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SEMIGROUPS CONTAINING COVERED ONE-SIDED IDEALS

IMRICH FABRICI

In [3] a notion of a covered ideal was introduced. The aim of the present paper is to show some other properties and the mutual relation between covered ideals and bases of semigroups.

Definition 1. A proper left ideal \( L \) of \( S \) is called a covered left ideal (briefly a CL-ideal) if \( L \subset S(S - L) \). Analogously a covered right ideal (CR-ideal) is defined. The case of two-sided ideals will be treated later.

Clearly if \( S \) contains a zero element \( 0 \) and \( \text{card } |S|^2 \), then \( 0 \) is a CL-ideal. Note that by definition \( S \) itself is not a CL-ideal.

Lemma 1. If \( S \) contains two different left ideals \( L_1 \) and \( L_2 \) such that \( L_1 \cup L_2 = S \), then none of the ideals \( L_1, L_2 \) is a CL-ideal.

Proof. If \( L_1 \cup L_2 = S \), then \( S - L_2 \subset L_1 \), and \( S - L_1 \subset L_2 \). Now \( L_1 \subset S(S - L_1) \) implies \( L_1 \subset S \), \( L_2 \subset S \), \( L_2 \subset S \), \( L_1 \subset L_2 \), hence \( L_1 = L_2 \) a contradiction.

Corollary. If \( S \) contains more than one maximal left ideal, then none of them is a CL-ideal of \( S \).

If \( L \) is a left ideal of \( S \) and \( L \subseteq S_a \), then \( L \) is certainly a CL-ideal. (For, in this case we have \( a \in S - L \).) In particular if \( L = S_a \cap S_b \) is a proper subset of \( S_a \) or \( S_b \) then \( L \) is a CL-ideal of \( S \).

A semigroup in which \( a \) is not contained in \( S_a \) (i.e. \( a \in S - S_a \)) contains CL-ideals, since for the left ideal \( L = S_a \) we have \( L = S_a \subset S(S - L) \).

In a semigroup which does not contain a CL-ideal, the ideal \( S_a \) cannot contain a proper left ideal of \( S \), hence \( S_a \) is a minimal left ideal for every \( a \in S \). In such a semigroup for any \( a \neq b \) we have either \( S_a = S_b \) or \( S_a \cap S_b = \emptyset \). Moreover, \( a \in S_a \) for every \( a \in S \).

Lemma 2. A semigroup \( S \) with \( \text{card } |S| > 1 \) contains no CL-ideals iff \( S \) is a union of (disjoint) minimal left ideals.

Proof. 1. It has been just remarked that such a semigroup is necessarily of the form: \( S = \bigcup_{i \in I} S_{a_i} \), where each summand is a minimal left ideal.
2. Conversely, let be $S = \bigcup_{i \in I} L_i$, where each $L_i$ is a minimal left ideal of $S$. Any left ideal of $S$ is a union of some minimal left ideals. Write $A = \bigcup_{i \in K} L_i$, $B = \bigcup_{i \in I \setminus K} L_i$, then $S = A \cup B$. By Lemma 1 neither $A$ nor $B$ is a CL-ideal of $S$.

If $S$ is a union of its minimal left ideals, it is known that $S$ is simple.

In the following when speaking about CL-ideals we shall suppose that such ideals exist i.e. $S$ is not a simple semigroup (without zero) containing a minimal left ideal.

**Lemma 3.** If $L_1$ and $L_2$ are two CL-ideals of $S$, then $L_1 \cup L_2$ is a CL-ideal of $S$.

**Proof.** We have to show that $L_1 \cup L_2 \subseteq S[ S - (L_1 \cup L_2)]$. Note that by Lemma 1, $S - (L_1 \cup L_2) \neq \emptyset$. Let $x$ be any element from $L_1$. $L_1 \subseteq S(S - L_1)$ implies that there is $a \in S - L_1$ such that $x \in Sa$.

1. If $a \in S - L_1 - L_2$, then $x \in S(S - L_1 - L_2)$
2. If $a \in (S - L_1) \cap L_2$, we have $a \in L_2 \subseteq S(S - L_2)$. Hence there is $k_2 \in S - L_2$ such that $a \in Sk_2$. The element $k_2$ cannot be contained in $L_1$ since otherwise we would have $a \in Sk_2 \subseteq SL_1 \subseteq L_1$, a contradiction with $a \in S - L_1$. Hence $k_2 \in (S - L_1 \cap (S - L_2)) = S - (L_1 \cup L_2)$. Therefore, $x \in Sa \in SSk_2 \subseteq Sk_2 \subseteq S[S - (L_1 \cup L_2)]$.

