Ivan Chajda; Gábor Czédli; Eszter K. Horváth
Trapezoid lemma and congruence distributivity


Persistent URL: [http://dml.cz/dmlcz/130492](http://dml.cz/dmlcz/130492)

**Terms of use:**

© Mathematical Institute of the Slovak Academy of Sciences, 2003

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
TRAPEZOID LEMMA AND
CONGRUENCE DISTRIBUTIVITY

IVAN CHAJDA* — GÁBOR CZÉDLI** — ESZTER K. HORVÁTH**

(Communicated by Tibor Katriňák)

ABSTRACT. Motivated by Gumm's (rectangular) Shifting Lemma, in our con­
text a condition rather than a statement, and Shifting Principle, which play a
key role in his treatment of congruence modularity and the theory of modular
commutator, the present paper relates analogous triangular and trapezoid lemmas
and principles to the distributivity of congruence lattices of single algebras
and varieties. For varieties, the Trapezoid Lemma is equivalent to congruence
distributivity. As a byproduct, congruence distributivity is characterized by a
Mal’cev condition with a very clear connection with Day terms characterizing
congruence modularity. Some results presented here were previously announced
by J. Duda.

H.-P. G u m m [11] defined a certain condition by a rectangular scheme for
congruences, respectively congruences and tolerances of an algebra under the
name Shifting Lemma and Shifting Principle. In a variety, each of these two
conditions is equivalent to congruence modularity. Keeping congruence distribu­
tivity rather than congruence modularity in mind our goal is to study some other
schemes which can be defined by triangles or trapezes. Following G u m m ’s style
of [11; Corollary 3.6], schemes for congruences will be called lemmas although
they are just conditions, and we keep the word principle for schemes where toler­
ances also occur. We are going to study our conditions for congruences of single
algebras and for congruences of varieties. As it will be detailed at the end of the
paper, some of our results have previously been announced by D u d a [7].

2000 Mathematics Subject Classification: Primary 08B10; Secondary 08B05.
Keywords: congruence distributivity, Trapezoid lemma, Shifting lemma, Triangular lemma,
Mal’cev condition = Mal’tsev condition.

This research was partially supported by the project J98:MSM153100011 of the Czech Gov­
ernment, by the NFSR of Hungary (OTKA), Grant No. T034137, T026243 and T037877, and
also by the Hungarian Ministry of Education, Grant No. FKFP 0169/2001.
Now we give some definitions.

An algebra $A$ is said to satisfy the Shifting Lemma (in other words, Rectangular Lemma) if for any $\alpha, \beta, \gamma \in \text{Con} A$ if $\alpha \cap \beta \subseteq \gamma$, $(x,u),(y,v) \in \alpha$, $(x,y),(u,v) \in \beta$ and $(u,v) \in \gamma$, then $(x,y) \in \gamma$, cf. Gumm [11]. Pictorially, the Rectangular Lemma is the condition given by Figure 1.

Similarly, $A$ is said to satisfy the Triangular Lemma if for any $\alpha, \beta, \gamma \in \text{Con} A$, $(x,y) \in \gamma$, $(x,z) \in \alpha$ and $(y,z) \in \beta$ if $\alpha \cap \beta \subseteq \gamma$, then $(y,z) \in \gamma$, cf. [1], [3] and Duda [8]. The Triangular Lemma is depicted in Figure 2.

Now we introduce a new condition under the name Trapezoid Lemma as follows: for any $\alpha, \beta, \gamma \in \text{Con} A$ if $\alpha \cap \beta \subseteq \gamma$, $(x,u),(y,v) \in \alpha$, $(x,y) \in \beta$ and $(u,v) \in \gamma$, then $(x,y) \in \gamma$. The Trapezoid Lemma is depicted in Figure 3.
Three corresponding conditions called *Shifting* (or Rectangular) Principle (cf. Gumm [11]), *Triangular Principle* (cf. [3]) and *Trapezoid Principle* are defined similarly, the only difference is that $\alpha$ should be replaced by $\Phi$, which stands for an arbitrary *tolerance* (i.e., compatible, reflexive and symmetric binary relation) of $A$.

Our figures follow the tradition that parallel edges have the same label. Sometimes we do not require the above-defined conditions for all triplets $(\alpha, \beta, \gamma)$ just for a single triplet $(\alpha_0, \beta_0, \gamma_0)$; in this case we will say so.

