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ON RADICALS OF THE SEMIGROUP OF TRIANGULAR MATRICES

FRANTIŠEK KMEŤ

Let $S$ be the multiplicative semigroup of all $m \times m$ upper triangular matrices over a ring $T$, i.e. matrices of the form

$$A = \begin{bmatrix}
a_1 & a_{12} & \cdots & a_{1m} \\
0 & a_{22} & \cdots & a_{2m} \\
0 & \cdots & 0 & a_{mm}
\end{bmatrix}$$

where $a_{ik} \in T$, $i, k = 1, 2, \ldots, m$ and $a_{ik} = 0$ for $i > k$.

Let $U$ be a semigroup and $J$ be a two-sided ideal of $U$. Denote by $R_J(U)$, $M_J(U)$, $L_J(U)$, $R^*(U)$, $C_J(U)$ and $N_J(U)$ respectively the radical of Schwarz, McCoy, Ševrín, Clifford, Luh and the set of all nilpotent elements of $U$ with respect to $J$.

R. Šulka [5, Theorem 7] proved that in a commutative semigroup $U$ we have:


The same results were obtained by J. Kuczkwowski [2] for $C_2$-semigroups and by H. Lai [3] for quasi-commutative semigroups.

J. Bosák [1, Theorem 2] proved that for the radicals of an arbitrary semigroup $U$ we have:

$$R_J(U) \subseteq M_J(U) \subseteq L_J(U) \subseteq R^*(U) \subseteq N_J(U) \subseteq C_J(U).$$

The purpose of this paper is to prove that in the semigroup $S$ of all $m \times m$ upper triangular matrices over a commutative ring $T$ we have: $R(S) = M(S) = L(S) = R^*(S) = N(S)$.

We introduce some definitions. Let $U$ be a semigroup with a zero $O$ and all ideals considered in the following two-sided.

An element $x \in U$ is called nilpotent (nilpotent with respect to $J$) if for some positive integer $n$: $x^n = O(x^n \in J)$.

An ideal (subsemigroup) $I$ of $U$ is called nilpotent (nilpotent with respect to $J$) if for some positive integer $n$: $I^n = O$ ($I^n \subseteq J$).
An ideal $P$ of $U$ is called prime if for any two ideals $A$ and $B$ of $U$, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

An ideal of $U$, each element of which is nilpotent (nilpotent with respect to $J$), is called a nilideal (nilideal with respect to $J$).

An ideal $I$ of $U$, each finitely generated subsemigroup of which is nilpotent (with respect to $J$), is called locally nilpotent.

The set of all nilpotent elements of $U$ (with respect to $J$) will be denoted by $N(U)$ ($N_J(U)$).

The union $R(U)$ ($R_J(U)$) of all nilpotent ideals of $U$ (with respect to $J$) is called the Schwarz radical of $U$ (with respect to $J$).

The union $L(U)$ ($L_J(U)$) of all locally nilpotent ideals of $U$ (with respect to $J$) is called the Ševrin radical of $U$ (with respect to $J$).

The intersection $M(U)$ ($M_J(U)$) of all prime ideals of $U$ (which contain $J$) is called the McCoy radical of $U$ (with respect to $J$).

The union $R^*(U)$ ($R_J^*(U)$) of all nilideals of $U$ (with respect to $J$) is called the Clifford radical of $U$ (with respect to $J$).

Denote by $N_0$ the set of all matrices of $S$ with a zero diagonal.

**Lemma 1.** Let $S$ be the semigroup of all $m \times m$ upper triangular matrices over a ring $T$. Then $N_0$ is a nilpotent ideal of $S$ and $N_0^m = O$.

**Proof.** Let $A \in N_0$, $B, C \in S$ be arbitrary matrices. Then $BA$, $AC$ are matrices with the zero diagonal, i.e. $BA \in N_0$, $AC \in N_0$ and hence $N_0$ is an ideal of $S$. The set $N_0^0$ is evidently an ideal of $S$. If $A, B \in N_0$ are two arbitrary matrices, then in the $n$th row ($n = 1, 2, \ldots, m - 1$) of the matrix $AB$ the first possible non-zero element is equal to $a'_{n,n+2} = a_{n,n+1}b_{n+1,n+2}$. Therefore the ideal $N_0^2$ is contained in the set of matrices of the form:

$$
\begin{pmatrix}
0 & 0 & a_{13} & \cdots & a_{1m} \\
0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & a_{m-2,m} \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$

Similarly, the ideal $N_0^s$ for $s = 3, \ldots, m - 1$ is contained in the set of matrices of the form:

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & a_{1,s+1} & \cdots & a_{1m} \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & 0 & a_{m-s,m} \\
0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0
\end{pmatrix}
$$

and so $N_0^m = O$. 366
Lemma 2. Let $S$ be the semigroup of all upper triangular $m \times m$ matrices over a commutative ring $T$ and let $A$ be a matrix of $S$. Then $A$ is a nilpotent matrix if and only if all its diagonal elements $a_{11}, \ldots, a_{mm}$ are nilpotent elements of $T$.

