George Szeto; T. O. To

The p. p. ring and the Pierce sheaf representation of non-commutative rings


Persistent URL: [http://dml.cz/dmlcz/130418](http://dml.cz/dmlcz/130418)

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1980

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://project.dml.cz](http://project.dml.cz)
THE P. P. RING AND THE PIERCE SHEAF REPRESENTATION OF NON-COMMUTATIVE RINGS

GEORGE SZETO—T. O. TO

1. Introduction. A. Grothendieck and J. Dieudonne [6] showed that a commutative ring is isomorphic with the ring of sections of local rings. More sheaf representations of algebraic systems were given by J. Dauns and K. Hofmann [4], R. Pierce [11], J. Lambek [9], K. Koh [8] and others. A lot of applications of the sheaf representation theory have been found by R. Pierce [11], O. Villamayor and D. Zelinsky [14], A. Magid [10], F. DeMeyer [5], and G. Bergman [2]. We note that most applications are in homological aspect of commutative rings. For example, when $R$ is a commutative ring with identity, G. Bergman ([2], Lemma 3.1) showed that $R$ is a p.p. ring (that is, every principal ideal is projective) if and only if the stalks of the Pierce sheaf are integral domains and the support of each element is both open and closed in the Boolean spectrum of the ring. In the present paper, we ask what kind of rings $R$ have stalks of the Pierce sheaf induced by $R$ being domains (non-commutative), prime rings or semiprime rings. For a non-commutative ring $R$, G. Szeto [12] claimed that if $R$ is a left almost hereditary ring in which the left annihilator of any element is the left annihilator of a central idempotent, then the stalks are domains. This fact will be studied for non-commutative p.p. rings. It will be shown that the class of p.p. rings in which the left annihilator of any element is the left annihilator of a central idempotent, is precisely the class of strongly p.p. rings (p.p. rings in which the left annihilator of any element is a two sided ideal), and that such a class of rings is also the class of rings in which the support of any element is both open and closed and the stalks of the Pierce sheaf are domains. Moreover, when the ring $R$ does not have an identity, we shall show that if $R$ is a left p.p. ring then every nilpotent element $r$ is square $0$ ($r^2 = 0$) in case either all idempotents of $R$ are central or $R$ is a stably p.p. ring (see Section 2 for definition). At the end of the paper, some topological characterizations shall be given in terms of the Boolean spectrum of the ring whose stalks are domains, prime rings or semiprime rings.

2. Preliminaries. Let $R$ be a ring with identity $1$, $B(R)$ the set of central idempotents of $R$. Then $B(R)$ is a Boolean algebra under the joint $e \lor f =$
\[ e + f - ef, \text{ the meet } e \land f = ef, \text{ and the complement of } e = 1 - e \text{ for all } e \text{ and } f \text{ in } B(R). \]

Denote the Boolean spectrum of \( B(R) \) by \( \text{Spec } B(R) \). Now let us recall the Pierce sheaf of rings [11]: Let \( T \) be the disjoint union of rings \( R/xR \) for all \( x \) in \( \text{Spec } B(R) \). Then each \( r \) in \( R \) induces a mapping \( f_r \) from \( \text{Spec } B(R) \) to \( T \) such that \( f_r(x) = r \) in \( R/xR \) for each \( x \) in \( \text{Spec } B(R) \), that \( f_x = 0 \) for all \( x \) in \( \text{Spec } B(R) \) implies that \( r = 0 \). Thus the mapping \( r \rightarrow f_r \) is one-to-one from \( R \) to the set of mappings: \( \text{Spec } B(R) \rightarrow T \). Next, \( T \) can be topologized so that \( f_r \) is continuous. In fact, we take \( \{f_x(\Gamma(e))/r \text{ in } R \text{ and } e \text{ in } B(R)\} \) as a system of basic open sets, where \( \Gamma(e) \) are basic open sets for \( \text{Spec } B(R) \).

Next, \( T \) can be topologized so that \( f_r \) is continuous. In fact, we take \( \{f_x(\Gamma(e))/r \text{ in } R \text{ and } e \text{ in } B(R)\} \) as a system of basic open sets, where \( \Gamma(e) \) are basic open sets for \( \text{Spec } B(R) \).

