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MARTINGALE CONVERGENCE THEOREM
IN QUANTUM LOGICS

JÁN BÁN

Introduction

It is clear that both martingales and quantum logics belong to the topical themes of today's probability theory. The central result of the martingale theory is the martingale convergence theorem. This theorem was generalized into quantum logics (by supposing the distributivity) by B. Harman and B. Riečan in [3].

O. Nánásiová and S. Pulmannová have shown in [5] the possibility of a further generalization of the conditional expectation, the construction of which is fundamentally used in the proof of the martingale convergence theorem.

This paper follows the mentioned works and shows the possibility of the generalization of the martingale convergence theorem into quantum logics without supposing the distributivity, which means the generalization in line with the main purpose of the quantum -- logics -- theory.

The terms in this paper are used according to [7]; some preliminaries are even verbatim quoted.

Preliminaries

In the mathematical description of quantum experiments, a generalization of the classical probability theory is needed. While in the classical probability theory the set of all "experimentally verifiable propositions" of the physical system (or, equivalently, the set of all random events) can be mathematically described as a Boolean σ-algebra, in the quantum case a more general algebraic structure is needed. The reason for this is the fact that there exist pairs of physical quantities (e.g. position and momentum of a particle) which cannot be measured simultaneously with an arbitrary accuracy — as it can be seen in the case of the well-known Heisenberg uncertainty principle. In quantum logics approach to quantum mechanics the basic concepts are the set $L$ of all experimentally verifiable propositions of the physical system and the set $M$ of the
physical states. \( L \) is usually supposed to be a partially ordered set with the greatest element \( 1 \) and the least element \( 0 \), with the orthocomplementation \( \perp : L \to L \) such that

(i) \( (a^\perp)^\perp = a \), \( a \in L \)

(ii) \( a \leq b \) if and only if \( b^\perp \leq a^\perp \), \( a, b \in L \)

(iii) \( a \lor a^\perp = 1 \) for all \( a \in L \)

(where we denote by \( x \land y \), resp. \( x \lor y \), the infimum, resp. supremum of \( x \) and \( y \) of \( L \) if they exist); and with the orthoriodular property

(iv) \( a \leq b \) (\( a, b \in L \)) implies \( \exists c \in L : c \leq c^\perp \) and \( b = a \lor c ; \) and which is closed under the formation of the suprema \( \lor a , \) for any sequence \( \{a_i\} \subset L \) such that \( a_i \leq a_i^\perp , i \neq j . \) A set \( L \) with the properties just described is called a logic.

The elements \( a, b \in L \) are called orthogonal (\( a \perp b \)) if \( a \perp b^\perp \), and they are compatible (\( a \leftrightarrow b \)) if there exist elements \( a_1, b_1, c \in L \) mutually orthogonal and such that \( a = a_1 \lor c \) and \( b = b_1 \lor c \). The sets \( A \subset L, B \subset L \) are called compatible (\( A \leftrightarrow B \)) if \( a \leftrightarrow b \) for all \( a \in A , b \in B ; A \) is \( \varepsilon \) comparable set (\( A \leftrightarrow \)) if \( a \leftrightarrow b \) for all \( a , b \in A \).

If for any \( 0 \neq a \in L \) and a set \( M \subset L \) there holds:

(i) \( M \leftrightarrow a \)

(ii) \( (M \land a) \leftrightarrow \),

we define \( M \) as partial compatible with \( a \) (\( M \) p.c. \( [a] \)).

A state on the logic \( L \) is a probability measure on \( L \), i.e. a map \( m: L \to [0, 1] \) such that:

(i) \( m(1) = 1 \)

(ii) \( m(a \lor a) = \sum m(a) \)

for any sequence \( \{a_i\} \) of mutually orthogonal elements of \( L \).