We have proved $L_1 \subseteq S[S - (L_1 \cup L_2)]$ and by the same argument $L_2 \subseteq S[S - (L_1 \cup L_2)]$, so that

$$L_1 \cup L_2 \subseteq S[S - (L_1 \cup L_2)].$$

**Lemma 4.** If $L_1$, $L_2$ are two CL-ideals of $S$ and $L_1 \cap L_2 \neq \emptyset$ then $L_1 \cap L_2$ is a CL-ideal of $S$.

**Proof.** $L_1 \subseteq S(S - L_1)$ implies $L_1 \cap L_2 \subseteq S(S - L_1) \subseteq S[S - (L_1 \cap L_2)]$.

If we consider the empty set $\emptyset$ as a CL-ideal, we may state:

**Theorem 1.** The set of all CL-ideals of $S$ (including $\emptyset$) is a sublattice of the lattice of all left ideals of $S$ (including $\emptyset$).

**Example 1.** Let $S = \{a, b, c, d\}$ with the multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
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<td>$a$</td>
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<td>$b$</td>
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<td>$c$</td>
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<td>$d$</td>
<td>$a$</td>
<td>$b$</td>
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<td>$d$</td>
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</table>

$S$ has the lattice of all left ideals given in fig. 1, while fig. 2 gives the lattice of all CL-ideals.
Definition 2. A left ideal $L$ of $S$ and $L \neq S$ is called the greatest left ideal of $S$ if $L$ contains any proper left ideal of $S$.

Example 2. Let $S_0 = \langle 0, 1 \rangle$ with the usual multiplication of real numbers and $S_1 = \{a_i, 0\}$, $a_i^2 = a_i$ and 0 having the properties of a zero. Let $S$ be the 0-direct union of $S_0$ and $S_1$. Then $S$ contains a unique maximal ideal, namely $S_0$. But $S_0$ is not the greatest ideal of $S$, since $S_0$ does not contain the ideal $\{0, a_1\}$.

If $S$ contains the greatest left ideal of $S$, this ideal will be denoted by $L^*$. Clearly if $S$ contains $L^*$, then $L^*$ is a maximal left ideal of $S$.

If $S$ contains the greatest left ideal of $S$, then $L^*$ will be denoted by $L^*$. Clearly if $S$ contains $L^*$, then $L^*$ is a maximal left ideal of $S$.

\[ \begin{array}{c}
(a, b, c, d) \\
(a, b, c) \\
(a, b) \\
(a) \\
\emptyset
\end{array} \]

Fig. 2

Fig. 1

Theorem 2. A maximal left ideal $L$ of $S$ is a CL-ideal of $S$ iff $S$ contains $L^*$ and in this case $L = L^*$.

Proof. 1. By Lemma 1 a maximal left ideal $L$ of $S$ can be CL-ideal only if for any left ideal $l$ of $S$ we have $l \subset L$ (For otherwise $L \cup l$ would be equal to $S$). Since $L$ is maximal, necessarily $L = L^*$.

2. Conversely, suppose that $L^*$ exists. We prove that $L^* \subset S(S - L^*)$. Since $S(S - L^*)$ is a left ideal of $S$ we have either $S(S - L^*) = S$, or $S(S - L^*) \subset L^*$. In the first case $L^* \subset S = S(S - L^*)$.

In the second case $S(S - L^*) \subset L^*$ and $L^* \subset SL^*$ imply $S^2 = S[(S - L^*) \cup L^*] \subset L^*$. If $S - S^2 = \{a, b, c, \ldots\}$, then any set $S - a, S - b, \ldots$ is a left ideal of $S$. Hence, since $L^*$ exists we have $S - S^2 = \{a\}$. Denote $S - S^2 = \{a\}$. Then $L^* = S - a$ and $S = L^* \cup \{a\}$. Now $a \cup Sa$ is a left ideal of $S$ and since it is not contained in $L^*$, we have $a \cup Sa = S$. The equalities $a \cup L^* = a \cup Sa = S$ (since $a \in L^*$ and $a \in Sa$) imply $L^* = Sa$, so that $L^* \subset S(S - L^*)$. This proves our statement.

We now treat the case that $S$ contains more than one maximal left ideal.

Definition 3. A CL-ideal $L$ is called a greatest covered left ideal of $S$ if $L$ contains every covered left ideal of $S$. 