Then the map \( f_r \) is continuous. Thus, we have a sheaf (Pierce) \( T \) of rings \( R/xR \) for \( x \) in \( \text{Spec } B(R) \) [11], where a sheaf \( T \) of rings \( R/xR \) for \( x \) in a topological space \( X (= \text{Spec } B(R)) \) is a disjoint union of \( R/xR \) such that (1) for each \( x \) in \( X \), a ring \( R_x \) is given with identity \( 1_x \), (2) \( R_x \cap R_y = 0 \), a void set for \( x \neq y \) in \( X \), (3) the projection \( P \) from \( T \) to \( X \) maps \( r \) in \( R_x \) to \( x \) for each \( r \), (4) a topology is imposed on \( T \) such that 1) if \( r \) is in \( T \), there exists an open set \( U \) in \( T \) with \( r \) in \( U \) and \( N \subset X \) such that \( P \) maps \( U \) homeomorphically on an open set \( N \), 2) let \( T + T \) denote \( \{(r, s)/P(r) = P(s)\} \), with the product topology in \( T \times T \), then the inverse map \( r \rightarrow -r \), the addition map \( (r, s) \rightarrow r + s \) and the product map \( (r, s) \rightarrow rs \) are continuous, and 3) the constant map \( x \rightarrow 1_x \) is continuous on \( X \) to \( T \). The rings \( R_x \) are called stalks of the sheaf \( T \). For a subset \( U \) of \( X \), the collection of all continuous functions from \( U \) to \( T \) are called sections from \( U \) to \( T \). Then one can show that \( R \) is isomorphic with the ring of sections of the sheaf \( T \) of \( R/xR \) under the mapping \( r \rightarrow f_r \).

An ideal \( I \) of \( R \) is called completely prime if \( R/I \) is a domain, the ring \( R \) is called reduced if it has no nonzero nilpotent elements, and \( R \) is called left (right) p.p. if each left (right) principal ideal is projective. We note that \( R \) is left p.p. if and only if the left annihilator \( A(r) \) or an element \( r \) in \( R \) is the left annihilator \( A(e) \) of some idempotent \( e \). Such an idempotent \( e \) is called an associated idempotent of \( r \), which is denoted by \( e_r \). A strongly felt p.p. ring is a left p.p. ring such that \( A(r) \) is an ideal for each \( r \) in \( R \). A stably left p.p. ring is a left p.p. ring such that \( A(e_r) = A(re_r) \) for each \( r \) in \( R \). A ring \( R \) is called almost reduced if every nilpotent element \( r \) is square zero \( (r^2 = 0) \).

Throughout, we assume that \( R \) is a ring, and that a p.p. ring means a left p.p. ring. The left annihilator of an element \( r \) is denoted by \( A(r) \).

3. p.p. rings. In this section, we shall show that the following two classes of p.p. rings are almost reduced: (1) for each \( r \) in \( R \), \( e \) is central, and (2) \( R \) is stably p.p.. Several characterizations of a strongly p.p. ring are then given. A condition is also obtained for a reduced ring being strongly p.p.. We begin with p.p. rings without identity.

**Lemma 3.1.** Let \( R \) be a p.p. ring such that \( e \) is central for an \( r \) in \( R \). Then \( Rr \cap A(r) = \{0\} \).

338
Proof. Let \( tr \) be in \( A(r) \) for some \( t \) in \( R \). Then \( trr = 0 \). Hence \( tre = 0 \). Since \( e_r \) is central, \( tre = te_r = 0 \) (for \( A(r) = A(e_r) \)). Thus \( tr = 0 \).

**Theorem 3.2.** Let \( R \) be a p.p. ring such that \( e_r \) is central for each \( r \) in \( R \). Then, \( r \) is nilpotent if and only if \( A(r) = R \). Consequently, \( R \) is almost reduced.

Proof. The sufficiency is clear. Conversely, if \( r = 0 \), clearly, \( A(r) = R \). If \( r \neq 0 \) such that \( r^n = 0 \) for some \( n > 2 \), then \( r^{n-1} = r^{n-2} r \) is in \( Rr \cap A(r) \). By Lemma 3.1, \( r^{n-1} = 0 \). Thus, by mathematical induction principle, \( r^2 = 0 \). Since \( e_r \) is central, \( re_r = e_r r = 0 \) (\( r^2 = 0 \)); and so \( e_r e_r = e_r = 0 \). This implies that \( A(r) = A(e_r) = A(0) = R \).

Next we give another class of almost reduced p.p. rings.

**Theorem 3.3.** Let \( R \) be a stably p.p. ring. Then, \( r \) is a nilpotent element of \( R \) if and only if \( A(r) = R \). Consequently, \( R \) is almost reduced.