Let \( L_1 \) and \( L_2 \) be two logics. The mapping \( h: L_1 \to L_2 \) is called a \( \sigma \)-homomorphism if

(i) \( h(1_1) = 1_2 \) (where \( 1_1 \) and \( 1_2 \) are the greatest elements in \( L_1 \) and \( L_2 \), respectively),

(ii) \( p \perp q, p, q \in L_1 \) implies \( h(p) \perp h(q) , \)

(iii) \( h(a) = a \) for any sequence \( \{a_i\} \) of mutually orthogonal elements of \( L_1 \).

With the help of the concept of \( \sigma \)-homomorphism we introduce observables (corresponding to physical quantities). If \( R \) is the real line and \( B(R) \) is the \( \sigma \)-algebra of all Borel sets, then any \( \sigma \)-homomorphism of \( B(R) \) into \( L \) is called observable on \( L \).

If \( x \) is an observable and \( f: R \to R \) is a Borel measurable function, then the map \( x \circ f^{-1} : E \mapsto x(f^{-1}(E)) \) is also an observable, which is called the function \( f \) of the observable \( x \).

Two observables \( x \) and \( y \) are called compatible if \( x(E) \leftrightarrow y(E) \) for all \( E, F \in B(R) \) (\( x \leftrightarrow y \)); \( x \) is compatible with \( a \) if \( x(E) \leftrightarrow a \) for all \( E \in B(R) \).
If \( x \) is an observable and \( m \) is a state on \( L \), then the map \( m_x : E \mapsto m(x(E)) \) is a probability measure on \( B(R) \). It is called the probability distribution of the observable \( x \). For \( a \in L \) we define

\[
\int_a x \, dm = \int_R t m(x(dt) \land a) = \int_R t \, dv
\]

where \( v : E \mapsto m(x(E) \land a) \) if such an integral exists.

Remark: For the existence of this integral we need firstly to make sure that \( m(x(E) \land a) \) is a measure; the sufficient condition for that is \( x \leftrightarrow a \).

The expectation of the observable \( x \) in the state \( m \) can be defined by

\[
m(x) = \int_R t m_x(dt)
\]

if the integral exists. The subset \( L_o \) of a logic \( L \) is called a sublogic of \( L \) if

(i) \( a \in L_o \) implies \( a^\perp \in L_o \)
(ii) \( a_1, a_2, \ldots \in L_o \) with \( a_i \perp a_i \) implies \( \lor a_i \in L_o \).

It can be easily verified that a sublogic of a logic is a logic itself. A sublogic \( L_o \subset L \) is called a lattice sublogic provided that \( a, b \in L_o \) implies \( a \lor b \) exists in \( L \) and is in \( L_o \). In this case \( L_o \) is a lattice. If a lattice sublogic is distributive, then it is a Boolean \( \sigma \)-algebra and it is called a sub-\( \sigma \)-algebra of \( L \). A logic \( L \) is called separable if for any Boolean sub-\( \sigma \)-algebra \( B_o \) of \( L \) there exists a countable subset \( D \) of \( L \) such that \( B_o \) is the smallest Boolean sub-\( \sigma \)-algebra of \( L \) containing \( D \).

Some important statements

Let us mention some of the generally known statements, which we shall use further in a relevant way. Their proof is to be found in [9] or [5].

**Theorem 1:** Let \( x \) be an observable associated with a logic \( L \). The range of the observable \( x(R(x)) \) is always a Boolean sub-\( \sigma \)-algebra of the logic \( L \). If \( L_o \) is a sublogic of a logic \( L \), then \( x^{-1}(L_o) \) is a Boolean sub-\( \sigma \)-algebra \( B(R) \) as well.

**Theorem 2:** Let \( L \) be a logic which is simultaneously a lattice. Then a necessary and sufficient condition for the existence a Boolean sub-\( \sigma \)-algebra \( B \) such that \( A \subset B \subset L \) is that \( A \) is a compatible set (in \( L \)).

**Theorem 3:** Two observables are compatible if and only if such a Boolean sub-\( \sigma \)-algebra \( B \) exists that \( R(x) \subset B \) and also \( R(y) \subset B \).

