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APPLICATION OF ROTHE'S METHOD TO PARABOLIC VARIATIONAL INEQUALITIES

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Introduction. We shall be concerned with the existence, uniqueness and approximation of the solution $u(t)$ for parabolic variational inequalities of the form:

$$u(t)\in K \quad \text{for a.e.} \quad t\in(0, T) \quad \text{and}$$

$$\left(\frac{du(t)}{dt},\ v-u(t)\right)+\langle Au(t), v-u(t)\rangle \geq (f(t), v-u(t)) \tag{1}$$

$$\text{holds for all} \quad v\in K \quad \text{and a. e.} \quad t\in(0, T)$$

where $A\colon V\to V^*$ is a monotone, coercive operator, $T<\infty$ and K is a closed convex subset in a reflexive space V. Together with (1) we assume the initial condition

$$u(0)=u_0. \tag{2}$$

The problem (1), (2) has first been studied by Brezis in [1 – 2] and by Lions in [3] in the case $A\colon L_p(\langle 0, T\rangle, V) \to L_q(\langle 0, T\rangle, V^*)$. The problem (1), (2) has been solved by the method of penalization and regularization. Duvaut, Lions in [4] considered a more general inequality than (1) but with the linear operator A.

Our concept of treating the problem (1), (2) is based on Rothe's method developed recently in [5—10]. A solution of the given problem is transformed into the solution of the sequence of elliptic variational inequalities. By a simple method we obtain the solution $u(t)$ which is regular in t.

Formulation of the main result

Let V be a reflexive Banach space with the norm $\|\cdot\|_V$, V^* its dual space with the norm $\|\cdot\|_*$ and H a real Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. We denote by $\langle\cdot,\cdot\rangle$ the duality between V^* and V. We assume that the space $V\cap H$ with the norm $\|\cdot\|_{V\cap H} = \|\cdot\|_V+\|\cdot\|$ is a dense set in V and H and K is a closed convex subset in $V\cap H$. Suppose $A\colon K\to V^*$ satisfies the following assumptions:

(3) $$A \text{ is demicontinuous};$$

(4) $$\langle Au - Av, u - v\rangle \geq 0 \quad \text{for all} \quad u, v \in K;$$

(5) $$\text{there exists} \quad v_0 \in K \quad \text{such that}$$
$$\langle Au, u - v_0\rangle/[u] \to \infty \quad \text{for} \quad [u] \to \infty;$$

where $[\cdot]$ is a seminorm on V with the properties: there exist $\lambda > 0$, $c > 0$ such that

(6) $$[u] + \lambda \|u\| \geq c\|u\|_V \quad \text{for all} \quad u \in V \cap H.$$

For u_0, f from (1), (2) we assume

(7) $$u_0 \in K, \quad Au_0 \in H;$$

(8) $$f \in C(\langle 0, T\rangle, H), \operatorname*{Var}_{\langle 0, T\rangle} f < \infty,$$

where $\operatorname*{Var}_{\langle 0, T\rangle} f = \sup_{\{t_i\}} \sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|$ for all finite division $\{t_i\}$ of $\langle 0, T\rangle$.

We apply the idea of Rothe in the following way: Successively for $i = 1, \ldots, n$ let u_i be the solution of the elliptic inequality

(9) $$\left(\frac{u_i - u_{i-1}}{h}, \; v - u_i\right) + \langle Au_i, v - u_i\rangle \geq (f_i, v - u_i)$$

for all $v \in K$, where $h = \frac{T}{n}$, n is a positive integer, $t_i = ih$, $f_i = f(t_i)$ and u_0 is from (2). The inequality (9) can be expressed in the form

(10) $$\langle A_h u_i, v - u_i\rangle \geq \left(f_i + \frac{u_{i-1}}{h}, v - u_i\right)$$

where $\langle A_h u, v\rangle = \langle Au, v\rangle + \frac{1}{h}(u, v)$. The operator $A + \frac{1}{h} I$: $K \to (V \cap H)^* = V^* + H$ is bounded, demicontinuous, strictly monotone and coercive. Hence and due to [3, Chap. 2, Theorems 8.2, 8.3] there exists a unique solution $u_i \in K$ of (10) which implies (9).

