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THE P. P. RING AND THE PIERCE SHEAF REPRESENTATION OF NON-COMMUTATIVE RINGS

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1. Introduction. A. Grothendieck and J. Dieudonne [6] showed that a commutative ring is isomorphic with the ring of sections of local rings. More sheaf representations of algebraic systems were given by J. Dauns and K. Hofmann [4], R. Pierce [11], J. Lambek [9], K. Koh [8] and others. A lot of applications of the sheaf representation theory have been found by R. Pierce [11], O. Villamayor and D. Zelinsky [14], A. Magid [10], F. DeMeyer [5], and G. Bergman [2]. We note that most applications are in homological aspect of commutative rings. For example, when R is a commutative ring with identity, G. Bergman ([2], Lemma 3.1) showed that R is a p.p. ring (that is, every principal ideal is projective) if and only if the stalks of the Pierce sheaf are integral domains and the support of each element is both open and closed in the Boolean spectrum of the ring. In the present paper, we ask what kind of rings R have stalks of the Pierce sheaf induced by R being domains (non-commutative), prime rings or semiprime rings. For a non-commutative ring R, G. Szeto [12] claimed that if R is a left almost hereditary ring in which the left annihilator of any element is the left annihilator of a central idempotent, then the stalks are domains. This fact will be studied for non-commutative p.p. rings. It will be shown that the class of p.p. rings in which the left annihilator of any element is the left annihilator of a central idempotent, is precisely the class of strongly p.p. rings (p.p. rings in which the left annihilator of any element is a two sided ideal), and that such a class of rings is also the class of rings in which the support of any element is both open and closed and the stalks of the Pierce sheaf are domains. Moreover, when the ring R does not have an identity, we shall show that if R is a left p.p. ring then every nilpotent element r is square 0 ($r^2=0$) in case either all idempotents of R are central or R is a stably p.p. ring (see Section 2 for definition). At the end of the paper, some topological characterizations shall be given in terms of the Boolean spectrum of the ring whose stalks are domains, prime rings or semiprime rings.

2. Preliminaries. Let R be a ring with identity 1, $B(R)$ the set of central idempotents of R. Then $B(R)$ is a Boolean algebra under the joint $e \vee f$ =

$= e+f-ef$, the meet $e\wedge f=ef$, and the complement of $e=1-e$ for all e and f in $B(R)$. Denote the Boolean spectrum of $B(R)$ by Spec $B(R)$. Now let us recall the Pierce sheaf of rings [11]: Let T be the disjoint union of rings R/xR for all x in Spec $B(R)$. Then each r in R induces a mapping f_r from Spec $B(R)$ to T such that $f_r(x)=\bar{r}$ in R/xR for each x in Spec $B(R)$. Since $\bigcap(xR)=0$ for all x in Spec $B(R)$, that $f_r(x)=\bar{0}$ for all x in Spec $B(R)$ implies that $r=0$. Thus the mapping $r\rightarrow f_r$ is one-to-one from R to the set of mappings: Spec $B(R)\rightarrow T$. Next, T can be topologized so that f_r is continuous. In fact, we take $\{f_r(\Gamma(e))/r$ in R and e in $B(R)\}$ as a system of basic open sets, where $\Gamma(e)$ are basic open sets for Spec $B(R)$. Then the map f_r is continuous. Thus, we have a sheaf (Pierce) T of rings R_x $(=R/xR)$ for x in Spec $B(R)$ [11], where a sheaf T of rings R_x (R/xR) for x in a topological space X ($=$ Spec $B(R)$) is a disjoint union of R_x such that (1) for each x in X, a ring R_x is given with identity 1_x, (2) $R_x\cap R_y=\emptyset$, a void set for $x\neq y$ in X, (3) the projection P from T to X maps r in R_x to x for each r, (4) a topology is imposed on T such that 1) if r is in T, there exists an open set U in T with r in U and $N\subset X$ such that P maps U homeomorphically on an open set N, 2) let $T+T$ denote $\{(r,s)/P(r) = P(s)\}$, with the product topology in $T\times T$, then the inverse map $r\rightarrow -r$, the addition map $(r,s)\rightarrow r+s$ and the product map $(r,s)\rightarrow rs$ are continuous, and 3) the constant map $x\rightarrow 1_x$ is continuous on X to T. The rings R_x are called *stalks* of the sheaf T. For a subset U of X, the collection of all continuous functions from U to T are called *sections* from U to T. Then one can show that R is isomorphic with the ring of sections of the sheaf T of R_x under the mapping $r\rightarrow f_r$.