**Theorem 4:** Let \( x : B(R) \to L_o \) be an observable associated with \( L_o \), where \( L_o \) is a countable generated Boolean sub-\( \sigma \)-algebra of the logic \( L \). Then there exists
an observable \( y : B(R) \to L \) and a Borel measurable function \( f : R \to R \) so that \( L_0 = \{ y(E) : E \in B(R) \} \) and \( x(E) = y \circ f^{-1}(E) \) for all \( E \in B(R) \).

**Theorem 5**: Let \( \{ x_n \} \) be maximally a countable system of mutually compatible observables associated with a lattice-logic \( L \). Then there exists an observable \( y \) associated with \( L \) and a Borel measurable functions \( f_n \) so that \( x_n = y \circ f_n^{-1} \).

**Theorem 6**: Let \( L \) be a logic, \( a \in L \). Then \( L_{[0, a]} = \{ b \in L : b \leq a \} \) is also a logic with the greatest element \( a \) and an ortho-complement \( b' = b^\perp \wedge a \).

**Theorem 7**: Let \( x \) be an observable associated with \( L \) and \( x \leftrightarrow a, a \in L \). Then \( x \wedge a (E \mapsto x(E) \wedge a) \) is also an observable associated with \( L_{[0, a]} \).

**Theorem 8**: Let \( m \) be a state on \( L \). Let \( a \in L \) be such an element that \( m(a) = 1 \). Then \( m \) is also a state on \( L_{[0, a]} \).

**Theorem 9**: Let \( L \) be a logic, \( M \in L, M \leftrightarrow a \). Then \( M \wedge a \) is a compatible set in \( L \) if and only if \( M \wedge a \) is a compatible set in \( L_{[0, a]} \).

**Theorem 5** enables us to define the *calcul* of compatible observables. If \( x \) and \( y \) are compatible observables, then let us define \( y = z \circ (f_1 - f_2)^{-1} \), where \( y = z \circ f_1^{-1} \), \( x = z \circ f_2^{-1} \).

In the same way we can define the *sum* and the *product* of compatible observables. Another way how to find the calcul is available in [2].

Let \( \{ x_n \} \) be a sequence of the observables associated with \( L \), \( x \) be an observable associated with \( L \), \( m \) be a state and \( x \leftrightarrow x \). We can say that \( \{ x_n \} \) converges almost everywhere to \( x \) according to the state \( m \) \( (x_n \to x \ a.e. \ [m]) \) if

\[
m(\lim_n \sup(x_n - x)(-\varepsilon, \varepsilon)) = 0 \quad \text{for all } \varepsilon > 0.
\]

**Conditional expectation of an observable**

Before we state and prove the martingale convergence theorem, it is necessary to realize the construction of the conditional expectation of observable.

For \( a, b \in L \) put \( a \Delta b = (a^\perp \wedge b) \lor (a \wedge b^\perp) \). For the observables \( x \) and \( y \) associated with \( L \) we shall write \( x \approx y(m) \) if

\[
m(x(E) \Delta y(E)) = 0 \quad \text{for any } \ E \in B(R).
\]

In [5] is proved the following lemma:

**Lemma 1**: Let \( x, y, z \) be observables associated with the logic \( L \) such that \( (R(x) \cup R(y) \cup R(z)) \) p.c. \( [a] \) for some \( a \in L \) and let \( m(a) = 1 \). Then \( x \approx y(m), y \approx z(m) \) implies \( x \approx z(m) \).

**Proof**: First we prove the lemma in the special case \( a = 1 \). If \( b, c, d \) are compatible elements of \( L \), then \( b \Delta d \leq (b \Delta c) \lor (c \Delta d) \), so that
\[ m(x(E) \Delta y(E)) = m(z(E) \Delta y(E)) = 0 \quad \text{implies} \quad m(x(E) \Delta z(E)) = 0 \]
for all \( E \in B(R) \).

