$ is a nonempty set partially ordered by $\leq$, 1 is the largest element of $D$, and $\ominus$ is the difference operation which defines for every $a, b \in D$, $a \leq b$, an element $b \ominus a$ in such a way that the following conditions are true:

i) $b \ominus a \leq b$,

ii) $b \ominus (b \ominus a) = a$,

iii) $a \leq b \leq c$ implies $c \ominus b \leq c \ominus a$, and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

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It can easily be seen that in any D-poset, \(0 = 1 \ominus 1\) is the smallest element of \(D\).

**Definition 1.2.** An orthomodular poset (OMP) is a triple \((P, \leq, \bot)\), where \(P\) is a nonempty set partially ordered by \(\leq\), possessing the largest element 1 and the smallest element 0, and \(\bot: P \rightarrow P\) is a map with properties:

i) \(a \leq b\) implies \(b \bot \leq a \bot\),

ii) \((a \bot) \bot = a\),

iii) \(a \lor a \bot = 1\),

iv) \(a \leq b\) implies \(b = a \lor (b \land a \bot)\).

Two elements \(a, b\) of \(P\) are called orthogonal (written \(a \bot b\)) if \(a \leq b \bot\). For \(a, b\) orthogonal, there exists \(a \lor b\) in \(P\). An OMP which is a \(\sigma\)-lattice is called quantum logic.

It is clear that every OMP becomes a D-poset if we put for every \(a, b \in P\),

\[a \leq b\] if \(a \leq b \bot\).

In the following, \(D\) denotes always a D-poset. Let us write \(G = \{(a, b) \in D \times D \mid a \leq b\}\). A net \(a_\alpha\) of elements of \(D\) is called increasing (decreasing) if \(\alpha \leq \beta\) implies \(a_\alpha \leq a_\beta\) \((a_\alpha \geq a_\beta\)\). Increasing and decreasing nets are called monotone.

**Definition 1.3.** A function \(\mu: D \rightarrow R\) is called a signed measure if for every \(a, b \in D\), \(a \leq b\), \(\mu(b) = \mu(a) + \mu(b \ominus a)\). If \(\mu(a) \geq 0\) for every \(a \in D\), we say that \(\mu\) is a measure.

A set \(\mathcal{M}\) of signed measures on \(D\) is called separating if for every \(a, b \in D\), \(a \neq b\), there exists \(\mu \in \mathcal{M}\) such that \(\mu(a) \neq \mu(b)\).

**2. D-poset as a topological space**

It is well known in classical measure theory that, if \(\mu\) is a finite measure on the \(\sigma\)-algebra \(\mathcal{S}\) of subsets of some set \(X\), then the function \(q_\mu(A, B) = \mu(A \Delta B\), \(A, B \in \mathcal{S}\), \((A \Delta B = (A \setminus B) \cup (B \setminus A)\) is the symmetric difference of \(A\) and \(B\)\) is a pseudometric on \(\mathcal{S}\) (see [4]). Quantum logics as topological spaces were investigated, for example, in [8], [9], [10], [11], [12], [13] and [14]. Some considerations of these papers are extendable also to D-posets.

If a D-poset \(D\) with a topology \(\mathcal{T}\) form a topological space \((D, \mathcal{T})\), and \(\mathcal{T} \times \mathcal{T}\) is the usual product topology on \(D \times D\), let \(\mathcal{T}_0\) be the relative topology on \(G\) induced by \(\mathcal{T} \times \mathcal{T}\). If \(\mathcal{T}\) is an uniform topology induced by an uniformity \(\mathcal{U}\), \(\mathcal{U} \times \mathcal{U}\) is the product of uniformities, and \(\mathcal{U}_0\) is the relative uniformity on \(G\) induced by \(\mathcal{U} \times \mathcal{U}\), then, of course, \(\mathcal{U}_0\) induces \(\mathcal{T}_0\).
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**DEFINITION 2.1.** \((D, T)\) is called a topological \(D\)-poset if \(\Theta : (G, T_0) \to (D, T)\) is continuous. If \(T\) is induced by the uniformity \(\mathcal{U}\), then \((D, \mathcal{U})\) is called a uniform \(D\)-poset if \(\Theta : (G, U_0) \to (D, \mathcal{U})\) is uniformly continuous.