An ideal I of R is called completely prime if R/I is a domain, the ring R is called reduced if it has no nonzero nilpotent elements, and R is called left (right) p.p. if each left (right) principal ideal is projective. We note that R is left p.p. if and only if the left annihilator $A(r)$ or an element r in R is the left annihilator $A(e)$ of some idempotent e. Such an idempotent e is called an associated idempotent of r, which is denoted by e_r. A *strongly* felt *p.p. ring* is a left p.p. ring such that $A(r)$ is an ideal for each r in R. A *stably left p.p. ring* is a left p.p. ring such that $A(e_r) = A(re_r)$ for each r in R. A ring R is called *almost reduced* if every nilpotent element r is square zero $(r^2=0)$.

Throughout, we assume that R is a ring, and that a p.p. ring means a left p.p. ring. The left annihilator of an element r is denoted by $A(r)$.

3. p.p. rings. In this section, we shall show that the following two classes of p.p. rings are almost reduced: (1) for each r in R, e_r is central, and (2) R is stably p.p.. Several characterizations of a strongly p.p. ring are then given. A condition is also obtained for a reduced ring being strongly p.p.. We begin with p.p. rings without identity.

Lemma 3.1. *Let R be a p.p. ring such that e_r is central for an r in R. Then $Rr\cap A(r)=\{0\}$.*

Proof. Let tr be in $A(r)$ for some t in R. Then $trr = 0$. Hence $tre_r = 0$. Since e_r is central, $tre_r = te_r r = 0$. But then $te_r e_r = te_r = 0$ (for $A(r) = A(e_r)$). Thus $tr = 0$.

Theorem 3.2. *Let R be a p.p. ring such that e_r is central for each r in R. Then, r is nilpotent if and only if $A(r) = R$. Consequently, R is almost reduced.*

Proof. The sufficiency is clear. Conversely, if $r = 0$, clearly, $A(r) = R$. If $r \neq 0$ such that $r^n = 0$ for some $n > 2$, then $r^{n-1} = r^{n-2}r$ is in $Rr \cap A(r)$. By Lemma 3.1, $r^{n-1} = 0$. Thus, by mathematical induction principle, $r^2 = 0$. Since e_r is central, $re_r = e_r r = 0$ $(r^2 = 0)$; and so $e_r e_r = e_r = 0$. This implies that $A(r) = A(e_r) = A(0) = R$.

Next we give another class of almost reduced p.p. rings.

Theorem 3.3. *Let R be a stably p.p. ring. Then, r is a nilpotent element of R if and only if $A(r) = R$. Consequently, R is almost reduced.*

Proof. The sufficiency is clear. For the necessity, let $r^n = 0$ with $n > 2$. Since R is stably p.p., $A(r) = A(re_r)$; and so $r^n = r^{n-1}r = 0$ implies $r^{n-1}e_r = 0$. That is, $r^{n-2}re_r = 0$. Hence $r^{n-2}e_r = 0$. Thus, by mathematical induction, $r^2 = 0$. This gives that $re_r = 0$. Therefore $A(r) = A(0) = R$.

We recall that R is a strongly p.p. ring if $A(r)$ is a two sided ideal of R. When R has an identity 1, it can be shown that the class of strongly p.p. rings is precisely the class of p.p. rings R in which e_r is central for each r in R. Moreover, strongly p.p. rings can be characterized in terms of the Pierce sheaf, and this characterization is a non-commutative generalization as given by G. Bergman ([2], Lemma 3.1). From now on, we assume that R always has an identity 1.

Theorem 3.4. *The following statements are equivalent:*

(1) *R is a strongly p.p. ring.*

(2) *R is p. p. in which e_r is central for each r in R.*

(3) *For each r in R,* supp (r) *is both open and closed in* Spec $B(R)$, *and R_x is a domain for all x in* Spec $B(R)$.

Proof. (1)$\to$(2). It suffices to show that all idempotents are central. Let e be an idempotent. Then $A(e) = R(1-e)$ which is an ideal by hypothesis, so $R(1-e)R = R(1-e)$. Similarly, Re is an ideal. Hence $R \cong Re \oplus R(1-e)$ as a direct sum of ideals. Since $1 = e + (1-e)$, for any r in Re, $r = re$ and $r = er + (1-e)r$. But $(1-e)Re = 0$, then $(1-e)re = (1-e)r = 0$. Thus $r = re = er$ for all r in Re. Noting that $eR(1-e) = 0$, we conclude that e is central.