Let \( 0 < a < 1 \). Then \( x \wedge a, y \wedge a, z \wedge a \) are mutually compatible observables associated with \( L_{[0,a]} \) so that, by the above part of proof, \( x \approx y(m), y \approx z(m) \) implies \( m((x \wedge a)(E) \Delta (z \wedge a)(E)) = 0 \) for all \( E \in B(R) \). But

\[ m(x(E) \Delta z(E)) = m((x(E) \Delta z(E)) \wedge a) = m((x \wedge a)(E) \Delta (z \wedge a)(E)) = 0. \]

**Definition 1**: Let \( x \) be an observable associated with the logic \( L \). Let \( L_o \) be a sublogic of the logic \( L \), \( m \) be a state on \( L \).

The conditional expectation of the observable \( x \) according to the sublogic \( L_o \) (we shall write \( E(x/L_o) \)) can be defined as such an observable \( u \), for which there holds:

(i) \( u(E) \in L_o \) for all \( E \in B(R) \)

(ii) \( \int_a u \ dm = \int_a x \ dm \) for all \( a \in L_o \) i.e. both sides exist and are equal.

In case when \( L_1 \) is a Boolean sub-\( \sigma \)-algebra of the logic \( L \) and \( x(B(R)) \cup \cup L_o \subset L_1 \), the existence and almost everywhere uniqueness (in accordance \( \approx (m) \)) of the conditional expectation of \( x \) is guaranteed by the following theorem.

**Theorem 10**: Let \( L_1 \) be a countably generated Boolean sub-\( \sigma \)-algebra of the logic \( L \), \( m \) be a state on \( L \). Let \( L_o \) be a subalgebra of \( L_1 \). Let \( x: B(R) \rightarrow L_1 \) be an integrable observable. Then there exists an integrable observable \( u: B(R) \rightarrow L_1 \) such that there holds:

(i) \( \int_a u \ dm = \int_a x \ dm \) for all \( a \in L_o \)

(ii) \( u(E) \in L_o \) for all \( E \in B(R) \).

**Proof**: see [2].

**Definition 2**: Let \( L \) be a logic, \( \{L_n\}^\infty \) be a sequence of sublogics of the logic \( L \), \( \{x_n\}^\infty \) be a sequence of observables, \( m \) be a state on \( L \). The sequence \( (x_n, L_n)_{n=1}^\infty \) is called the martingale on the logic \( L \) if:

(i) \( L_n \subset L_{n+1} \subset L \), \( n = 1, 2, \ldots \)

(ii) \( x_n \) is associated with \( L_n \), \( n = 1, 2, \ldots \)

(iii) \( E(x_{n+1}/L_n) \approx x_n(m) \), \( n = 1, 2, \ldots \)

Theorem 5 and 10 allow to make the following statement:

**Theorem 11**: Let \( L \) be a logic and \( \{L_n\}_{n=1}^\infty \) a sequence of a countable generated Boolean sub-\( \sigma \)-algebras of the logic \( L \). Let \( (x_n, L_n)_{n=1}^\infty \) be a martingale on \( L \) and \( \sup m(|x_n|) < \infty \), \( m \) is a state on \( L \). Then there exists such an observable \( x \)
associated with \( \mathcal{L} = \sigma_B \left( \bigcup_{n=1}^{\infty} L_n \right) \) (the smallest Boolean \( \sigma \)-algebra such that \( \mathcal{L} \supset \bigcup_{n=1}^{\infty} L_n \)) that \( x_n \to x \) a.e. \([m]\).

Proof: see [3].

But this understanding of the conditional expectation is not too generalized, because it supposes the distributivity of \( L \). Our purpose is, however, to avoid this presumption. Another conditional expectation version of an observable can be found by the following construction. Let \( L \) be a separable lattice-logic, \( L_o \) its sublogic, \( x \) an integrable observable associated with \( L \), \( m \) a state on \( L \). Let us have an element \( 0 \neq a \in L \), so that

(i) \( m(a) = 1 \)
(ii) \( (R(x) \cup L_o) \) p.c. \([a]\).