If \(T\) is uniform, we do not distinguish in the notation between \((D, T)\) and \((D, U)\).

It is obvious that the discrete topology on any \(D\) forms a Hausdorff uniform \(D\)-poset.

The following lemma is routine.

**LEMMA 2.2.** Let \(B\) be a prebase of \(\mathcal{U}\). Then \((D, \mathcal{U})\) is a uniform \(D\)-poset if and only if for every \(U \in B\) there exists \(V \in \mathcal{U}\) such that \((x_1, x_2) \in V\), \((y_1, y_2) \in V\), \(x_1 \leq y_1\), \(x_2 \leq y_2\) implies \((y_1 \ominus x_1, y_2 \ominus x_2) \in U\).

We exhibit several uniform \(D\)-posets. All of them are Hausdorff.

**EXAMPLE 1.** If \(S\) is the \(\sigma\)-algebra of subsets of \(X\), \(\mu\) a finite measure on \(S\), let us define an equivalence relation on \(S\) via: \(A \sim B\) if \(\mu(A \triangle B) = 0\). Let us denote by \(\mathcal{S}\) the system of all equivalence classes \([A]\), \(A \in S\), and by \(\leq\), the partial ordering on \(\mathcal{S}\), where \([A] \leq [B]\) if \(A_1 \subset B_1\) for some \(A_1 \in [A]\), \(B_1 \in [B]\). If we define the orthocomplementation \(\perp\) on \(\mathcal{S}\): \([A] \perp = [X \setminus A]\), then \(\mathcal{S}\) becomes a Boolean algebra and, hence, a \(D\)-poset. If we define the metric \(\varrho_\mu\) on \(\mathcal{S}\) by \(\varrho_\mu([A], [B]) = \mu(A \triangle B)\), and \(\mathcal{T}_\mu\) is the topology induced by \(\varrho_\mu\), then \((\mathcal{S}, \mathcal{T}_\mu)\) is a uniform \(D\)-poset.

**EXAMPLE 2.** Let \(\mathcal{L}(H)\) be the set of all closed subspaces of the separable Hilbert space \(H\) (complex or real) with \(\dim H \geq 3\). If \(\mathcal{L}(H)\) is partially ordered by inclusion, and, for \(M \in \mathcal{L}(H)\), \(M \perp\) is the usual orthogonal complement of \(M\), then \(\mathcal{L}(H)\) is a complete orthomodular lattice. Then the sets \(U_\varphi, \varepsilon = \{(M, N) \in \mathcal{L}(H) \times \mathcal{L}(H) ; \|P_M \varphi - P_N \varphi\| < \varepsilon\}, \varphi \in H, \varepsilon > 0, (P_M\) denotes the orthogonal projector corresponding to \(M\)\) form a prebase of a uniformity \(\mathcal{U}\). Let us denote by \(\tau_{\text{strong}}\) the topology induced by \(\mathcal{U}\). We can also obtain \(\tau_{\text{strong}}\) as the relative topology induced by the strong topology on the space of all bounded linear operators operating from \(H\) to \(H\) (identifying closed subspaces with orthogonal projectors projecting on them). \((\mathcal{L}(H), \tau_{\text{strong}})\) is a uniform \(D\)-poset. Every increasing (decreasing) net \(M_\alpha\) converges in this space to \(M = \bigvee M_\alpha (\bigwedge M_\alpha)\).

**EXAMPLE 3.** If we define a metric \(d\) on \(\mathcal{L}(H)\) by \(d(M, N) = \|P_M - P_N\|\), \(M, N \in \mathcal{L}(H)\), where \(\|\cdot\|\) denotes the usual operator norm, and \(\tau_{\text{unif}}\) is the topology induced by \(d\), then \((\mathcal{L}(H), \tau_{\text{unif}})\) is uniform \(D\)-poset.