(2)$\to$(1) is clear. For (2)$\to$(3), since e_r is central for each r in R, $R \cong Re_r \oplus R(1-e_r)$ in which e_r is an identity of Re_r. Also, $r = e_r r + (1-e_r)r = e_r r = re_r$ (for $(1-e_r)r = 0$), so r is not a right zero-divisor of Re_r. Noting that Spec $B(R) =$ Spec $B(Re_r) \cup$ Spec $B(R(1-e_r))$, we have supp $(r) =$ supp $(e_r) =$ Spec $B(Re_r)$, which is both open and closed in Spec $B(R)$. Next, we claim that

R_x is a domain. Let $r_x \neq 0_x$ in R_x. Then x is in supp (r) which is Spec $B(Re_r)$, where r is a preimage of r_x in R. From the decomposition of R, $R \cong Re_r \oplus R(1-e_r)$, we have known that r $(=re_r)$ is not a right zero-divisor of Re_r, so r_x is not a right zero-divisor of R_x $(=(Re_r)_x)$.

(3)$\rightarrow$(2). Since supp (r) is both open and closed for each r in R, supp (r) = = supp (e) = Spec $B(Re)$ for some e in $B(R)$. Since $R \cong Re \oplus R(1-e)$, r is in Re. Noting that R_x is a domain for each x in Spec $B(R)$, we have that r_x is not a zero-divisor in R_x for each x in Spec $B(Re)$; and so r is not a zero-divisor in Re. Thus $A(r) = A(e)$ for an e in $B(R)$.

We remark that the hypothesis that R_x are domains cannot be dropped in (3)$\rightarrow$(2). For example, let $R = Z/(p^2)$ for a prime integer p in the ring of integers Z. Then R is a ring with no idempotents but 0 and 1 and with $\bar{p}$ a zero-divisor. Hence it is not a p.p. ring.

Corollary 3.5. *If* supp (r) *is both open and closed for each* r *in* R, *and if* R *is a reduced ring such that* R_x *is a prime ring for each* x *in* Spec $B(R)$, *then* R *is strongly p.p..*

Proof. Since R_x is a prime ring, (xR) is a minimal prime ideal for each x. The ring R is reduced, so $(xR) = 0_{(xR)}$ which is the set, $\{r$ in $(xR) \mid$ there is some s not in (xR) with $res = 0\}$. Hence R_x is also reduced ([8]). Any prime reduced ring is a domain, so R_x is a domain for each x. Thus R is strongly p.p. by Theorem 3.4.

4. Topological characterisations. Theorems 3.4 and 3.5 characterize certain classes of rings whose stalks are domains or prime rings. In this section, we shall characterize the rings R whose stalks are domains, prime rings or semiprime rings in terms of the support of a set in Spec $B(R)$.

Theorem 4.1. *The stalks* R_x *are domains if and only if* sup $(r) \cap$ supp $(A(r)) = \emptyset$ *for each* r *in* R.

Proof. We first show that supp $(r) \cap$ supp $(A(r)) = \emptyset$, a void set. Let x be a point such that $\bar{r} \neq 0_x$ in R_x and $\bar{A}(r) \neq 0_x$. But $A(r)r = 0$, so $\bar{A}(r)\bar{r} = 0_x$ implies that either $\bar{A}(r) = 0_x$ or $\bar{r} = 0_x$ in R_x. This is a contradiction.

Conversely, let $\bar{s}\bar{r} = 0_x$ in R_x with $\bar{r} \neq 0_x$. Then there exists an idempotent e in $B(R)$ such that $esr = 0$ with $\bar{e} = 1_x$ in R_x. Hence (es) is in $A(r)$, and so $\bar{e}\bar{s} = 0_x$ (for x is in supp (r)). Thus $\bar{s} = \bar{e}\bar{s} = 0_x$. This implies that R_x is a domain for each x.

Theorem 4.2. *The stalks* R_x *are prime rings for all* x *in* Spec $B(R)$ *if and only if* supp $(r) \cap$ supp $(A(RrR)) = \emptyset$ *and* $\bar{A}(RrR) = A(\bar{R}\bar{r}\bar{R})$ *in* R_x *for each* x *and* r *in* R.