Proof. The sufficiency is clear. For the necessity, let \( r^n = 0 \) with \( n > 2 \). Since \( R \) is stably p.p., \( A(r) = A(re_r) \); and so \( r^n r = 0 \) implies \( r^{n-1} e_r = 0 \). That is, \( r^{n-1} e_r = 0 \). Hence \( r^{n-2} e_r = 0 \). Thus, by mathematical induction, \( r^2 = 0 \). This gives that \( re_r = 0 \). Therefore \( A(r) = A(0) = R \).

We recall that \( R \) is a strongly p.p. ring if \( A(r) \) is a two sided ideal of \( R \). When \( R \) has an identity 1, it can be shown that the class of strongly p.p. rings is precisely the class of p.p. rings \( R \) in which \( e_r \) is central for each \( r \) in \( R \). Moreover, strongly p.p. rings can be characterized in terms of the Pierce sheaf, and this characterization is a non-commutative generalization as given by G. Bergman ([2], Lemma 3.1).

From now on, we assume that \( R \) always has an identity 1.

**Theorem 3.4.** The following statements are equivalent:

(1) \( R \) is a strongly p.p. ring.
(2) \( R \) is p. p. in which \( e_r \) is central for each \( r \) in \( R \).
(3) For each \( r \) in \( R \), \( \text{supp}(r) \) is both open and closed in \( \text{Spec} \ B(R) \), and \( R_r \) is a domain for all \( x \) in \( \text{Spec} \ B(R) \).

Proof. (1) \( \rightarrow \) (2). It suffices to show that all idempotents are central. Let \( e \) be an idempotent. Then \( A(e) = R(1 - e) \) which is an ideal by hypothesis, so \( R(1 - e)R = R(1 - e) \). Similarly, \( Re \) is an ideal. Hence \( R \cong Re \oplus R(1 - e) \) as a direct sum of ideals. Since \( 1 = e + (1 - e) \), for any \( r \) in \( Re \), \( r = re \) and \( r = er + (1 - e)r \). But \( (1 - e)Re = 0 \), then \( (1 - e)re = (1 - e)r = 0 \). Thus \( r = re = er \) for all \( r \) in \( Re \). Noting that \( eR(1 - e) = 0 \), we conclude that \( e \) is central.

(2) \( \rightarrow \) (1) is clear. For (2) \( \rightarrow \) (3), since \( e_r \) is central for each \( r \) in \( R \), \( R \cong Re_r \oplus R(1 - e_r) \) in which \( e_r \) is an identity of \( Re_r \). Also, \( r = e_r r + (1 - e_r)r = e_r r \) (for \( (1 - e_r)r = 0 \)), so \( r \) is not a right zero-divisor of \( Re_r \). Noting that \( \text{Spec} \ B(R) = \text{Spec} \ B(Re_r) \cup \text{Spec} \ B(R(1 - e_r)) \), we have \( \text{supp}(r) = \text{supp}(e_r) = \text{Spec} \ B(Re_r) \), which is both open and closed in \( \text{Spec} \ B(R) \). Next, we claim that
**Corollary 3.5.** If $\text{supp}(r)$ is both open and closed for each $r$ in $R$, and if $R$ is a reduced ring such that $R_x$ is a prime ring for each $x$ in $\text{Spec } B(R)$, then $R$ is strongly p.p.

**Proof.** Since $R_x$ is a prime ring, $(xR)$ is a minimal prime ideal for each $x$. The ring $R$ is reduced, so $(xR) = 0_{(xR)}$ which is the set, $(r \in (xR) | \text{there is some } s \not\in (xR) \text{ with } res = 0)$. Hence $R_x$ is also reduced ([8]). Any prime reduced ring is a domain, so $R_x$ is a domain for each $x$. Thus $R$ is strongly p.p. by Theorem 3.4.

### 4. Topological characterisations.

Theorems 3.4 and 3.5 characterize certain classes of rings whose stalks are domains or prime rings. In this section, we shall characterize the rings $R$ whose stalks are domains, prime rings or semiprime rings in terms of the support of a set in $\text{Spec } B(R)$.

**Theorem 4.1.** The stalks $R_x$ are domains if and only if $\text{supp}(r) \cap \text{supp}(A(r)) = \emptyset$ for each $r$ in $R$.

