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ON SLA–IDEALS

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ABSTRACT. The aim of this paper is to study left A-ideals which are at the same time semigroups, and to give an outline of the extent to which this notion is useful. The notion has a very close relation to known notions such as quasi-zeros, mild-ideals, and directed groups. There are also some connections with left-simple semigroups having no idempotents. The main result is a description of minimal semigroup left A-ideals in the commutative case.

Left A-ideals appear in various areas of mathematics and unify several notions. They are a generalization of left ideals in semigroups because any left ideal is at the same time a left A-ideal of a semigroup. We will deal with left A-ideals which are also subsemigroups of a given semigroup. In the theory of semigroups one question arises naturally: "Does there exist a minimal left A-ideal in the class of all left A-ideals which are at the same time subsemigroups of a given semigroup?" We give, in Theorem 12, a complete answer in the commutative case. The situation in the noncommutative case is discussed at the end of the paper. There exists also a very close relation to directed groups. First we briefly recall some notions.

DEFINITION 1. ([6]) A nonempty subset G_L of a semigroup S is called a *left A-ideal* of S (*LA-ideal*) if $sG_L \cap G_L \neq \emptyset$ for any $s \in S$. A nonempty subset G_R of a semigroup S is called a *right A-ideal* of S (*RA-ideal*) if $G_R s \cap G_R \neq \emptyset$ for any $s \in S$. By *two-sided A-ideal*, or simply *A-ideal*, we mean a subset of S which is both a left and a right A-ideal of S.

Let us now proceed to the notion of a semigroup left A-ideal of a semigroup S.

DEFINITION 2. Let a left A-ideal G_L of a semigroup S be a subsemigroup of S. Then it is called a *semigroup left A-ideal* (*SLA-ideal*) of S.

A *semigroup right A-ideal* (*SRA-ideal*) and a *semigroup A-ideal* (*SA-ideal*) are defined analogously. All these notions coincide in the commutative case.

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Now we recall some special cases of the notions just introduced, which we have found in the literature.

a) The notion of an A-ideal is a generalization of the well-known notion of a difference set on a finite group.

b) Putcha [9] introduced the notion of a mild-ideal. In fact this notion is equivalent to our SLA-ideal. One of the main results was that, in the case where S is a commutative, idempotent-free and archimedean semigroup, there exists a proper mild-ideal.

c) Rankin and Reis [10] have studied semigroups with so called quasi-zeros: An element $a \in S$ is said to be a *right quasi-zero* if

$$s\langle a\rangle \cap \langle a\rangle \neq \emptyset \qquad \text{for any} \quad s \in S\,,$$

where $\langle a\rangle = \{a, a^2, a^3, \dots\}$.

Obviously, in the case when $a \in S$ is a right quasi-zero, $G_L = \langle a\rangle$ is an SLA-ideal of S. The authors, in their paper, have described the structure of semigroups consisting entirely of right quasi-zeros.

Several papers deal with a connection between the concept of the ideal theory and LA-ideal theory of semigroups ([7], [11], [12]). We deal with A-ideals which are at the same time semigroups and study some relations between such A-ideals and ideals in a semigroup. The next theorem shows the difference between these two concepts.

THEOREM 1. *Let G_L be an SLA-ideal of a semigroup S, and let $H_L \subseteq G_L$ be an SLA-ideal of the semigroup G_L. Then H_L is an SLA-ideal of S.*

Proof. To any $s \in S$ we have to find an element $h \in H_L$ such that $sh \in H_L$. First, since G_L is an SLA-ideal, there exists $g_1 \in G_L$ such that $sg_1 \in G_L$. Analogously, for $sg_1 \in G_L$ there exists $h_1 \in H_L$ such that $(sg_1)h_1 \in H_L$. Similarly, for $g_1h_1 \in G_L$ we can find $h_2 \in H_L$ which satisfies $(g_1h_1)h_2 \in H_L$. Finally, the fact that H_L is a semigroup forces $\big((sg_1)h_1\big)h_2 = s\big((g_1h_1)h_2\big) \in H_L$, and the element $h = (g_1h_1)h_2 \in H_L$. □

It is known that an analogous statement does not hold for left ideals of a semigroup.