\(D\)-posets in Examples 4–9 are systems of fuzzy sets, i.e., systems of functions defined on some set \(A\) with values in the interval \((0, 1)\). In all these \(D\)-posets,
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\( \leq \) is defined via: \( f \leq g \) if \( f(t) \leq g(t) \) for every \( t \in A \). Then \((g \ominus f)(t) = g(t) - f(t)\), \( t \in A \). The largest element is the constant function equal to 1.

On these D-posets, we can define two topologies in a natural way. The first one is the uniform topology of pointwise convergence, where \( f_\alpha \to f \) if and only if \( f_\alpha(t) \to f(t) \) for every \( t \in A \). Let us denote it by \( \tau_{pc} \). The second is the topology induced by the metric \( d(f,g) = \sup\{|f(t) - g(t)|; t \in A\} \), denoted by \( \tau_{sup} \). Then in Examples 4–9, we have further Hausdorff uniform D-posets.

**Example 4.** \((D, \tau_{pc})\), where \( D \) is the set of all functions defined on an arbitrary set \( A \) with values in \( \langle 0, 1 \rangle \).

**Example 5.** \((D, \tau_{sup})\), where \( D \) is the same as in E4.

**Example 6.** \((D, \tau_{pc})\), where \( D \) is the set of all continuous functions defined on \( \langle 0, 1 \rangle \) with values in \( \langle 0, 1 \rangle \).

**Example 7.** \((D, \tau_{sup})\), where \( D \) is the same as in E6.

**Example 8.** \((D, \tau_{pc})\), where \( D \) is the set of all convergent sequences with values in \( \langle 0, 1 \rangle \).

**Example 9.** \((D, \tau_{sup})\), where \( D \) is the same as in E8.

**Example 10.** Let \( D \) be the subset of the normed space \( L^p(\langle 0, 1 \rangle) \), \( p \geq 1 \), such that \( [f] \in D \) if \( 0 \leq f_1(t) \leq 1 \), \( t \in \langle 0, 1 \rangle \), for some \( f_1 \in [f] \). For \( [f],[g] \in D \), \( [f] \leq [g] \) if \( f_1(t) \leq g_1(t) \), \( t \in \langle 0, 1 \rangle \), for some \( f_1 \in [f] \), \( g_1 \in [g] \). If for \( [f] \leq [g] \), \( [g] \ominus [f] = [g - f] \), then \( D \) is a D-poset. If \( T \) is the topology induced by the metric \( d([f],[g]) = \left( \frac{1}{0} \int |f(t) - g(t)|^p dt \right)^{\frac{1}{p}} \), then \((D,T)\) is a uniform D-poset.

On any D-poset with a separating set of signed measures, it is possible to define some nontrivial Hausdorff uniform topologies.

If \( \mathcal{M} \) is a separating set of signed measures on \( D \), let \( \tau(\mathcal{M}) \) be the uniform topology with a prebase containing sets \( U_{m, \varepsilon} = \{(a,b) \in D \times D; |m(a) - m(b)| < \varepsilon\} \), \( \varepsilon > 0 \), \( m \in \mathcal{M} \). Let \( T(\mathcal{M}) \) be the topology induced by the metric \( \varrho_{\mathcal{M}}(a,b) = \sup\{|m(a) - m(b)|; m \in \mathcal{M}\} \).

**Theorem 2.3.** If \( \mathcal{M} \) is a separating set of signed measures on \( D \), then \((D, \tau(\mathcal{M}))\) and \((D, T(\mathcal{M}))\) are Hausdorff uniform D-posets.

All uniform D-posets in Examples 1–10 are equal to \((D, \tau(\mathcal{M}))\) or \((D, T(\mathcal{M}))\) for some \( \mathcal{M} \).