Proof. Assume R_x are prime rings. We claim that supp $(r) \cap$ supp $(A(RrR)) = \emptyset$ for any r in R. Let x be a point such that $\bar{r} \neq 0_x$ and $\bar{A}(RrR) \neq 0_x$ in R_x. Since

$(A(RrR))RrR=0$, there is some $s\neq 0$ in $A(RrR)$ such that $\bar{s}\neq 0_x$; and so $\bar{s}\bar{R}\bar{r}\bar{R}=0_x$. This contradicts to that R_x is a prime ring. Moreover, $\bar{A}(RrR)\subset A(\bar{R}\bar{r}\bar{R})$ is clear. Now let $\bar{s}\neq 0_x$ in $A(\bar{R}\bar{r}\bar{R})$ with s in R. Then $\bar{s}\bar{R}\bar{r}\bar{R}=0_x$. Since $\bar{R}$ $(=R_x)$ is a prime ring, either $\bar{s}=0_x$ or $\bar{r}=0_x$. Since $\bar{s}\neq 0_x$, we have $\bar{r}=0_x$. Thus we have some e in $B(R)$ such that $er=0$; and so $esRrR=0$. Therefore we have (es) such that $es(RrR)=0$ with $\bar{es}=\bar{s}$ in $A(\bar{R}\bar{r}\bar{R})$.

Conversely, let $\bar{s}\bar{R}\bar{r}=0_x$ in R_x for s and r in R, and x in $\operatorname{Spec} B(R)$. By hypothesis, $\bar{A}(RrR)=A(\bar{R}\bar{r}\bar{R})$, so there is an s' in $A(RrR)$ such that $\bar{s}'=\bar{s}$ in $A(\bar{R}\bar{r}\bar{R})$. Hence $s'RrR=0$. Assume $\bar{s}\neq 0_x$. Then $\bar{s}'=\bar{s}\neq 0_x$ implies that x is in $\operatorname{supp}(A(RrR))$. By hypothesis, x is not in $\operatorname{supp}(r)$, so $\bar{r}=0_x$. Assume $\bar{r}\neq 0_x$. By hypothesis again, $\operatorname{supp}(r)\cap\operatorname{supp}(A(RrR))=\emptyset$, $\bar{s}'=0_x=\bar{s}$. Therefore, R_x is a prime ring for each x.

For a ring finitely generated over its center as a ring, the lifting condition of the annihilator of a principal ideal ($\bar{A}(RrR) = A(\bar{R}\bar{r}\bar{R})$) can be dropped.

Corollary 4.3. *Let R be a ring finitely generated over its center as a ring. Then R_x are prime rings for all x in* $\operatorname{Spec} B(R)$ *if and only if* $\operatorname{supp}(r)\cap\operatorname{supp}(A(RrR))=\emptyset$ *for each r in R.*

Proof. By Theorem 4.2, it suffices to show that $\bar{A}(RrR) = A(\bar{R}\bar{r}\bar{R})$. $\bar{A}(RrR)\subset A(\bar{R}\bar{r}\bar{R})$ is clear. To show the other inclusion, let $\bar{s}$ be an element in $A(\bar{R}\bar{r}\bar{R})$ with s in R, and $\{r_1, \ldots, r_n\}$ a generating set for R. Then $\bar{s}\bar{r}_i\bar{r}\bar{r}_j=0_x$ for all i and j. By a basic property of sheaf theory, there is a basic open set $\Gamma(e)$ of $\operatorname{Spec} B(R)$ containing x such that the system of the above equations hold over $\Gamma(e)$ (where $\Gamma(e) = \{x \text{ in } \operatorname{Spec} B(R) \mid (1-e) \text{ is in } x\}$), so $esr_irr_j=0$ for all i and j. Hence $esRrR=0$ since $\{r_1, \ldots, r_n\}$ generate R over its center. So, (es) is in $A(RrR)$ such that $\bar{es}=\bar{s}$ in $A(\bar{R}\bar{r}\bar{R})$. This completes the proof.

It is not difficult to show the following characterization of a semiprime ring: R_x are semiprime for all x in $\operatorname{Spec} B(R)$ if and only if for each r in R, $rRr\subset xR$ implies that there exists some e in x such that $rRr\subset eR$.