Because \( (R(x) \cup L_o) \wedge a \) is the compatible set, there exists such a Boolean \( \sigma \)-algebra \( B \) that \( (R(x) \cup L_o) \wedge a \subseteq B \in \mathcal{L}_{[0, a]} \), an observable \( y: B(R) \to B \) and a Borel measurable function \( f: R \to R \) so that \( x \wedge a = y \circ f^{-1} \). To \( L_o \wedge a \) there exists a Boolean \( \sigma \)-algebra \( B_o \) so that \( L_o \wedge a \subseteq B_o \subseteq B \). Let us define \( S_o = \{E \in B(R): y(E) \in B_o\} \). It is clear that \( S_o \) is a sub-\( \sigma \)-algebra of the \( \sigma \)-algebra \( S = y^{-1}(B) \). Therefore there exists \( g = Em_y(f/S_o) \) (the "classical" conditional expectation of the function \( f \) according to \( S_o \)). Now let us define \( z^+ = y \circ g^{-1} \).

It is clear that \( z^+ \) is an observable associated with \( L_{[0, a]} \) whereby \( R(z^+) \subseteq B_o \) (because \( g \) is \( S_o \)-measurable). In order to "extend" \( z^+ \) to the whole logic \( L \), let us define

\[
z = y \circ g^{-1} \vee (w \wedge a^\perp) \quad \text{where} \quad w(E) = \begin{cases} 0 & \text{if } E \neq E \\ 1 & \text{if } 0 \in E \end{cases}
\]

Now it is still valid that

\[
z(R) = z^+(R) \vee (w(R) \wedge a^\perp) = z^+(R) \vee a^\perp = a \vee a^\perp = 1.
\]

Let us show that \( z \) is a conditional expectation version of the observable \( x \) according to \( L_o \). Let \( b \in L_o \).

Because \( m(a) = 1 \)

\[
\int_b x \, dm = \int tm(x(dt) \wedge b) =
\]

\[
= \int tm(x \wedge a(dt) \wedge (b \wedge a)) = \int tm(y \circ f^{-1}(dt) \wedge y(A)) =
\]

\[
= \int tm_y(f^{-1}(dt) \cap A) = \int_\omega f(\omega) m_y(d\omega) = \int_\omega g(\omega) m_y(d\omega) =
\]

\[
= \int tm_y(g^{-1}(dt) \cap A) = \int tm(y \circ g^{-1}(dt) \wedge y(A)) =
\]
Let us prove yet the uniqueness (in accordance with \( \approx (m) \)) of this conditional expectation.

Let \( z_1, z_2 \) be two versions of a conditional expectation \( E(x/L_o) \) for the same required element \( a \in L \). We have \( z_1 \leftrightarrow a, z_2 \leftrightarrow a \) and \( R(z_1) \land a \in L_o \land a, R(z_2) \land \land a \in L_o \land a \).

As \( L_o \land a \subset B_o \subset L_{[0,a]} \) and \( B_o \) is a Boolean sub-\( \sigma \)-algebra, then \( z_1 \land a \leftrightarrow z_2 \land a \). Let \( g_1 : \Omega \to R, g_2 : \Omega \to R \) be \( S_o \)-measurable functions such that \( z_1 = y \circ g_1^{-1} \) and \( z_2 = y \circ g_2^{-1} \). Then, as \( z_1(E) \Delta z_2(E) \leftrightarrow a \) for any \( E \in B(R) \),

\[
m((z_1 \land a)(E) \Delta (z_2 \land a)(E)) = m(y \circ g_1^{-1}(E) \Delta y \circ g_2^{-1}(E)) =
\]

\[
= m(y \circ (g_1^{-1}(E) \Delta g_2^{-1}(E))) = m_y(g_1^{-1}(E) \Delta g_2^{-1}(E))
\]

From that there holds for \( g_1 \) and \( g_2 \int_F g_1 \ dm_y = \int_F g_2 \ dm_y \), for all \( F \in S_o \) we get

\[
m_y(g_1^{-1}(E) \Delta g_2^{-1}(E)) = 0 \text{ for any } E \in B(R). \text{ Indeed, put}
\]

\[
F_1 = \{ \omega \in \Omega ; \ g_1(\omega) > g_2(\omega) \},
\]

\[
F_2 = \{ \omega \in \Omega ; \ g_1(\omega) < g_2(\omega) \}.
\]