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If $S$ contains the greatest covered left ideal of $S$, this ideal will be denoted by $L^a$. Suppose that $S$ contains maximal left ideals and \( \{L_\alpha/\alpha \in I\} \) is the totality of all such ideals. Denote $\hat{L} = \bigcap_{\alpha \in I} L_\alpha$ and suppose $\hat{L} \neq \emptyset$ (i.e. $S$ is not a simple semigroup containing a minimal left ideal).

If $L^a$ exists, we have necessarily $L^a \subset \hat{L}$. For if there is at least one $L_\alpha$ such that $L^a$ is not contained in $L_\alpha$, then $L_\alpha \cup L^a = S$ and by Lemma 1 $L^a$ cannot be a CL-ideal.

Unfortunately $\hat{L}$ need not be a covered left ideal.

Example 3. Let $S_0$ be the multiplicative semigroup of real numbers from the half-open interval $(0, 1)$ and $S_1 = \{0, a_1\}$, $S_2 = \{0, a_2\}$, $a_1^2 = a_1$, $a_2^2 = a_2$, the element $0$ having the usual properties of multiplicative zero. The 0-direct union $S = S_0 \cup S_1 \cup S_2$ contains two maximal ideals, namely $L_1 = S - \{a_1\}$, $L_2 = S - \{a_2\}$. The ideal $S_1 \cup S_2$ is not contained in a maximal ideal of $S$. $\hat{L} = S_0$, $S(S - \hat{L}) = \{0, a_1, a_2\}$ so $\hat{L} \neq S(S - \hat{L})$.

Example 4. Modify the foregoing example by taking for $S_0$ the closed interval $(0, 1)$. Then $S$ contains a further maximal ideal, namely $L_3 = S - \{1\}$, and $\hat{L} = \langle 0, 1 \rangle$. In this case $S - \hat{L} = \{a_1, a_2, 1\}$ and $S(a_1, a_2, 1) = S$, so that $\hat{L} \subset S(S - \hat{L})$. Hence $\hat{L}$ is a covered left ideal.

An $\mathcal{L}$-class (the set of all elements of $S$ generating the same principal left ideal) containing a given element $a$ will be denoted by $L^a$.

An $\mathcal{L}$-class $L^a$ is a maximal one, if $\langle a \rangle_L$ is not a proper subset of any principal left ideal of $S$.

In [1] it is proved that a complement of a maximal left ideal is a maximal $\mathcal{L}$-class. We shall denote maximal left ideals by $L_\alpha$ and corresponding maximal $\mathcal{L}$-classes by $L^a$.

Now we introduce a partial ordering $\preceq$ between $\mathcal{L}$-classes namely: $L^a \preceq L^b$ if $\langle a \rangle_L \subset \langle b \rangle_L$.

A non-empty subset $A$ of $S$ is a right base of $S$ if

1. $A \cup SA = S$
2. there is no proper subset $B \subsetneq A$ such that $B \cup Sb = S$

Consider a quasi-ordering in $S$, namely: $a \preceq b$ means $\langle a \rangle_L \subset \langle b \rangle_L$.

**Lemma 4** [6]. A non-empty subset $A$ of $S$ is a right base of $S$ iff

1. for any $x \in S$ there is $a \in A$ such that $x \preceq a$,
2. for any two distinct elements $a_1, a_2 \in A$ neither $a_1 \preceq a_2$, nor $a_2 \preceq a_1$.

**Remark.** Lemma 4 implies that a right base $A$ consists of elements from all maximal $\mathcal{L}$-classes.

**Lemma 5** [5]. Let $S$ contain maximal left ideals. Then the intersection of all maximal left ideals $\bigcap_{\alpha \in I} L_\alpha = \emptyset$ iff $S$ is a simple semigroup (without zero) containing a minimal left ideal.
Theorem 3. A semigroup $S$ contains $L^q$ iff

1. $S$ is not a simple semigroup, containing a minimal left ideal,
2. $S$ contains a right base $A$.

Prof. (a) Suppose that $S$ satisfies (1), (2). Then (see [3], Theorem 1) $S$ contains maximal left ideals. Denote by $\hat{L} = \bigcap_{\alpha \in \lambda} L_{\alpha}$ the intersection of all maximal left ideals. $\hat{L} \neq \emptyset$ by (1). As we know from [4] $L_{\alpha} = S - L^a(\alpha \in \lambda)$ and $L^a$ is a maximal $\mathcal{L}$-class of $S$. Then $\hat{L} = \bigcap_{\alpha \in \lambda} L_{\alpha} = \bigcap_{\alpha \in \lambda} (S - L^a) = S - \bigcup_{\alpha \in \lambda} L^a$. So, $S - \bigcup_{\alpha \in \lambda} L^a = \hat{L}$. This implies that no element from $L^a(\alpha \in \lambda)$ and therefore from the right base $A$ belongs to $\hat{L}$.