By means of u_i $(i = 1, \ldots, n)$ we construct Rothe's function

$$u_n(t) = u_{i-1} + h^{-1}(t - t_{i-1})(u_i - u_{i-1}) \quad \text{for} \quad t_{i-1} \leq t \leq t_i,$$

$i = 1, \ldots, n$ and we prove that $u_n(t)$ converges for $n \to \infty$ to the solution $u(t)$ of (1), (2). Our main result is

Theorem 1. *Let (3)—(8) be satisfied. Then there exists the unique solution $u \in L_\infty(\langle 0, T\rangle, V \cap H)$ of (1), (2) with the following properties:*

$$u(t): \langle 0, T\rangle \to H \quad \textit{is Lipschitz continuous};$$

$$\frac{du}{dt} \in L_\infty(\langle 0, T\rangle, H), \; Au \in L_\infty(\langle 0, T\rangle, H);$$

$$u_n(t) \to u(t) \quad \textit{in } H \textit{ for } n \to \infty \textit{ uniformly on } \langle 0, T \rangle;$$

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} \xrightarrow[w^*]{} \frac{\mathrm{d}u}{\mathrm{d}t} \quad \textit{in} \quad L_\infty(\langle 0, T\rangle, H);$$

if $f\colon \langle 0, T\rangle \to H$ *is Lipschitz continuous then the estimate*

$$\|u_n(t) - u(t)\|^2 \leqslant \frac{C}{n} \quad \textit{is true.}$$

We first prove some lemmas.

Lemma 1. *There exists a constant* C *depending only on* T, u_0, f *such that*

$$\left\|\frac{u_i - u_{i-1}}{h}\right\| \leqslant C \tag{11}$$

$$\|u_i\|_{V \cap H} \leqslant C, \quad \textit{for all} \quad n, i = 1, \dots, n. \tag{12}$$

Proof. Putting $i = j$, $v = u_{j-1}$ and $i = j-1$, $v = u_j$ in (9) we obtain, after adding,

$$\frac{1}{h}\|u_j - u_{j-1}\|^2 \leqslant \left(\frac{u_{j-1} - u_{j-2}}{h}, u_j - u_{j-1}\right) -$$

$$- \langle Au_j - Au_{j-1}, u_j - u_{j-1}\rangle + (f_j - f_{j-1}, u_j - u_{j-1}).$$

Using the monotonicity of A we obtain the recurrent inequality

$$\left\|\frac{u_j - u_{j-1}}{h}\right\| \leqslant \left\|\frac{u_{j-1} - u_{j-2}}{2}\right\| + \|f_j - f_{j-1}\|, \quad j = 1, \dots, n. \tag{13}$$

Putting $i = 1$, $v = u_0$ in (9) we arrive at

$$\left\|\frac{u_1 - u_0}{h}\right\| \leqslant \|f_1\| + \|Au_0\|. \tag{14}$$

We obtain successively from (13), (14)

$$\left\|\frac{u_i - u_{i-1}}{h}\right\| \leqslant \operatorname*{Var}_{\langle 0, T\rangle} f + \|f_0\| + \|Au_0\| \leqslant C,$$

which is Conclusion (11). Directly from (11) we obtain

$$\|u_i\| \leqslant C, \quad i = 1, \dots, n \tag{15}$$

and from (9) we have $\langle Au_i, u_i - v_0\rangle \leqslant C$. The coercivity of A implies $[u_i] \leqslant C$ and the estimate (12) is then the result of (6) and (15), which concludes the proof.

We now construct the functions

$$\bar{u}_n(t) = u_j, \quad t_{j-1} \leqslant t \leqslant t_j, \quad \bar{u}_n(0) = u_0, \quad j = 1, \dots, n.$$

Similarly we construct $f_n(t)$ and $\bar{f}_n(t)$ by means of $f_i = f(t_i)$, $i = 1, \dots, n$.

Lemma 1 implies

$$(16) \qquad \|u_n(t)-\bar{u}(t)\| \leq \frac{C}{n}, \quad \text{for all} \quad t \in \langle 0, T\rangle$$

$$(17) \qquad \|u_n(t)-u_n(t')\| \leq C|t-t'| \quad \text{for all} \quad t, t' \in \langle 0, T\rangle .$$

Lemma 2. *There exists a function* $u \in L_\infty(\langle 0, T\rangle, V \cap H)$ *with the following properties:*

i) $u(t) \in K$ *for all* $t \in \langle 0, T\rangle$;

ii) $\frac{du}{dt} \in L_\infty(\langle 0, T\rangle, H)$;

iii) $u_n \to u$ *in the norm of the space* $C(\langle 0, T\rangle, H)$;

iv) $\frac{du_n}{dt} \xrightarrow[w^*]{} \frac{du}{dt}$ *in* $L_\infty(\langle 0, T\rangle, H)$.