Proof. If $A$ is a nilpotent matrix of $S$ and $A^r = O$ for a positive integer $r$, then we have $a_{11}^r = a_{22}^r = \ldots = a_{mm}^r = 0$. Conversely, if $a_{11}^r = a_{22}^r = \ldots = a_{mm}^r = 0$, then $A^r$ with $r = \max (r_1, r_2, \ldots, r_m)$ is a matrix with a zero diagonal, so $A^r \in N_0$ and hence by Lemma 1 we have $A^m = O$.

Lemma 3. Let $S$ be the semigroup of all upper triangular $m \times m$ matrices over a commutative ring $T$. Then the set $N(S)$ is a nilideal of $S$ and $R^*(S) = N(S)$.

Proof. We shall show that the set $N(S)$ of all nilpotent matrices of $S$ is an ideal of $S$.

Let $A \in N(S)$, $B, C \in S$ be arbitrary matrices and let $a_{11}^r = a_{22}^r = \ldots = a_{mm}^r = 0$. Then the diagonal elements $(b_{11}a_{11}^r), \ldots, (b_{mm}a_{mm}^r)$ of the matrix $(BA)^r$ are equal to zero and so $(BA)^r \in N_0$. But $(BA)^r \in N_0$ implies by Lemma 1 that $(BA)^m = O$ and hence $BA \in N(S)$. Analogously $(AC)^r \in N_0$ implies $AC \in N(S)$.

Then from the definition of the Clifford radical we obtain $N(S) \subseteq R^*(S)$. Conversely $R^*(S) \subseteq N(S)$ is true for any semigroup with zero and therefore $N(S) = R^*(S)$ holds.

Remark. In the semigroup $S$ of all upper triangular $m \times m$ matrices over a non-commutative ring the set $N(S)$ in general does not form an ideal and only $R^*(S) \subseteq N(S)$ is true.

For example, let $K_2$ be the ring of all $2 \times 2$ matrices over a commutative ring $K$ and let $U$ be the multiplicative semigroup of all $m \times m$ triangular matrices over $K_2$.

Consider the matrices

$$C_1 = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \in K_2, \quad C_2 = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in K_2,$$

with $c^2 = c \neq 0$, $c \in K$. Then the product $AB$ of two nilpotent matrices

$$A = \begin{pmatrix} C_1 & 0 & \ldots & 0 \\ 0 & C_1 & \ldots & 0 \\ 0 & 0 & \ldots & C_1 \end{pmatrix}, \quad B = \begin{pmatrix} C_2 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ 0 & 0 & \ldots & C_2 \end{pmatrix}, \quad (0 \in K_2)$$

of $U$ is equal to the matrix

$$AB = \begin{pmatrix} C & 0 & \ldots & 0 \\ 0 & C & \ldots & 0 \\ 0 & 0 & \ldots & C \end{pmatrix}, \quad 0 \in K_2, \quad C = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix},$$

and $AB \in N(U)$. The set $N(U)$ of all nilpotent matrices of the semigroup $U$ cannot be an ideal of $U$ and therefore by [1] we have only $R^*(U) \subseteq N(U)$. 367
Lemma 4. Let $S$ be the semigroup of all upper triangular $m \times m$ matrices over a commutative ring $T$. Then $R(S) = R^*(S)$.

Proof. In an arbitrary semigroup with zero we have $R(S) \subseteq R^*(S)$. We shall prove that $R^*(S) \subseteq R(S)$.

It is sufficient to show that for each $A \in R^*(S)$ the principal ideal $(A)$ is a nilpotent ideal of $S$. If $A \in R^*(S)$ and $A^r = 0$ for a positive integer $r$, then there exists a least positive integer $s \leq r$ such that $A^s$ is a matrix with a zero diagonal. Then an arbitrary matrix $C = B_1AB_2 \ldots B_nAB_{n+1}$ of the ideal $(A)'$ (where some of $B_i$ can be empty) is a matrix with a zero diagonal and hence $(A)' \subseteq N$. Since by Lemma 1 we have $(A)^m = 0$, this implies $(A) \subseteq R(S)$ and hence $A \in R(S)$.

From the inclusions between radicals in an arbitrary semigroup: $R(S) \subseteq M(S) \subseteq L(S) \subseteq R^*(S) \subseteq N(S)$, from Lemma 3 and Lemma 4 we obtain

Theorem. Let $S$ be the semigroup of all upper triangular $m \times m$ matrices over a commutative ring $T$. Then

$$R(S) = M(S) = L(S) = R^*(S) = N(S).$$

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РАДИКАЛЫ ПОЛУГРУППЫ ТРЕУГОЛЬНЫХ МАТРИЦ

Франишиек Кметь

Резюме

В мультипликативной полугруппе треугольных матриц над коммутативным кольцом радикалы Шварца, Маккойа, Шеврина, Клиффорда и Луга равны множеству всех нильпотентных элементов полугруппы.