Let $x \in \hat{L}$ by any element. By (1) of Lemma 4 there is $a \in A$ such that $x \leq a$, i.e. $(x)_L \subset (a)_L$, or in another form:

$$\bigcup_{x \in L} [x \cup Sx] \subset \bigcup_{a \in A} [a \cup Sa] = S.$$ 

Hence, we have $\hat{L} \subset SA \subset S(S - \hat{L})$, so $\hat{L}$ is a $\mathcal{C}$-ideal of $S$. It remains to show that any $\mathcal{C}$-ideal is contained in $\hat{L}$. Let $L$ be any left ideal of $S$, which is not contained in $\hat{L}$, so $L \cap (\bigcup_{\alpha \in \lambda} L^a) \neq \emptyset$, i.e. $L^a \subset L$ at least for one $\alpha \in \lambda$. Let $L^b \subset L$ ($L^b$ is a maximal $\mathcal{L}$-class of $S$). We shall show that $L$ is not a $\mathcal{C}$-ideal of $S$. Let $b \in L^b \subset L$, so $(b)_L \subset L$. In $S - L$ are $\mathcal{L}$-classes either from $\hat{L}$, or from $S - \hat{L}$, except $L^a$. Therefore, there is no $\mathcal{L}$-class $L^a$ in $S - L$ such that $L^b < L^a$. So we have proved that any left ideal which is not contained in $\hat{L}$ cannot be a $\mathcal{C}$-ideal of $S$. Since $\hat{L}$ is a $\mathcal{C}$-ideal, we conclude that $L^a$ exists and $\hat{L} = L^a$.

(b) Now suppose that $S$ contains $L^a$. We show that (1) and (2) are satisfied.

It is known that any left ideal of $S$ is a union of certain $\mathcal{L}$-classes of $S$, so its complement must be a union of the remaining $\mathcal{L}$-classes. Let us construct a subset $A$ in the following way: exactly one element is chosen into $A$ from each $\mathcal{L}$-class in $S - L^a$. We show that $A$ satisfies (1) and (2) of Lemma 4.

Let $x \in S$ be any element. Then either $x \in L^a$, or $x \in S - L^a$. If $x \in L^a$, then $L^a \subset S(S - L^a)$ implies that there is $a \in S - L^a$ such that $x \in Sb$ and $b \in L^a$. From $x \in Sb$ we have $(x)_L \subset (b)_L = (a)_L$, so $x \leq a$. If $x \in S - L^a$, then $x \in L^b$ and $x \leq b$. Therefore, (1) is satisfied in both cases.

Let $a, b \in A$, $a \neq b$. We shall show that neither $a \leq b$ nor $b \leq a$ holds. If $a \leq b$, then $a \cup Sa \subset b \cup Sb$. Since $a \neq b$, we have $a \in Sb$. This implies $(a)_L \subset Sb (b \in (a)_L)$, therefore $(a)_L$ is a $\mathcal{C}$-ideal of $S$. Then $L^a \cup (a)_L$ is a $\mathcal{C}$-ideal of $S$, properly containing $L^a$, which is a contradiction. Similarly we can prove that $b \leq a$ does not hold. Hence $A$ satisfies the condition (2) of Lemma 4. We have proved that $S$ contains a right base.
It remains to show that \( S \) is not simple, containing a minimal left ideal. According to Lemma 5 it suffices to show that the intersection of all maximal left ideals is non-empty. This follows from our assumption that \( S \) contains \( L^q \) and from the fact that we always have \( L^a \subseteq \bar{L} \).

**Corollary.** If \( S \) contains \( L^q \), then \( L^q \) is of the form: \( L^q = \bigcap_{a \in \lambda} L_a \), i.e. \( L^q \) is the intersection of all maximal left ideals of \( S \).

**Theorem 4.** Every left ideal of a semigroup \( S \) is covered if and only if either there is a chain of principal left ideals such that the union of its elements is \( S^* \), or \( S \) contains \( L^* \).