One of the most important facts in a description of structural properties of semigroups is the existence of a minimal left ideal. In the following, our main task is to find all semigroups having minimal SLA-ideals in the class of all SLA-ideals.

DEFINITION 3. By an *SLA-simple semigroup* we shall mean any semigroup without proper SLA-ideals.

From Theorem 1, we immediately have

THEOREM 2. *Let G_L be an SLA-ideal of S. Then it is a minimal SLA-ideal of S if and only if it is an SLA-simple semigroup.*

Hence, to solve our problem, we will concentrate on SLA-simple semigroups. Since any left ideal of a semigroup is also an SLA-ideal, any SLA-simple semigroup is necessarily left simple. We will discuss these semigroups in two steps. First we shall study left simple semigroups without idempotents. Then we shall turn to left simple semigroups having idempotents (left groups).

THEOREM 3. *Let S be a left simple semigroup without idempotents. Let g be an arbitrary fixed chosen element from S. Then the set $M_g = \{s \in S \mid sx \neq g \text{ for all } x \in S\}$ is a proper right ideal of S.*

Proof. Let us start by showing that M_g is a proper subset of S. Since the semigroup S has no idempotents, S is an infinite semigroup. Thus there exists an element x_0 different from g. Since S is left simple, $Sx_0 = S$ holds. Therefore, there exists an element $y \in S$ so that $yx_0 = g$. Hence $y \notin M_g$ and $M_g \neq S$. [3; Vol. II, Lemma 8.3] states $yx \neq y$ for any $x, y \in S$. By letting $y = g$, we have $g \in M_g$. Hence M_g is a nonempty proper subset of S. Now we show M_g is a right ideal of S. Really, let $m \in M_g$ and $s \in S$. If the equation $(ms)x = g$ has a solution for suitably chosen x, then we get the solution $y = sx$ for $my = g$, which is a contradiction with $m \in M_g$. Hence $ms \in M_g$ for any $m \in M_g$ and any $s \in S$. □

THEOREM 4. *Let S be a left simple semigroup having no idempotents. Then the set M_g, from Theorem 3, is a proper SLA-ideal.*

Proof. As M_g is a semigroup, the only thing which we must prove is: for any $s \in S \setminus M_g$ there exists $x \in M_g$ such that $sx \in M_g$. The fact $s \notin M_g$ implies $sx = g$ for suitably chosen $x \in S$. If $x \notin M_g$, then, again by the definition of M_g, there exists $y \in S$ such that $xy = g$. Now, using the fact that M_g is a right ideal, we conclude that $sg = s(xy) = (sx)y = gy \in M_g$. □

Remark 1. In other words, Theorem 4 states that any left simple semigroup which is not a left group possesses a proper SLA-ideal. Thus an SLA-simple semigroup must be a left group.

Any left group S can be viewed in the form EG, where G is any maximal subgroup of S and E is a left-zero semigroup of all idempotents of S. Here, any $e \in E$ is a right identity element of EG.

THEOREM 5. *Let a left group $G_L \subset S$ be an SLA-ideal of a semigroup S. Suppose further that G_L is an SLA-simple semigroup. Then G_L is a minimal left ideal of S.*

Proof. Let $s \in S$, and let $ey \in G_L = EG$ be arbitrarily chosen elements. Since G_L is an SLA-ideal, one can find an element $fg \in EG$ such that $s(ey)fg =$

$seyg \in G_L$. Now, $sey = (sey)f = (sey)(fgg^{-1}) = (seyg)g^{-1}$ is an element of G_L. □

We can reformulate our results in the following way:

THEOREM 6. *An SLA-ideal G_L of a semigroup S is a minimal SLA-ideal of S if and only if G_L is both a minimal left ideal of S and an SLA-simple left group.*

The next theorem shows that SLA-simplicity of a left group EG is an attribute of the group G.