**Proof.** We first show that $\text{supp}(r) \cap \text{supp}(A(r)) = \emptyset$, a void set. Let $x$ be a point such that $\tilde{r} \neq 0_x$ in $R_x$ and $\hat{A}(r) \neq 0_x$. But $A(r)\tilde{r} = 0$, so $\hat{A}(r)\tilde{r} = 0$, implies that either $\hat{A}(r) = 0_x$ or $\tilde{r} = 0$ in $R_x$. This is a contradiction.

Conversely, let $\tilde{s}\tilde{r} = 0_x$ in $R_x$, with $\tilde{r} \neq 0_x$. Then there exists an idempotent $e$ in $B(R)$ such that $esr = 0$ with $\tilde{e} = 1_x$ in $R_x$. Hence $(es)$ is in $A(r)$, and so $\tilde{e}s = 0_x$ (for $x$ is in $\text{supp}(r)$). Thus $\tilde{s} = \tilde{e}s = 0_x$. This implies that $R_x$ is a domain for each $x$.

**Theorem 4.2.** The stalks $R_x$ are prime rings for all $x$ in $\text{Spec } B(R)$ if and only if $\text{supp}(r) \cap \text{supp}(A(RrR)) = \emptyset$ and $\hat{A}(RrR) = \hat{A}(R\tilde{rR})$ in $R_x$ for each $x$ and $r$ in $R$.

**Proof.** Assume $R_x$ are prime rings. We claim that $\text{supp}(r) \cap \text{supp}(A(RrR)) = \emptyset$ for any $r$ in $R$. Let $x$ be a point such that $\tilde{r} \neq 0_x$ and $\hat{A}(RrR) \neq 0_x$ in $R_x$. Since
(A(RrR))RrR = 0, there is some \( s \neq 0 \) in \( A(RrR) \) such that \( \hat{s} \neq 0 \); and so \( \hat{s} \hat{R} \hat{r} \hat{R} = 0 \). This contradicts to that \( R_\pi \) is a prime ring. Moreover, \( \hat{A}(RrR) \subset A(\hat{R} \hat{r} \hat{R}) \) is clear. Now let \( \hat{s} \neq 0 \) in \( A(\hat{R} \hat{r} \hat{R}) \) with \( s \) in \( R \). Then \( \hat{s} \hat{R} \hat{r} \hat{R} = 0 \). Since \( \hat{R} (= R_\pi) \) is a prime ring, either \( \hat{s} = 0 \) or \( \hat{r} = 0 \). Since \( \hat{s} \neq 0 \), we have \( \hat{r} = 0 \). Thus we have some \( \epsilon \) in \( B(R) \) such that \( \epsilon \hat{r} = 0 \); and so \( \epsilon s R r R = 0 \). Therefore we have \((\epsilon s)\) such that \( \epsilon s(RrR) = 0 \) with \( \epsilon \hat{s} = \hat{s} \) in \( A(\hat{R} \hat{r} \hat{R}) \).

Conversely, let \( \hat{s} \hat{R} \hat{r} = 0 \) in \( R_\pi \) for \( s \) and \( r \) in \( R \), and \( x \) in \( \text{Spec} B(R) \). By hypothesis, \( \hat{A}(RrR) = A(\hat{R} \hat{r} \hat{R}) \), so there is an \( s' \) in \( A(RrR) \) such that \( \hat{s}' = \hat{s} \) in \( A(\hat{R} \hat{r} \hat{R}) \). Hence \( s' \hat{R} \hat{r} = 0 \). Assume \( \hat{s}' \neq 0 \). Then \( \hat{s}' = \hat{s} \neq 0 \) implies that \( x \) is in \( \text{supp} (A(RrR)) \). By hypothesis, \( x \) is not in \( \text{supp} (r) \), so \( \hat{r} = 0 \). Assume \( \hat{r} \neq 0 \). By hypothesis again, \( \text{supp} (r) \cap \text{supp} (A(RrR)) = \emptyset \), \( \hat{s}' = 0 = \hat{s} \). Therefore, \( R_\pi \) is a prime ring for each \( x \).

For a ring finitely generated over its center as a ring, the lifting condition of the annihilator of a principal ideal \((\hat{A}(RrR) = A(\hat{R} \hat{r} \hat{R})\)) can be dropped.

**Corollary 4.3.** Let \( R \) be a ring finitely generated over its center as a ring. Then \( R_\pi \) are prime rings for all \( x \) in \( \text{Spec} B(R) \) if and only if \( \text{supp} (r) \cap \text{supp} (A(RrR)) = \emptyset \) for each \( r \) in \( R \).