Given a direct product $A = A_1 \times A_2$, a congruence $\gamma \in \text{Con } A$ is called *directly decomposable* if $\gamma = \gamma_1 \times \gamma_2$ for appropriate $\gamma_1 \in \text{Con } A_1$ and $\gamma_2 \in \text{Con } A_2$. One of the motivations for introducing the Trapezoid Lemma is revealed by the following statement, which strengthens [1; Assertion 1].

**Proposition 1.** Let $\gamma \in \text{Con } (A_1 \times A_2)$ and let $\pi_i$ denote the kernel of the projection $A_1 \times A_2 \to A_i$, $(x_1, x_2) \mapsto x_i$, $i = 1, 2$. Then the following three conditions are equivalent:

(a) $\gamma$ is directly decomposable;

(b) the Trapezoid Lemma holds for $(\pi_1, \pi_2, \gamma)$ and $(\pi_2, \pi_1, \gamma)$;

(c) both the Rectangular Lemma and the Triangular Lemma hold for $(\pi_1, \pi_2, \gamma)$ and $(\pi_2, \pi_1, \gamma)$.

**Proof.** The equivalence of (a) and (b), in a slightly different formulation, is proved by Fraser and Horn [9; Theorem 1(1),(3)], cf. also the trapezes in [2; p. 128, Fig. 31]. The implication (b) $\implies$ (c) is evident; this will also be clear from the forthcoming Proposition 2. Proving (c) $\implies$ (b) is obvious, too: if $(x, u), (y, v) \in \pi_1$, $(x, y) \in \pi_2$ and $(u, v) \in \gamma$, then with $w := (y_1, u_2) = (v_1, u_2)$ the Triangular Lemma gives $(u, w) \in \gamma$, whence the Rectangular Lemma yields $(x, y) \in \gamma$.

The following statement presents some connections among our conditions in case of a single algebra; for varieties of algebras we will soon state more.

**Proposition 2.** Let $A$ be an algebra.

(1) If $A$ satisfies the Trapezoid Lemma respectively the Trapezoid Principle, then it satisfies the Rectangular Lemma and the Triangular Lemma respectively. Moreover, each of the three principles implies the corresponding lemma.

(2) If $\text{Con } A$ is distributive, then $A$ satisfies the Trapezoid Lemma (and therefore the other two lemmas as well).

(3) If $A$ satisfies the Trapezoid Principle, then $\text{Con } A$ is distributive.

(4) If $A$ satisfies the Rectangular Principle, then $\text{Con } A$ is modular (cf. Gumm [11; Lemma 3.2]).
(5) If $A$ is congruence permutable, then $\text{Con} A$ is distributive if and only if $A$ satisfies the Triangular Lemma (cf. [3; Corollary 2]).

Proof. (1) is trivial. (2) comes easily from the fact that a lattice is distributive if and only if it satisfies the Horn sentence

$$\alpha \land \beta \leq \gamma \implies \beta \land (\alpha \lor \gamma) \leq \gamma,$$

(*)

cf. [1]. Hence only (3) needs a proof. Suppose $A$ is an algebra satisfying the Trapezoid Principle and $\alpha, \beta, \gamma \in \text{Con} A$ with $\alpha \land \beta \leq \gamma$. According to (*) it suffices to show $\beta \land (\alpha \lor \gamma) \leq \gamma$. Borrowing the idea from the proof of [11; Lemma 3.2] by Gumm define tolerances $\Phi_0 = \alpha$ and $\Phi_{n+1} = \Phi_n \circ \gamma \circ \alpha$, $n \in \mathbb{N}$. Via induction on $n$ we want to show that $\beta \cap \Phi_n \subseteq \gamma$. For $n = 0$ this is clear. Now suppose $\beta \cap \Phi_n \subseteq \gamma$ and let $(x, y)$ be an arbitrary pair in $\beta \cap \Phi_{n+1}$. Then $(x, y) \in \beta \cap \Phi_{n+1} = \beta \cap (\Phi_n \circ \gamma \circ \alpha) \subseteq \beta \cap (\Phi_n \circ \gamma \circ \Phi_n)$, so there are $u, v \in A$ such that $(x, u), (y, v) \in \Phi_n$, $(x, y) \in \beta$ and $(u, v) \in \gamma$. Hence the induction hypothesis $\beta \cap \Phi_n \subseteq \gamma$ and the Trapezoid Principle gives $(x, y) \in \gamma$. This shows $\beta \cap \Phi_{n+1} \subseteq \gamma$, completing the induction. Finally,