**Proof.** If \((a_1,a_2), (b_1,b_2) \in U_{m, \frac{\varepsilon}{3}}, a_1 \leq b_1, a_2 \leq b_2, \) then \((b_1 \ominus a_1, b_2 \ominus a_2) \in U_{m, \varepsilon} \). Hence, by Lemma 2.2, \((D, \tau(\mathcal{M}))\) is a uniform D-poset. Similarly, if \( \varrho_{\mathcal{M}}(a_1,a_2) < \frac{\varepsilon}{3}, \varrho_{\mathcal{M}}(b_1,b_2) < \frac{\varepsilon}{3}, a_1 \leq b_1, a_2 \leq b_2, \) then \( \varrho_{\mathcal{M}}(b_1 \ominus a_1, b_2 \ominus a_2) < \varepsilon \).
Hence, \((D, T(\mathcal{M}))\) is a uniform D-poset. Since \(\mathcal{M}\) is separating, \(\tau(\mathcal{M})\) and \(T(\mathcal{M})\) are Hausdorff.

Let us prove that all topologies in Examples 1–10 are special cases of \(\tau(\mathcal{M})\) or \(T(\mathcal{M})\).

E1. \(\overline{T}_\mu\) is equal to \(\tau(\mathcal{M})\), where \(\mathcal{M}\) is the family of all measures \(m_A\) on \(\overline{S}\) of the form \(m_A([E]) = \mu(A \cap E), [E] \in \overline{S}, A \in S\).

E2. A measure \(\mu\) on \(\mathcal{L}(H)\) is called Gleason measure if \(\mu\) is of the form \(\mu(M) = \text{tr} TP^M, m \in \mathcal{L}(H)\) (\(\text{tr} TP^M\) is the trace of \(TP^M\)), where \(T\) is a nonnegative hermitean trace class operator. Let \(\mathcal{M}\) be the set of all Gleason measures \(\mu\) such that \(\mu(H) = 1\). It was proved in [8] that \(\tau_{\text{strong}} = \tau(\mathcal{M})\).

E3. Let \(\mathcal{M}\) be the same as in E2. It was proved in [2] that \(\|P^M - P^N\| = \sup\{|m(M) - m(N)|; m \in \mathcal{M}\}\). Hence, \(\tau_{\text{unif}} = T(\mathcal{M})\).

E4, E6, E8. Every topological D-poset \((D, T)\) in these examples is a topological subspace of some linear norm space \(X\). In E5, \(X\) is the space of all bounded real functions defined on \(A\), in E7, \(X\) is the space of all real continuous functions defined on \((0,1)\), in E9, \(X\) is the space of all real convergent sequences. The norm of \(X\) in these examples is the usual supremum norm. In E10, \(X\) is the space \(L^p((0,1))\) with the usual norm. Let \(X'\) be the dual space to \(X\), i.e., the space of all bounded functionals defined on \(X\), and \(X'' = (X')'\) be the second dual space. If \(J: X \rightarrow X''\) is the canonical mapping, i.e., for \(x \in X\), \(Jx = x''\), where \(x''(x') = x'(x), x' \in X'\), then \(\|x\| = \|Jx\|\) (see [15]). Hence, we have

\[
\|x\| = \|Jx\| = \sup\{|x''(x')|; x' \in X', \|x'\| \leq 1\}
= \sup\{|x'(x)|; x' \in X', \|x'\| \leq 1\}.
\]

Then for every net \(a_\alpha \in D, a \in D\),

\[
\|a_\alpha - a\| = \sup\{|x'(a_\alpha - a)|; x' \in X', \|x'\| \leq 1\}
= \sup\{|x'(a_\alpha) - x'(a)|; x' \in X', \|x'\| \leq 1\},
\]

and, hence, \(a_\alpha \rightarrow a\) in \((D, T)\) if and only if \(a_\alpha \rightarrow a\) in \((D, T(\mathcal{M}))\), where \(\mathcal{M}\) contains restrictions of all bounded linear functionals \(x', \|x'\| \leq 1\), from \(X\) to \(D\). Hence, \(T = T(\mathcal{M})\).

\[\square\]