THEOREM 7. *Let $G_L = EG$ be a left group. Then G_L is SLA-simple if and only if G is SLA-simple.*

Proof.

a) Let M be a proper SLA-ideal of G. We show that EM is an SLA-ideal of G_L. For any element $fg \in EG$ there exists a $g_1 \in M$ such that $gg_1 \in M$. Then $(fg)g_1 = (fg)(eg_1) \in EM$, where e is the identity element of G. Since $EM \cap G = M$, EM is a proper SLA-ideal of EG.

b) Conversely, let M be a proper SLA-ideal of the left group G_L. We will consider the set eM, where $e \in G$ is the identity element of G. Because of multiplication in G_L, obviously $eM \subset G$.

We show eM is an SLA ideal of G. For given $g \in G$, there exists $m \in M$ such that $gm \in M$. Now we get

$$gm = (eg)m = e(gm) \in eM\,, \qquad \text{or}$$
$$gm = (ge)m = g(em) \in eM \qquad \text{with} \quad em \in eM\,.$$

This implies that eM is an SLA-ideal of G.

To finish the proof, we must prove that eM is a proper SLA-ideal of G. Clearly, the proof is trivial if $e \notin eM$. In the case $e \in eM$, we have to consider two possibilities.

First, $eM \neq G$, in which case eM is a proper SLA-ideal of G. Or $eM = G$. We prove in this case $E \subset M$, which forces $M = G_L$. Let $f \in E$ be an idempotent of G_L. Since M is an SLA-ideal of G_L, there exists $m \in M$ such that $fm \in M$. If we assume m in the form $m = f_1g$, where $f_1 \in E$ and $g \in G$, then $fm = ff_1g = fg \in M$. Since $eM = G$, also $g^{-1} \in eM$. Thus $(fg)g^{-1} \in MeM$. Since e is a right identity in G_L, we have $MeM = MM \subset M$. Hence, $fgg^{-1} = fe = f \in M$; thus $E \subset M$. We finish with $G_L = EG \subset MG$ $MeM = MM \subset M$, which is a contradiction with M being a proper SLA-ideal of G_L. □

If we combine our results with known results on completely simple semigroups (see [3; Vol. I, Corollary 2.52.b] and [13]), we obtain our *first main theorem*

THEOREM 8. *A semigroup S possesses a minimal SLA-ideal if and only if S possesses a kernel which is a rectangular band of SLA-simple groups.*

Now we focus our attention on SLA-simple groups. A group cannot possess a proper SLA-simple SLA-ideal, say M, because, by Theorem 6, M would be at the same time a left ideal of a group. Thus a group cannot contain a proper minimal SLA-ideal.

THEOREM 9. *Let G be a group, and H be an SLA-ideal of G. If H is a group, then $H = G$.*

Proof. Let $g \in G \setminus H$. Under suppositions of our theorem, there exists $h \in H$ such that $gh = x \in H$. Since H is a group, there exists $h_1 \in H$ such that $h_1 h = x$. Thus $gh = x = h_1 h$. By right reduction in G, we have $g = h_1$, which is a contradiction to $g \in G \setminus H$. Thus $H = G$ □

It is known that any subsemigroup of a periodic group is a subgroup. Thus we have

COROLLARY 1. *Any periodic group is SLA-simple.*

Now we turn our attention to groups which contain elements of infinite order. Among these, partially ordered groups play a significant role. The partial order on a group G is given by the semigroup P of nonegative elements. This semigroup is determined by the following conditions: $P \cap P^{-1} = e$, $PP \subset P$, $xpx^{-1} \in P$ for any $x \in G$ and $p \in P$. Then the partial order is defined by $a \preceq b$ if and only if $a^{-1}b \in P$. We recall that $a \preceq b$ if and only if $ax \preceq bx$ and $xa \preceq xb$ for any $x \in G$. Obviously, any $p \in P \setminus \{e\}$ is of infinite order. Only a trivial partial order exists for periodic groups.

A partially ordered group G is *directed* if for arbitrary $a, b \in G$ there exists $c \in G$ such that $a \preceq c$, $b \preceq c$.

Let G be a directed group, and $a \in G$. Let $U(a) = \{c \in G \mid \ a \preceq c\}$. Then any $b \in G$ can be written in the form $b = yz^{-1}$, where $y, z \in U(a)$ (see [4]). Thus for any $b \in G$ there exist $y, z \in U(a)$ such that $bz = y$. This immediately implies that $U(a)$ is an LA-ideal of G.