As \( F_1 \cup F_2 \in S_o \), \( \int_{F_1} (g_1 - g_2) \ dm_y = 0 \) for any \( F_o \subset F_1 \cup F_2, F_o \in S_o \), hence

\[
m_y(F_1 \cup F_2) = m_y(\{ \omega ; \ g_1(\omega) \neq g_2(\omega) \}) = 0.
\]

As \( g_1^{-1}(E) \Delta g_2^{-1}(E) \subset F_1 \cup F_2 \), we obtain

\[
m_y(g_1^{-1}(E) \Delta g_2^{-1}(E)) = 0
\]

for any \( E \in B(R) \).

Now \( 0 = m((z_1 \land a)(E) \Delta (z_2 \land a)(E)) = m(z_1(E) \Delta z_2(E)) \) and this concludes the proof of uniqueness.

Let us notice the conditions we require for the element \( 0 \neq a \in L \). We have to recognize that if we have to use \( 1 \in L \) instead of this element, then the second condition actually says that \( L_o \) has to be a Boolean \( \sigma \)-algebra, which means that we get the same conditional expectation version as in theorem 10. On the other hand, it is clear that the conditions for the required \( a \in L \) are "strong enough". One of the possibilities, how to seek such an \( a \in L \) is to use the set-comutator qualities (see [5]). Furthermore we shall require \( a \in L_o \), which is a sufficient condition for the \( L_o \)-measurability of the conditional expectation.

In concluding this part of the paper let us make a generalization of the previously defined calculus of observables.
Let \(x, y\) be two observables associated with the logic \(L\), \(m\) is a state. Let there exist \(0 = a \in L\) such that \((I? (x) \cup I? (y))\) p.c. \([a]\) and \(m(a) = 1\). Then we can define \(x - y = z \circ f_1^{-1} \lor (w \land a)\) where \(x \land a = z \circ f_1^{-1}\) and \(y \land a = z \circ f_2^{-1}\) and \(w\) is the observable \(w = \begin{pmatrix} 0 & 0 \not{\in} E \\ 1 & 0 \in E \end{pmatrix}\).

The martingale convergence theorem

**Theorem 12:** Let \((x_n, L_n)\) be a martingale on the separable lattice-logic \(L\). Let there exist such a \(0 \neq a \in L\) that

(i) \(m(a) = 1\)
(ii) \((R(x_{n+1}) \cup L_n)\) p.c. \([a]\), \(n = 1, 2, \ldots\)

Let \(\sup \{m(x_n)\} < \infty\).

Then there exists such an observable \(x\) that \(x_n \to x\) a.e. \([m]\).

**Proof:** For each \(n = 1, 2, \ldots\) \((R(x_{n+1}) \cup L_n) \land a\) are compatible sets and \((R(x_{n+1}) \cup L_n) \land a \in L_{[0, a]}\), which means that there exists the Boolean sub-\(\sigma\)-algebras \(B_1, B_2, \ldots\) such that

\((R(x_{n+1}) \cup L_n) \land a \in B_n, B_n \subseteq B_{n+1} \subseteq L_{[0, a]}, \quad n = 1, 2, \ldots\)

Because \(\{B_n\}\) is an upper bounded sequence by \(L_{[0, a]}\), there exists a Boolean sub-\(\sigma\)-algebra \(B\) of \(L_{[0, a]}\) such that for each \(n\) there holds \((R(x_{n+1}) \cup L_n) \land a \subseteq B \subseteq L_{[0, a]}\). Therefore the observables \(x_n \land a, \quad n = 1, 2, \ldots\) are compatible (because \(R(x_n) \land a \subseteq R(x_n) \land a \subseteq B\)) and this means that there exists an observable \(y: B(R) \to B\) and Borel measurable functions \(f_n: R \to R\) that \(x_n \land a = y \circ f_n^{-1}\). Similarly \(L_n \land a, \quad a = 1, 2, \ldots\) are compatible sets, too and therefore there exists a sequence \(B_1^+, B_2^+, \ldots\) of the Boolean sub-\(\sigma\)-algebras of \(L_{[0, a]}\) such that \(L_n \land a \subseteq B_n^+\).