3. Uniform lattice D-posets

If \((D, T)\) is a topological D-poset and \(D\) is a lattice, then the continuity of the lattice operations \(\lor\) and \(\land\) is not guaranteed, in general. \((\mathcal{L}(H), \tau_{\text{strong}})\)
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and \((L(H), \tau_{\text{unif}})\) are uniform D-posets, but \(\lor\) and \(\land\) are not continuous. For a nonzero vector \(\varphi \in H\), let \([\varphi]\) be the one dimensional subspace generated by \(\varphi\). If \(\varphi_n \to \varphi\), \(\psi_n \to \psi\), \(\|\varphi_n\| = \|\psi_n\| = \|\varphi\| = 1\) and \((\varphi_n, \psi_n) = 0\), \(n = 1, 2, \ldots\), then \([\varphi_n] \to [\varphi]\), \([\psi_n] \to [\psi]\) in \((L(H), \tau_{\text{strong}})\) and in \((L(H), \tau_{\text{unif}})\) as well, but \([\varphi_n] \lor [\psi_n]\) does not converge to \([\varphi] \lor [\varphi] = [\varphi]\) in any of these topologies. However, on orthomodular posets, a topology giving a D-poset guarantees at least partial continuity of \(\lor\) (and hence, also of \(\land\)). The following lemma is true:

**Lemma 3.1.** If \(D\) is an OMP, then \((D, T)\) is topological D-poset if and only if the following conditions are true:

i) if \(a_\alpha \to a\) in \((D, T)\), then \(a_\alpha \to a_\alpha \lor a\) in \((D, T)\);

ii) if \(a_\alpha \to a\), \(b_\alpha \to b\) in \((D, T)\), \(a_\alpha \bot b_\alpha\), \(a \bot b\), then \(a_\alpha \lor b_\alpha \to a \lor b\).

**Proof.** For \(a \leq b\), \(b \alpha \ominus a = b \land a^\perp = (b^\perp \lor a)^\perp\), where \(a \bot b^\perp\). So, if i) and ii) are true, then \(\ominus\) is continuous. Conversely, \(a^\perp = 1 \ominus a\), and for \(a, b \in D\), \(a \bot b\), \(a \lor b = 1 \ominus ((1 \ominus a) \ominus b)\). Hence, the continuity of \(\ominus\) implies i), ii). \(\square\)

Orthomodular posets with topologies with properties i) and ii) were studied in [8] before D-posets were introduced.

In the following, we introduce topologies on lattice D-posets, which also guarantee the continuity of lattice operations. Such topologies were studied in the last decade on orthomodular lattices (see [10], [11], [12], [14]). In the following, \(D\) is assumed to be a lattice.

**Definition 3.2.** We say that \((D, T)\) is a topological lattice D-poset if

i) the mapping \(\ominus : (G, T_0) \to (D, T)\) is continuous,

ii) the mappings \(\lor, \land : (D \times D, T \times T) \to (D, T)\) are continuous.

If \(T\) is uniform, we say that \((D, T)\) is an uniform lattice D-poset if the mappings \(\ominus, \lor\) and \(\land\) are uniformly continuous.

The following lemma is routine.

**Lemma 3.3.** If \(T\) is induced by a uniformity \(U\), then \((D, T)\) is a uniform lattice D-poset if and only if for every \(U \in U\) there exists \(V \in U\) such that \((x_1, x_2), (y_1, y_2) \in V\) implies \((x_1 \lor y_1, x_2 \lor y_2) \in U\), \((x_1 \land y_1, x_2 \land y_2) \in U\), and, if moreover \(x_1 \leq y_1\) and \(x_2 \leq y_2\), then also \((y_1 \ominus x_1, y_2 \ominus x_2) \in U\).

It can be easily seen that D-posets in E1 and E4–E9 are uniform lattice D-posets.

It was proved in [14] that the topological completion of a Hausdorff uniform orthomodular lattice in which all monotone nets are Cauchy is also an orthomodular lattice, which is complete. As we shall see, this result is true also for D-posets.
Let \((D, \mathcal{U})\) be a uniform lattice \(D\)-poset. Two nets \(a_\alpha, b_\beta\) of elements of \(D\) are called equivalent \((a_\alpha \sim b_\beta)\) if for every \(U \in \mathcal{U}\) there exist indices \(\alpha_0, \beta_0\) such that \((a_\alpha, b_\beta) \in U\) for \(\alpha \geq \alpha_0, \beta \geq \beta_0\).