**Proof.** By Theorem 4.2, it suffices to show that \( \hat{A}(RrR) = A(\hat{R} \hat{r} \hat{R}) \). \( \hat{A}(RrR) \subset A(\hat{R} \hat{r} \hat{R}) \) is clear. To show the other inclusion, let \( \hat{s} \) be an element in \( A(\hat{R} \hat{r} \hat{R}) \) with \( s \) in \( R \), and \( \{r_1, \ldots, r_n\} \) a generating set for \( R \). Then \( \hat{s} \hat{r}_i \hat{r}_j = 0 \) for all \( i \) and \( j \).

By a basic property of sheaf theory, there is a basic open set \( \Gamma(e) \) of \( \text{Spec} B(R) \) containing \( x \) such that the system of the above equations hold over \( \Gamma(e) \) (where \( \Gamma(e) = \{x \in \text{Spec} B(R) | (1 - e) \text{ is in } x\} \)), so \( e r_i r_j = 0 \) for all \( i \) and \( j \). Hence \( e R r R = 0 \) since \( \{r_1, \ldots, r_n\} \) generate \( R \) over its center. So, \( (\epsilon s) \) is in \( A(RrR) \) such that \( \epsilon \hat{s} = \hat{s} \) in \( A(\hat{R} \hat{r} \hat{R}) \). This completes the proof.

It is not difficult to show the following characterization of a semiprime ring: \( R_\pi \) are semiprime for all \( x \) in \( \text{Spec} B(R) \) if and only if for each \( r \) in \( R \), \( r R r \subset x R \) implies that there exists some \( e \) in \( x \) such that \( r R e r R \subset e R \).

We conclude the paper with a characterization of semiprime rings under a hypothesis on the lifting property of the annihilator of a principal ideal.

**Theorem 4.4.** Assume \( \hat{A}(E e E) = A(\hat{R} \hat{r} \hat{R}) \) in \( R_\pi \) for each \( x \) in \( \text{Spec} B(R) \) and \( r \) in \( R \). Then \( R \) is semiprime if and only if \( R_\pi \) is semiprime for each \( x \) in \( \text{Spec} (B(R)) \).

**Proof.** The sufficiency is clear. For the necessity, let \( \hat{r} \neq 0 \) in \( R_\pi \) for an \( x \) in \( \text{Spec} B(R) \). We claim \( \hat{r} \hat{R} \hat{r} = 0 \). Suppose not. We have \( \hat{r} \hat{R} \hat{r} = 0 \). Hence \( \hat{r} \) is in \( A(\hat{R} \hat{r} \hat{R}) \). By hypothesis, there is some \( \hat{s} \) in \( A(\hat{R} \hat{r} \hat{R}) \) such that \( \hat{s} = \hat{r} \). We then have an idempotent \( e \) in \( B(R) \) and not in \( x \) such that \( e s = e r \). Since \( e s R r R = 0 \), \( e r R e r R = 0 \), and so \( e r = 0 \) by hypothesis. But \( e r = e s \neq 0 \), so this gives a contradiction. Thus \( R_\pi \) is a semiprime ring.
From the proof of Corollary 4.3, we see that $\hat{A}(RrR) = \hat{A}(R\hat{r}\hat{R})$ holds for a ring finitely generated over its center. Thus we have:

**Corollary 4.5.** If $R$ is a semiprime ring finitely generated over its center as a ring, then $R_x$ is semiprime for each $x$ in Spec $B(R)$.

Remarks. 1. The sufficiency of Theorem 4.4 does not need the assumption that $\hat{A}(RrR) = \hat{A}(R\hat{r}\hat{R})$.
2. $\text{supp}(r) \cup \text{supp}(RrR) = \text{Spec } B(R)$ for any $r$ in a ring $R$.
3. For reduced rings, it is easy to see that $R$ is reduced if and only if so is $R_x$ for each $x$ in Spec $B(R)$.
4. There is a class of reduced rings called almost hereditary rings which are p.p. rings (see [12] and [13]).

**REFERENCES**


Received July '11, 1972
Кольцо называется п.п. кольцом, если каждый его правый идеал проективен. В работе изучаются пучки Пирса некоммутативных колец и особенно п.п. колец. В частности, характеризуются кольца $K$, у которых каждый слой $K$, является областью целостности (первичным или полупервичным кольцом).