Proof. We can rewrite (9) in the form

$$(18) \qquad \left(\frac{du_n(\tau)}{d\tau}, v-\bar{u}_n(\tau)\right) + \langle A\bar{u}_n(\tau), v-\bar{u}_n(\tau)\rangle \geq (\bar{f}_n(\tau), v-\bar{u}_n(\tau))$$

for all $v \in K$ and for a.e. $\tau \in (0, T)$. Putting $n=r$, $v=\bar{u}_s(\tau)$ and then $n=s$, $v=\bar{u}_r(\tau)$ in (18) and adding up we obtain

$$\left(\frac{d(u_r(\tau)-u_s(\tau))}{d\tau}, \bar{u}_r(\tau)-\bar{u}_s(\tau)\right) +$$
$$+\langle A\bar{u}_r(\tau)-A\bar{u}_s(\tau), \bar{u}_r(\tau)-\bar{u}_s(\tau)\rangle \leq (\bar{f}_r(\tau)-\bar{f}_s(\tau), \bar{u}_r(\tau)-\bar{u}_s(\tau)).$$

Integrating in $(0, t)$ and using the monotonicity of A we have

$$\|u_r(t)-u_s(t)\|^2 \leq 2\int_0^t \left(\left\|\frac{du_r(\tau)}{d\tau}\right\| + \left\|\frac{du_s(\tau)}{d\tau}\right\|\right)(\|u_r(\tau)-\bar{u}_r(\tau)\| +$$
$$+\|u_s(\tau)-\bar{u}_s(\tau)\|)\,d\tau + C\int_0^t \|\bar{f}_r(\tau)-\bar{f}_s(\tau)\|d\tau\Big).$$

The estimates (11) and (16) imply

$$\|u_r(t)-u_s(t)\|^2 \leq C\left(\frac{1}{r}+\frac{1}{s}+\int_0^t \|\bar{f}_r(\tau)-\bar{f}_s(\tau)\|d\tau\right).$$

Then we obtain f is uniformly continuous in $\langle 0, T\rangle$ and hence there exists $u \in C(\langle 0, T\rangle, H)$ such that $u_n \to u$ in the norm of the space $C(\langle 0, T\rangle, H)$. The inequality (17) implies

$$\|u(t)-u(t')\| \leq C|t-t'| \quad \text{for all} \quad t, t' \in \langle 0, T\rangle .$$

Then we obtain from the result of Komura [11] (see also [9, Lemma 1]) that there exists the strong (in the norm of H) derivative $\frac{du}{dt} \in L_\infty(\langle 0, T\rangle, H)$. Moreover

$u \in L_\infty(\langle 0, T\rangle, V \cap H)$, which is a consequence of (12) and reflexivity of $V \cap H$. Since K is weakly closed in $V \cap H$, we conclude $u(t) \in K$ for all $t \in \langle 0, T\rangle$. We can rewrite (11) in the form

$$\left\|\frac{du_n}{dt}\right\| \leq C \quad \text{for a.e.} \quad t \in (0, T). \tag{19}$$

Using (19) we have

$$\frac{du_n}{dt} \rightharpoonup \frac{du}{dt} \quad \text{in} \quad L_2(\langle 0, T\rangle, H)$$

(see [9, Lemma 5]) and moreover

$$\frac{du_n}{dt} \xrightarrow[w^*]{} \frac{du}{dt} \quad \text{in} \quad L_\infty(\langle 0, T\rangle, H) \tag{20}$$

which concludes the proof.

Proof of the Theorem. Let $u(t)$ be the function from Lemma 2. Setting $v(t) = u(t)$ in (18) we obtain with the help of (16), iii), (19) that

$$\lim_{n\to\infty} \langle A\bar{u}_n(\tau), \bar{u}_n(\tau) - u(\tau)\rangle \leq 0.$$

The operator A is pseudomonotone (see [3]), which implies that

$$\langle Au(\tau), u(\tau) - v\rangle \leq \liminf_{n\to\infty} \langle A\bar{u}_n(\tau), \bar{u}_n(\tau) - v\rangle \tag{21}$$

for all $v \in K$. Using the monotonicity of A and the boudedness of $\bar{u}_n$ in $L_\infty(\langle 0, T\rangle, v \cap H)$ we obtain

$$\langle A\bar{u}_n(\tau), \bar{u}_n(\tau) - v\rangle \geq -C(\|v\|).$$

By means of Fatou lemma we obtain from (21) that

$$\int_{t_2}^{t_1} \langle Au(\tau), u(\tau) - v\rangle \, d\tau \leq \liminf_{n\to\infty} \int_{t_2}^{t_1} \langle A\bar{u}_n(\tau), \bar{u}_n(\tau) - v\rangle \, d\tau \tag{22}$$

for arbitrary $t_1, t_2 \in (0, T)$ and $v \in K$. Integrating (18) we can see, taking into account (22), that

$$\int_{t_2}^{t_1} \langle Au(\tau), u(\tau) - v\rangle \, d\tau \leq \liminf_{n\to\infty} \int_{t_2}^{t_1} \left(\bar{f}_n(\tau) - \frac{du_n(\tau)}{d\tau}, \bar{u}_n(\tau) - v\right) d\tau.$$