**Remark 3.4.** If \(a_\alpha \sim a'_\alpha, b_\alpha \sim b'_\alpha\), then \(a_\alpha \lor b_\alpha \sim a'_\alpha \lor b'_\alpha\) and \(a_\alpha \land b_\alpha \sim a'_\alpha \land b'_\alpha\). Specially, if \(a_\alpha \leq b_\alpha, b'_\alpha \leq a'_\alpha\), we obtain \(b_\alpha = b_\alpha \lor a_\alpha \sim b'_\alpha \lor a'_\alpha = a'_\alpha \sim a_\alpha\).

**Lemma 3.5.** Let \((D, \mathcal{U})\) be a uniform lattice \(D\)-poset. If \(a_\alpha, b_\alpha, b'_\alpha, c_\alpha\) are nets in \(D\) and \(a_\alpha \leq b_\alpha, b'_\alpha \leq c_\alpha, b_\alpha \sim b'_\alpha\), then there exist nets \(b''_\alpha \sim b_\alpha\) and \(c' \sim c_\alpha\) such that \(a''_\alpha \leq b''_\alpha \leq c'\).

**Proof.** Put \(b''_\alpha = b_\alpha \lor b'_\alpha \sim b''_\alpha \lor b'_\alpha \sim b_\alpha\). Then we put \(c' = c_\alpha \lor b''_\alpha \sim c_\alpha \lor b'_\alpha = c_\alpha\). Clearly, \(a''_\alpha \leq b''_\alpha \leq c'\).

We can embed every uniform space \((X, \mathcal{U})\) into a complete uniform space \((\hat{X}, \hat{\mathcal{U}})\) in a standard way (see [5]).

**Theorem 3.6.** If \((D, \mathcal{U})\) is a Hausdorff uniform lattice \(D\)-poset in which all monotone nets are Cauchy, then there exist extensions of \(\leq\) and \(\ominus\) on \(\hat{D}\) such that \((\hat{D}, \mathcal{U})\) is also a Hausdorff uniform lattice \(D\)-poset, and \(\hat{D}\) is a complete lattice.

**Proof.**

I. In this first part, we define extensions of \(\leq\) and \(\ominus\). (For the extension of the partial ordering we shall use the same symbol \(\leq\).) For \(a, b \in \hat{D}\) we put \(a \leq b\) if there exist nets \(\{x_\delta\} \subseteq \mathcal{U}, \{y_\delta\} \subseteq \mathcal{U}\) of elements of \(D\) such that \(x_\delta \leq y_\delta, x_\delta \rightarrow a, y_\delta \rightarrow b\). The reflexivity of \(\leq\) is clear, the antisymmetry follows from Remark 3.4, and Lemma 3.5 implies the transitivity. Obviously, this partial ordering is the extension of that on \(D\).

Let us define the difference operation \(\ominus\) on \(\hat{D}\). For \(a, b \in \hat{D}, a \leq b\), there exist nets \(x_\delta, y_\delta \in D, x_\delta \leq y_\delta, x_\delta \rightarrow a, y_\delta \rightarrow b\). The net \(y_\delta \ominus x_\delta\) is Cauchy, let us denote its limit by \(b \ominus a\). Clearly, \(\ominus\) is the extension of \(\ominus\), and the uniform continuity of \(\ominus\) implies that the definition of \(\ominus\) is correct.

We shall prove that \(\ominus\) is a difference operation. Obviously, \(y_\delta \ominus x_\delta \leq y_\delta\) implies \(b \ominus a \leq b\). Since \(y_\delta \ominus (y_\delta \ominus x_\delta) = x_\delta\) and \(y_\delta \ominus (y_\delta \ominus x_\delta) \rightarrow b \ominus (b \ominus a)\), we have \(b \ominus (b \ominus a) = a\). Let \(a \leq b \leq c\). By Lemma 3.5, there exist nets \(x_\delta \leq y_\delta \leq z_\delta\) converging to \(a, b, c\), respectively. Then \(z_\delta \ominus y_\delta \leq z_\delta \ominus x_\delta\) implies \(c \ominus b \leq c \ominus a\). Moreover, \((z_\delta \ominus x_\delta) \ominus (z_\delta \ominus y_\delta) = y_\delta \ominus x_\delta\), and this implies \((c \ominus a) \ominus (c \ominus b) = b \ominus a\). Hence, \(\ominus\) is a difference operation, and \((\hat{D}, \leq, \ominus, 1)\) is a \(D\)-poset.