$$\beta \land (\alpha \lor \gamma) = \beta \cap \bigcup_{n=0}^{\infty} \Phi_n = \bigcup_{n=0}^{\infty} (\beta \cap \Phi_n) \subseteq \gamma,$$

proving (*) and (3). $\square$

We do not know if the implication in (1), (2), (3) and (4) of Proposition 2 can be reversed, but we guess the answer is negative in each case. However, for varieties rather than single algebras much more can be said. Of course, a condition is said to hold in a variety if it holds in all algebras of the variety. Part (a) $\iff$ (c) of the following theorem was announced by Duda [7].

**Theorem 1.** Let $\mathcal{V}$ be a variety of algebras. Then the following five conditions are equivalent.

(a) $\mathcal{V}$ is congruence distributive;
(b) the Trapezoid Principle holds in $\mathcal{V}$;
(c) the Trapezoid Lemma holds in $\mathcal{V}$;
(d) the Rectangular Lemma and the Triangular Lemma hold in $\mathcal{V}$;
(e) there is a positive integer $n$ and there are quaternary terms $d_0, d_1, \ldots, d_n$ such that the identities

(e1) $d_0(x, y, u, v) = x$, $d_n(x, y, u, v) = y$,
(e2) $d_i(x, y, x, y) = d_{i+1}(x, y, x, y)$ for $i$ even,
(e3) $d_i(x, y, z, z) = d_{i+1}(x, y, z, z)$ for $i$ odd,
(e4) $d_i(x, x, y, z) = x$ for all $i$

hold in $\mathcal{V}$. 

250
Remark 1. Congruence distributivity and congruence modularity of varieties are characterized by classical Mal'cev conditions, namely by the Jónsson terms, cf. Jónsson [13], and the Day terms, cf. Day [6]. Since distributivity implies modularity, one would expect that Jónsson terms trivially produce Day terms, but this is not the case. To fulfill this wish (and also to reduce the number of variables), Gumm [11], [12] characterizes congruence modularity with another Mal'cev condition, the Gumm terms, and he points out that Jónsson terms trivially produce Gumm terms. Now (e) of Theorem 1 gives an alternative way to meet the mentioned expectation. Namely, Day terms are quaternary terms satisfying (e1), (e2), (e3) and

\[(e4') \quad d_i(x, x, y, y) = x \text{ for all } i,\]

so our terms in (e) clearly produce (and in fact, constitute) Day terms. Notice that (e) is a byproduct of studying the Trapezoid Lemma; indeed, the proof of Theorem 1 is easier with (e) than with Jónsson terms. To reveal the connection between (e) and Jónsson terms, we mention that the $p_i(x, y, z) = d_i(x, z, y, z)$ are Jónsson terms provided the $d_i$ are (e) terms.

Remark 2. Theorem 1 and Proposition 2 clearly imply [3; Theorem 2], which says that congruence distributive varieties satisfy the Triangular Principle.

Proof of Theorem 1.
(a) $\implies$ (e) follows in the standard way of deriving Mal'cev conditions if we consider the principal congruences $\beta = \text{con}(u, v)$ and $\gamma = \text{con}(x, y)$, and the congruence $\alpha = \text{con}(x, u) \vee \text{con}(y, v)$ of the free algebra $F_\mathcal{V}(x, y, u, v)$.