Using Lemma 2 we obtain after limiting

$$\int_{t_2}^{t_1} \left[\left(\frac{du(t)}{dt}, v - u(t)\right) + \langle Au(t), v - u(t)\rangle - (f(t), v - u(t))\right] dt \geq 0,$$

for arbitrary $t_1, t_2 \in \langle 0, T\rangle$ and $v \in K$, which implies

(23) $$\left(\frac{du(t)}{dt}, v - u(t)\right) + \langle Au(t), v - u(t)\rangle \geq (f(t), v - u(t))$$

for all $v \in K$ and a.e. $t \in (0, T)$ and hence u is a solution of the problem (1), (2). There remains to verify the unicity of the solution. Let u_1, u_2 be two solutions of (1), (2). From (23), for $u = u_1$, $v = u_2$ and $u = u_2$, $v = u_1$, after adding and taking into account the monotonicity of A, we obtain the estimate

$$\left(\frac{d(u_1(t) - u_2(t))}{dt}, u_1(t) - u_2(t)\right) = \frac{1}{2}\frac{d}{dt}\|u_1(t) - u_2(t)\|^2 \leq 0.$$

As $\|u_1(0) - u_2(0)\| = 0$ we have $u_1(t) = u_2(t)$ a.e. on $(0, T)$, which concludes the proof.

Using the results of Kačur [8], an analogous result as Theorem 1 can be proved for the nonstationary parabolical inequalities

(1′) $$\left(\frac{du(t)}{dt}, v - u(t)\right) + \langle A(t)u(t), v - u(t)\rangle \geq (f(t), v - u(t)),$$

(2′) $$u(0) = u_0.$$

We formulate now the result.

Let $A(t)$ ($t \in \langle 0, T\rangle$) be a system of operators $A(t)\colon K \to V^*$ satisfying

(24) $A(t)$ is bounded and continuous for all $t \in (0, T)$;

(25) $\langle A(t)u - A(t)v, u - v\rangle \geq 0$ for all $u, v \in K$ and $t \in (0, T)$;

(26) $\langle A(t)u, u - v_0\rangle \geq \|u\|_V r(\|u\|_V)$ for all $u \in K$, $t \in (0, T)$,

where the function $r(s)$ is nondecreasing for $s \geq s_0$ bounded in $(0, s_0)$ and $r(s) \to \infty$ for $s \to \infty$, $v_0 \in K$;

(27) $A(t)u = \operatorname{grad}_u \Phi(t, u)$ for $u \in K$, $t \in (0, T)$

where $\Phi(t, u)$ for fixed t is a functional on V, i.e., $A(t)$ are potential operators. We assume that for each $u \in K$ there exist derivatives $\frac{d}{dt} A(t)u$, $\frac{d^2}{dt^2} A(t)u$ in V^* and the estimate

(28) $$\left\|\frac{d}{dt} A(t)u\right\|_* + \left\|\frac{d^2}{dt^2} A(t)u\right\|_* \leq C_1 + C_2 r(\|u\|_V)$$

takes place for all $t \in (0, T)$ and $u \in K$. For u_0, f we assume

(29) $\|f(t) - f(t')\| \leq C|t - t'|$ for all $t, t' \in \langle 0, T\rangle$;

(30) $$A(0)u_0 \in H.$$

In this case we solve successively the elliptic inequalities

$$\left(\frac{u_i - u_{i-1}}{h}, v - u_i\right) + \langle A(t_i)u_i, v - u_i \rangle \geqslant (f(t_i), v - u_i)$$

for all $v \in K$ where $u_i \in K$. By means of u_i $(i = 1, \ldots, n)$ we construct Rothe's function $u_n(t)$. The following theorem can be proved.

Theorem 2. *Let (24)—(30) be satisfied. Then there exists the unique solution* $u \in L_\infty(\langle 0, T \rangle, V \cap H)$ *of (1'), (2') with the properties*

$$\|u_n(t) - u_n(t)\|^2 \leqslant \frac{C}{n};$$

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} \xrightarrow[w^*]{} \frac{\mathrm{d}u}{\mathrm{d}t} \quad \textit{in} \quad L_\infty(\langle 0, T \rangle, H);$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in L_\infty(\langle 0, T \rangle, H), \quad Au \in L_\infty(\langle 0, T \rangle, H).$$

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ПРИЛОЖЕНИЕ МЕТОДА РОТЕ К ПАРАБОЛИЧЕСКИМ НЕРАВЕНСТВАМ

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Резюме

В работе исследуется решение начальной задачи для абстрактных параболических неравенств. С помощью метода Роте авторы свели задачу к решению последовательности эллиптических неравенств.