Let us prove that \(\hat{D}\) is a lattice. If \(a, b \in \hat{D}\) are given, then there exist nets \(x_\delta, y_\delta\) of elements of \(D\) such that \(x_\delta \rightarrow a, y_\delta \rightarrow b\). Then \(x_\delta \lor y_\delta \) is Cauchy, and it converges to some \(c \in \hat{D}, a \leq c, b \leq c\). If \(d \in \hat{D}\) is given, \(a \leq d, b \leq d\),
then there exist nets $a_{\hat{U}} \to a$, $d_{\hat{U}} \to d$, $b_{\hat{U}} \to b$, $d'_{\hat{U}} \to d$, $a_{\hat{U}} \leq d_{\hat{U}}$, $b_{\hat{U}} \leq d'_{\hat{U}}$. Then $d''_{\hat{U}} = d_{\hat{U}} \lor d'_{\hat{U}} \to d$, $a_{\hat{U}} \lor b_{\hat{U}} \leq d''_{\hat{U}}$. Then $c \leq d$ and, hence, $c = a \lor b$. We can prove the existence of $a \land b$ similarly. $\hat{D}$ is a lattice.

II. In this step, we shall prove that $(\hat{D}, \hat{U})$ is uniform lattice D-poset. We shall use the fact that closures of all $U \in \hat{U}$ in the product space $\hat{D} \times \hat{D}$ form a base of $\hat{U}$. Let $\hat{U} \in \hat{U}$ be given, $\hat{U} = \bar{U}$ for some $U \in \hat{U}$. Since $(D, \mathcal{U})$ was a uniform lattice D-poset, by Lemma 3.3, there exists $V \in \mathcal{U}$ such that $(x_1, x_2)$, $(y_1, y_2) \in V$ implies $(x_1 \lor y_1, x_2 \lor y_2)$ and $(x_1 \land y_1, x_2 \land y_2) \in U$, and, if moreover $x_1 \leq y_1$, $x_2 \leq y_2$, then also $(y_1 \ominus x_1, y_2 \ominus x_2) \in U$.

Let $V_1 \in \mathcal{U}$, $V_1 \circ V_1 \circ V_1 \subset V$. If $(x_1, x_2) \in \bar{V}_1$, $(y_1, y_2) \in \bar{V}_1$, $x_1 \leq y_1$, $x_2 \leq y_2$, then there exist nets $(x_{\hat{U}}^1, x_{\hat{U}}^2) \in V_1$, $(y_{\hat{U}}^1, y_{\hat{U}}^2) \in V_1$ converging to $(x_1, x_2)$ and $(y_1, y_2)$ in $\hat{D} \times \hat{D}$. By the definition of partial ordering in $\hat{D}$, there exist nets $x_{\bar{U}}^1, x_{\bar{U}}^2 \in \bar{D}$ converging to $x_1, x_2$, and nets $y_{\bar{U}}^1, y_{\bar{U}}^2 \in \bar{D}$ converging to $y_1, y_2$ such that $x_{\bar{U}}^1 \leq y_{\bar{U}}^1$ and $x_{\bar{U}}^2 \leq y_{\bar{U}}^2$. Then $y_{\bar{U}}^1 \sim y_{\bar{U}}^1 \lor y_{\bar{U}}^2 = z_{\bar{U}}^1$, and, starting from some index, we have $(y_{\bar{U}}^1, z_{\bar{U}}^1) \in V_1$. Similarly, if $z_{\bar{U}}^2 = y_{\bar{U}}^2 \lor y_{\bar{U}}^2$, we have $(y_{\bar{U}}^2, z_{\bar{U}}^2) \in V_1$, starting from a certain index. Then $(z_{\bar{U}}^1, z_{\bar{U}}^2) \in \bar{V}$, starting from a certain index. At the same time, we have $(x_{\bar{U}}^1, x_{\bar{U}}^2) \in \bar{V}$ and $x_{\bar{U}}^1 \leq z_{\bar{U}}^1$, $x_{\bar{U}}^2 \leq z_{\bar{U}}^2$. This implies $(z_{\bar{U}}^1 \ominus x_{\bar{U}}^1, z_{\bar{U}}^2 \ominus x_{\bar{U}}^2) \in \hat{U}$. Then $(y_{\bar{U}} \ominus x_1, y_2 \ominus x_2) \in \hat{U}$.