Let us take instead of \(\{B_n^+\}\) the smallest Boolean \(\sigma\)-algebras with this property.

Let us prove that \((x_n \land a, B_n^+)\) is a martingale on \(L_{[0, a]}\). Evidently:

(i) \(B_1^+ \subseteq B_2^+ \subseteq \ldots \subseteq B\) because \(L_1 \subseteq L_2 \subseteq \ldots\) and \(L_1 \land a \subseteq L_2 \land a \subseteq \ldots\)
(ii) because \(x_n(E) \in L_n\), therefore also \(x_n(E) \land a \in L_n \land a \subseteq B_n^+\) as well
(iii) it is also necessary the prove that \(E(x_{n+1} \land a/B_n^+) \approx x_n \land a(m)\).

Let us define:

\[ S_n = \{A \in B(R): y(A) \in B_n^+\} = y^{-1}(B_n^+) \quad \text{and} \quad g_n = E m_y(f_{n+1}/S_n). \]

From the assumption that \((x_n, L_n)\) is a martingale and from the conditional—expectation—construction of an observable we know that \(E(x_{n+1}/L_n) = y \circ g_n^{-1} \lor (w \land a) \approx x_n\), therefore \(E(x_{n+1} \land a) \approx y \circ g_n^{-1} \lor (w \land a) \land a \approx y \circ g_n^{-1}\) (since \(m(a) = 1\)). Because \(x_{n+1} \land a = y \circ f_{n+1}^{-1}\), from the construction \(g_n\) it is clear that \(E(x_{n+1} \land a/B_n^+) \approx x_n \land a\) (with respect to lemma 1, too).
Now we can finally use theorem 11, according to which \( x_n \land a \to h = y \circ g^{-1} \) a.e. \([m]\), where \( g \) is such a Borel measurable function that \( g_n \to g \) a.e. \([m]\).

Let us define \( x = y \circ g^{-1} \lor (w \land a^\perp) \).

It is also necessary to prove that

\[
m(\limsup_n (x_n \land (-\varepsilon, \varepsilon))) = 0.
\]

Indeed,

\[
m(\limsup_n (y \circ (g - g_n)^{-1}(-\varepsilon, \varepsilon)) \lor (w(-\varepsilon, \varepsilon)^c \land a^\perp)) =
\]

\[
= m \left( \bigcap_{i=1}^\infty \bigvee_{n=1}^\infty [(y \circ (g - g_n)^{-1}(-\varepsilon, \varepsilon)^c) \lor (w(-\varepsilon, \varepsilon)^c \land a^\perp)] \right) =
\]

\[
= \lim_i m \left( \bigvee_{n=1}^\infty [(y \circ (g - g_n)^{-1}(-\varepsilon, \varepsilon)^c) \lor (w(-\varepsilon, \varepsilon)^c \land a^\perp)] \right) =
\]

\[
= \lim_i m \left( \bigvee_{n=1}^\infty [y \circ (g - g_n)^{-1}(-\varepsilon, \varepsilon)^c] + m(w(-\varepsilon, \varepsilon)^c \land a^\perp) \right) =
\]

\[
= m(\limsup_n y \circ (g - g_n)^{-1}(-\varepsilon, \varepsilon)^c) + m(w(-\varepsilon, \varepsilon)^c \land a^\perp) = 0
\]

because \( g_n \to g \) a.e. \([m]\) and \( m(0) = 0, m(a^\perp) = 0 \) (from the condition \( m(a) = 1 \)), too.

This concludes the proof of the theorem.

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МАРТИНГАЛЬНАЯ СХОДИМОСТНАЯ ТЕОРЕМА В КВАНТОВЫХ ЛОГИКАХ

Ján Bán

Резюме

В работе приведено обобщение так называемой мартингальной сходимостной теоремы для квантовых логик.