(e) $\implies$ (b): Assuming that (e) holds in $\mathcal{V}$, let $A \in \mathcal{V}$, let $\Phi$ be a tolerance relation of $A$, let $\beta, \gamma \in \text{Con } A$ with $\Phi \cap \beta \subseteq \gamma$, let $x, y, u, v \in A$ and suppose $(x, u), (y, v) \in \Phi$, $(x, y) \in \beta$ and $(u, v) \in \gamma$. We have to show that $(x, y) \in \gamma$. Consider the elements $h_i = d_i(x, x, u, v), i = 0, \ldots, n$, where the terms $d_i$ are provided by (e). Then for $i$ odd, $h_i = d_i(x, y, u, v) \gamma d_i(x, y, u, u) \gamma d_{i+1}(x, y, u, v) = h_{i+1}$, i.e., $(h_i, h_{i+1}) \in \gamma$ for $i$ odd. For $i$ even we have to work a bit more. We start with $h_i = d_i(x, x, u, v) \Phi d_i(x, x, y)$ and $h_i = d_i(x, y, u, u) \beta d_i(x, x, u, v) = d_i(x, x, x, x) \beta d_i(x, y, x, y)$. Hence $(h_i, d_i(x, y, x, y)) \in \Phi \cap \beta \subseteq \gamma$. We obtain $(h_i, d_{i+1}(x, y, x, y)) \in \gamma$ similarly. But $d_i(x, y, x, y) = d_{i+1}(x, y, x, y)$, whence the transitivity of $\gamma$ gives $(h_i, h_{i+1}) \in \gamma$ for $i$ even. Now $(h_i, h_{i+1}) \in \gamma$ for all $i$, and we conclude $(x, y) = (h_0, h_n) \in \gamma$. I.e., $\mathcal{V}$ satisfies (b).

Observe that (b) $\implies$ (c) and (c) $\implies$ (d) are evident (or follow from Proposition 2).

(d) $\implies$ (a): Let $\mathcal{V}$ be a variety satisfying the Rectangular Lemma and the Triangular Lemma. The Rectangular Lemma in itself implies that $\mathcal{V}$ is congruence modular by Gumm [11; Corollary 3.6]. Now, by way of contradiction, assume that $\mathcal{V}$ is not congruence distributive. Then there is an algebra $A \in \mathcal{V}$
and there are congruences $\alpha, \beta, \gamma \in \text{Con } A$ generating a five-element nondistributive sublattice $M_3 = \{\alpha, \beta, \gamma, \omega, \iota\}$ of $\text{Con } A$ with $\omega < \alpha < \iota$, $\omega < \beta < \iota$ and $\omega < \gamma < \iota$. The theory of modular commutator says, cf. G u m m [11; Corollary 8.9] or F r e e s e and M c K e n z i e [10; Lemma 13.1], that any two elements of this $M_3$ permute. Since $\beta \not\subseteq \gamma$, we can pick a pair $(y, z) \in \beta \setminus \gamma$. Since $(y, z) \in \beta \subseteq \iota = \gamma \vee \alpha = \gamma \circ \alpha$, there is an element $x$ with $(y, x) \in \gamma$ and $(x, z) \in \alpha$, cf. the left hand side of Figure 2. Now $\alpha \cap \beta = \omega \subseteq \gamma$, so the Triangular Lemma yields $(y, z) \in \gamma$, a contradiction. This proves that $\mathcal{V}$ is congruence distributive. 


\begin{center}
\textbf{Comparison with subsequent and previous results.} While the Trapezoid Principle seems to be just a technical condition here, later it was proved to be an essential milestone towards [4], [5], which follow the present paper. Several parts of the present paper are in close connection with former results of J. D u d a. He also introduced the Trapezoid Lemma (under the name Upright Principle) and announced that conditions (a) and (c) of Theorem 1 are equivalent, cf. [7], and they are equivalent to the conjunction of congruence modularity and the Triangular Lemma, cf. [8]. (In virtue of G u m m ’s classical result, this conjunction is clearly equivalent to (d) of Theorem 1.) D u d a [7] also gave a Mal’cev condition to characterize the Trapezoid Lemma; his Mal’cev condition consists of 6-ary terms. Although we have never had access to his proofs, his highly sophisticated Mal’cev conditions for the Trapezoid Lemma and also for the Triangular Lemma, cf. [8], convinced us that our approach to his result is new and simpler than the original one.
\end{center}

\begin{center}
\textbf{REFERENCES}
\end{center}


TRAPEZOID LEMMA AND CONGRUENCE DISTRIBUTIVITY


Received September 25, 2001
Revised December 4, 2002

* Department of Algebra and Geometry
Palacký University Olomouc
Tomkova 40
CZ–779 00 Olomouc
CZECH REPUBLIC
E-mail: chajda@risc.upol.cz,

** University of Szeged
Bolyai Institute
Aradi vártauták teré 1
H–6720 Szeged
HUNGARY
E-mail: czedli@math.u-szeged.hu
horeszt@math.u-szeged.hu

253