Similarly, for any $(x_1, x_2) \in \bar{V}$, $(y_1, y_2) \in \bar{V}$, we have $(x_1 \lor y_1, x_2 \lor y_2)$ and $(x_1 \land y_1, x_2 \land y_2) \in \bar{U}$. Hence, $(\hat{D}, \hat{D}, \hat{D}, \hat{D}, \hat{D})$ is a Hausdorff uniform D-poset.

III. The proof of the completeness of $\hat{D}$ does not differ from the case where $D$ is an OML ([14]). First, we show that $\bigvee a_n$ exists in $\hat{D}$ for every increasing sequence $a_n \in \hat{D}$. If $a_n$ is given, let us prove that it is Cauchy. For $\hat{W} \in \hat{U}$ there exists a sequence $\hat{U}_n \in \hat{U}$ with the properties:

1) $\hat{U}_1 \circ \hat{U}_1 \circ \hat{U}_1 \subset \hat{W}$,

2) $(x_1, x_2), (y_1, y_2) \in \hat{U}_{n+1}$ implies $(x_1 \lor y_1, x_2 \lor y_2) \in \hat{U}_n$.

For every $n$ natural, there exists a net $a^n_\alpha \in \hat{D}$ converging to $a_n$. For $n$ there exists $\alpha_n$ such that $(a^n_\alpha, a_n) \in \hat{U}_{n+1}$. If we put $b_k = \bigvee \limits_{n=1}^k a^n_\alpha$, then $(b_k, a_k) \in \hat{U}_1, k = 1, 2, \ldots$. Since $b_k$ is Cauchy, we have $(a_n, a_m) \in \hat{U}_1 \circ \hat{U}_1 \circ \hat{U}_1 \subset \hat{W}$, starting from some index. Hence, $a_n$ is Cauchy in $\hat{D}$. Then there exists $a \in \hat{D}$, $a_n \to a$ in $(\hat{D}, \hat{U})$, and this implies $a = \bigvee a_n$. Hence, for every sequence $a_n$ (not only increasing) there exists $\bigvee a_n$ in $\hat{D}$.

Let us assume that $\hat{D}$ is not complete. Let $\alpha_0$ be the smallest ordinal number such that there exists $M \subset \hat{D}$ such that $M = \{ a_\alpha \}_{\alpha < \alpha_0}$, and $\bigvee M$ does not exist. For every $\alpha < \alpha_0$ put $b_\alpha = \bigvee \limits_{\beta \leq \alpha} a_\beta$ (by assumption, $b_\alpha$ exists). The net $b_\alpha$ is increasing. If it were not Cauchy, a non-Cauchy increasing subsequence
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of \( b_\alpha \) would exist, which is not possible. Hence, \( b_\alpha \) is Cauchy, and there exists \( b \in \hat{D}, b_\alpha \to b \), and this implies \( b = \bigvee b_\alpha = \bigvee M \), a contradiction. Hence, \( \hat{D} \) is a complete lattice.

Theorem is proved. □

**EXAMPLE 3.7.** In the Hausdorff uniform lattice D-poset \((D, \tau_{pc})\) from Example 8 all monotone sequences of elements of \( D \) are Cauchy. Then its completion \((\hat{D}, \hat{U})\) is the uniform lattice D-poset of Example 4, where \( A \) is the set of all natural numbers.

**REFERENCES**


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