If $e \prec a$, then $a \prec a^2$ and $a^2 \in U(a)$. Then, for any $b, c \in U(a)$ we have $a \preceq b$, $a \preceq c$, and thus $a \prec a^2 \preceq bc$, $bc \in U(a)$. Hence $U(a)$ is a semigroup and in fact an SLA-ideal of S. We formulate this result as

THEOREM 10. *Any directed group is not SLA-simple.*

A deep result of Shimbireva [14; Theorem III] states that: *Every non-periodic Abelian group can be directed.* Assuming this result with Corollary 1 and Theorem 10, we have our *second main result.*

THEOREM 11. *An Abelian group is SLA-simple if and only if it is periodic.*

By Theorems 8 and 11, we are ready to give the complete answer promised at the beginning of this paper.

THEOREM 12. *An Abelian semigroup S possesses a minimal SLA-ideal if and only if S contains a kernel which is a periodic group.*

Thus in the commutative case the existence of elements of infinite order is a necessary and sufficient condition for a group not to be SLA-simple. We show that this is not true for G in the noncommutative case.

Example. Let G be a semigroup generated by the set $\{a, b\}$ subject to the generating relations $a^2 = b^2 = e$, $ae = ea = a$, $be = eb = b$. Obviously this is a group with an identity element e and contains also elements of infinite order, for example $ab, ba, abab, \dots$.

Any element of G can be written in the *basic form* $\dots ababab \dots$, in which a and b do not repeat immediately. The number of a's and b's in the basic form of an element u is called the *length* of u. Next we show that G is SLA-simple. Let us suppose that $H \subset G$ is an SLA-ideal of G. We show $a, b \in H$, and thus $H = G$. If H is an SLA-ideal of G, it holds $aH \cap H \neq \emptyset$ and $bH \cap H \neq \emptyset$. Let us suppose $a \notin H$. There exists $u \in H$ such that $au \in H$. Evidently $u \neq e$. Also $u \neq b$ since $u = b$ implies $ab \in H$ and $b \in H$. Since H is a semigroup, $abb = a \in H$, which is a contradiction. Thus the length of u is greater than 1. Any element $u \in G$ of length greater than 1 belongs to one of the following four classes. In each of these classes we shall consider elements in the basic form of length greater than 1.

C_1 is the class of all elements of G of the form $ab \dots a$,
C_2 is the class of all elements of G of the form $ab \dots b$,
C_3 is the class of all elements of G of the form $ba \dots a$,
C_4 is the class of all elements of G of the form $ba \dots b$.

Obviously for any u from either C_1 or C_4 it holds that $u^2 = e$. Now consider the product au in each of these four cases. If $u = ab \dots a$, then $au = aab \dots a = b \dots a = v \in H$. Since H is a semigroup, we have $vu = auu = a \in H$, which is a contradiction. If $u = a \dots b$, then $au = aa \dots b = b \dots b = v \in H$. Then $u = av$ and $uv = avv = a \in H$, which is a contradiction. If $u = b \dots a$, then $au = ab \dots a = v \in H$. Then $u = av$ and $uv = avv = a \in H$, which is a contradiction. If $u = ba \dots b$, then $au = aba \dots b = v \in H$. Then $vu = auu = a \in H$, which is a contradiction. We have a contradiction in any of the four possible cases. Thus $a \in H$. Analogously, $b \in H$, and thus $G = H$. Therefore Theorem 11 is not valid in the noncommutative case.

A particular result can be obtained if we consider a factor-group by the commutant. It is known (see [14]) that *every group G whose factor-group by the commutant K is not periodic can be directed.* Therefore we can summarize our results for a general case.

THEOREM 13. *Let G be an arbitrary group.*

a) *If G is periodic, then it is SLA-simple.*
b) *If the factor-group G/K (K is the commutant) is not periodic, then G is not SLA-simple.*

We finish by stating two open problems which would enable to solve our problem in the noncommutative case:

1. Does there exist a nonperiodic group with a proper SLA-ideal whose factor-group by the commutant is periodic?
2. Does there exist a partially ordered group with a proper SLA-ideal which cannot be directed?

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