We conclude the paper with a characterization of semiprime rings under a hypothesis on the lifting property of the annihilator of a principal ideal.

Theorem 4.4. *Assume $\bar{A}(EeE)=A(\bar{R}\bar{r}\bar{R})$ in R_x for each x in* $\operatorname{Spec} B(R)$ *and r in R. Then R is semiprime if and only if R_x is semiprime for each x in* $\operatorname{Spec}(B(R)$.

Proof. The sufficiency is clear. For the necessity, let $\bar{r}\neq 0_x$ in R_x for an x in $\operatorname{Spec} B(R)$. We claim $\bar{r}\bar{R}\bar{r}\neq 0_x$. Suppose not. We have $\bar{r}\bar{R}\bar{r}\bar{R}=0_x$. Hence $\bar{r}$ is in $A(\bar{R}\bar{r}\bar{R})$. By hypothesis, there is some s in $A(RrR)$ such that $\bar{s}=\bar{r}$. We then have an idempotent e in $B(R)$ and not in x such that $es=er$. Since $esRrR=0$, $erRerR=0$, and so $er=0$ by hypothesis. But $er=es\neq 0$, so this gives a contradiction. Thus R_x is a semiprime ring.

From the proof of Corollary 4.3, we see that $\bar{A}(RrR) = A(\bar{R}\bar{r}\bar{R})$ holds for a ring finitely generated over its center. Thus we have:

Corollary 4.5. *If R is a semiprime ring finitely generated over its center as a ring, then R_x is semiprime for each x in* Spec $B(R)$.

Remarks. 1. The sufficiency of Theorem 4.4 does not need the assumption that $\bar{A}(RrR) = A(\bar{R}\bar{r}\bar{R})$.

2. supp $(r) \cup$ supp $(RrR) =$ Spec $B(R)$ for any r in a ring R.

3. For reduced rings, it is easy to see that R is reduced if and only if so is R_x for each x in Spec $B(R)$.

4. There is a class of reduced rings called almost hereditary rings which are p.p. rings (see [12] and [13]).

REFERENCES

[1] ARENS, R.—KAPLANSKY, I.: Topological representation of algebras. Trans. Amer. Math. Soc., *63*, 1949, 457—481.

[2] BERGMAN, G.: Hereditary commutative rings and centres of hereditary rings. Proc. London Math. Soc., *23*, 1971, 214—236.

[3] BURGESS, W.—STEPHENSON, W.: Pierce sheaves of non-commutative rings. Communications in Algebra, *4*, 1976, 51—75.

[4] DAUNS, J.—HOFMANN, K.: The representation of biregular rings by sheaves. Math. Z., *91*, 1966, 103—123.

[5] DeMEYER, F.: Separable polynomials over a commutative ring. Roc. Mtn. J. Math. *2*, 1972, 299—310.

[6] GROTHENDIECK, A.—DIEUDONNE, J.: Elements de Geometrie Algebriquue, I.H.E.S. Publications Math. Paris 1960.

[7] HERSTEIN, I.: Topics in ring theory. The University of Chicago Press, 1969.

[8] KOH, K.: On functional representations of a ring without nilpotent elements. Canadian Math. Bull., *14*, 1971, 349—352.

[9] LAMBEK, J.: On the representation of modules by sheaves of factor modules. Canad. Math. Bull., *14*, 1971, 359—368.

[10] MAGID, A.: Pierce's representation and separable algebras. Illinois J. Math. *15*, 1971, 114—121.

[11] PIERCE, R.: Modules over commutative regular rings. Mem. Amer. Math. Soc., *70*, 1967.

[12] SZETO, G.: On almost hereditary rings. J. Algebra, *34*, 1975, 97—104.

[14] VASCONCELOS, W.: Finiteness of projective ideals. J. Algebra, *25*, 1973, 269—278.

[14] VILLAMAYOR, O.—ZELINSKY, D.: Galois theory for rings with infinitely many idempotents. Nagoya J. Math., *35*, 1969, 83—98.

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П. П. КОЛЬЦО И ПУЧКЫ ПИРСА НЕКОММУТАТИВНЫХ КОЛЕЦ

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Резюме

Кольцо называется п.п. кольцом, если каждый его правый идеал проективен. В работе изучаются пучки Пирса некоммутативных колец и особенно п.п. колец. В частности, характеризуются кольца K, у которых каждый слой K_x является областю целостности (первичным или полупервичным кольцом).