**Proof.** (a) Let every left ideal of \( S \) be covered. Let \( L \) be any left ideal of \( S \), and \( a \in L \). Since every left ideal is covered, we have \( (a)_L \subseteq S(S(a)_L) \). It implies \( a \in Sb_L \), for \( b \in S(a)_L \), hence \( (a)_L \subseteq (b)_L \). So, we can construct a chain of principal left ideals. By Hausdorff Theorem any chain is contained in a maximal one. Denote by \( U_l \) \( (a_i)_L \) (\( i \in I \)) a maximal chain of proper principal left ideals of \( S \) and \( \bigcup_{i \in I} (a_i)_L = L_1 \). If \( L_1 = S \) there is nothing to prove more. \( L_1 \subseteq S \) we shall show that \( S \) contains \( L^* \). If \( L_1 \subseteq S \) holds, then \( S - L_1 \neq \emptyset \). \( L_1 \) is a left ideal of \( S \) and therefore (by supposition) a covered one, so \( L_1 \subseteq S(S - L_1) \). For every \( i \in I \), \( (a_i)_L \subseteq S(S - L_1) \). There is an element \( c \in S - L_1 \) such that \( a_i \in S c \), therefore \( (a_i)_L \subseteq (c)_L \). We shall show that \( (c)_L = S \). If this were not true, then \( (c)_L \subseteq S \) and since \( (a_i)_L \subseteq (c)_L \), then \( (c)_L \) would belong to the chain \( U \). But it is a contradiction with our assumption that \( U \) is a maximal chain. Hence \( c \cup S c = S \). The \( L \)-class containing \( c \) is a maximal one. Denote it by \( L^a \). Then \( S - L^a = L_{a1} \) is a maximal left ideal. Every left ideal \( T \) which is not contained in \( L_{a1} \) meets \( L^a \), hence \( T \cap L^a \neq \emptyset \), so that \( T = S \). It means that \( L_{a1} \) is such a maximal left ideal that every proper left ideal of \( S \) is contained in \( L \), hence \( L_{a1} = L^* \).

(b) If \( S \) contains \( L^* \), then \( L^* \) is a CL-ideal and for any proper left ideal \( L \) we have:

\[
L \subseteq L^* \subseteq S(S - L^*) \subseteq S(S - L).
\]

Hence \( L \) is a CL-ideal.

Let \( S \) contain a chain \( U \) of principal left ideals \( (a_i)_L \) \( j \in I \), and \( \bigcup_{j \in I} (a_i)_L = S \). Let \( L \) be any left ideal of \( S \). Recall that every left ideal is a union of principal left ideals generated by its elements. Let \( b \in S - L \). Since \( \bigcup_{j \in I} (a_i)_L = S \), then there exists an index \( i \in I \) such that \( b \in (a_i)_L \) and \( (b)_L \subseteq (a_i)_L \). The element \( a_i \in L \), since \( a \in L \) would imply \( (a_i)_L \subseteq L \) and \( (b)_L \subseteq (a_i)_L \subseteq L \) implies \( b \in L \), what is a contradiction with a choice of \( b \). Denote by \( K \) the set of indices of all elements of \( U \) that are
contained in \((a_i)_L\). Clearly \(\bigcup_{j \in I-K} (a_i)_L = S\). All elements \(a_i, j \in I - K\), belong to \(S - L\) and \(\bigcup_{j \in I-K} (a_i \cup S_a_i) = S\).

Now \(L \subseteq \bigcup_{j \in I-K} (a_i \cup S_a_i)\). But \(a_i \in S - L\) for \(j \in I - K\), hence \(L \subseteq \bigcup_{j \in I-K} S_a_i \subseteq S(S - L)\), so that \(L\) is a CL-ideal of \(S\). This proves Theorem 4.

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ПОЛУГРУППЫ СОДЕРЖАНИЕ ЗАКРЫТЫЕ ОДНОСТОРОННИЕ ИДЕАЛЫ

Имрих Фабрици

Резюме

Левый (правый) идеал \(L (R)\) называется закрытым, если

\[ L \subseteq S(S - L), \quad (R \subseteq (S - R)R). \]

В настоящей работе доказаны утверждения, касающиеся строения полугрупп, имеющих односторонние закрытые идеалы. Следующие утверждения являются главными:

1. Множество всех закрытых левых (правых) идеалов (включая 0) является подструктурой структуры всех левых (правых) идеалов (включая 0).
2. Приведено необходимое и достаточное условие для того, что бы:
   а) полугруппа содержала самый большой закрытый левый (правый) идеал
   б) всякий левый (правый) идеал полугруппы был закрытым.