Janusz Jerzy Charatonik

Homogeneous means and some functional equations

Mathematica Slovaca, Vol. 48 (1998), No. 4, 391--398

Persistent URL: http://dml.cz/dmlcz/130586

Math. Slovaca, **48** (1998), No. 4, 391–398

Mathematica
Slovaca

HOMOGENEOUS MEANS AND SOME FUNCTIONAL EQUATIONS

JANUSZ J. CHARATONIK

(*Communicated by Ľubica Holá*)

ABSTRACT. Conditions are shown concerning a continuous open surjection f on the closed unit interval $[0,1]$ under which the functional equation $f(\mu(x,y)) = \mu(f(x), f(y))$ has no solution $\mu\colon [0,1]\times[0,1] \to [0,1]$ among homogeneous means on $[0,1]$.

A *mean* on a nonempty topological Hausdorff space X is a (continuous) mapping $m\colon X \times X \to X$ such that $m(x,y) = m(y,x)$ and $m(x,x) = x$ whenever $x, y \in X$.

Various kinds of means and their basic properties have been discussed in Aumann's habilitation thesis [2] and [3]. Aumann [4], Bacon [5], Eckmann [8], Eckmann, Ganea and Hilton [9], and Sigmon [11] have shown that there are wide classes of spaces which do not admit any mean. And although the concept of a mean is defined for an arbitrary topological Hausdorff space, the most important space which admits a mean is the simplest one, viz. the closed unit interval $[0,1]$ of reals. For some open questions concerning means on $[0,1]$ see Bacon [5; p. 13] and Baker and Wilder [6; p. 103]. In the present paper just means m on $[0,1]$ will be discussed.

Functional equations of the type

$$f(\mu(x,y)) = \mu(f(x), f(y)), \tag{1}$$

with μ given and f unknown, have been studied extensively (see e.g. Aczél [1]). In this paper, however, we will consider (1) with f given and μ unknown, exactly as it is done in [6]. In this context, the equation is a functional equation in a single variable in the sense of the book by Kuczma [10]. But unlikely theorems concerning equation (1) in [10], we do not assume that f is injective.

AMS Subject Classification (1991): Primary 39B22, 26B35.
Key words: closed interval, functional equation, homogeneous, mapping, mean, open.

In what follows all mappings are assumed to be continuous. For shortness we let I to denote the interval $[0,1]$. We will consider a mapping $\nu\colon I\times I\to I$ such that

$$\nu(x,x)=x \qquad \text{for all} \quad x\in I \tag{2}$$

and that the functional equation

$$f\big(\nu(x,y)\big)=\nu\big(f(x),f(y)\big) \tag{3}$$

is satisfied for all $x,y\in I$, where $f\colon I\to I$ is a given mapping. Further, $g\colon I\to I$ will always denote a surjection defined by

$$g(x)=\begin{cases} 2x & \text{if } x\in[0,1/2]\,,\\ 2-2x & \text{if } x\in[1/2,1]\,. \end{cases} \tag{4}$$

Some relations between the functional equation (1) (or (3)) and the mapping $f=g$ were studied by Wilder [13] and by Baker and Wilder [6]. In particular, it is shown in [6] that if $f=g$, then the functional equation (1) has no solution μ among the means on $[0,1]$. This is a corollary to a more general result (see [6; p. 92, Theorem 5]) which runs as follows.

5. THEOREM. (Baker and Wilder) *If a mapping $\nu\colon I\times I\to I$ satisfies condition*

$$\nu(x,x)=x \qquad \textit{for all} \quad x\in I \tag{2}$$

and the functional equation

$$g\big(\nu(x,y)\big)=\nu\big(g(x),g(y)\big) \qquad \textit{for} \quad x,y\in I\,, \tag{6}$$

then

$$\begin{array}{llll} \textit{either} & \nu(x,y)=x & \textit{for all} & x,y\in I\,,\\ \textit{or} & \nu(x,y)=y & \textit{for all} & x,y\in I\,. \end{array} \tag{7}$$

We apply the above theorem to show that the conclusion (7) holds true in the case when equation (6) is replaced by (3), where f is a mapping from I into I, a restriction of which is similar to g in a sense that will be explained below, provided that the mapping ν satisfies some additional conditions. Next, the obtained result will be applied to get (7) in case when (3) holds not necessary for $f=g$ but for any member $f\colon I\to I$ of a countable family of open mappings, provided that ν is homogeneous. Recall that a mapping $f\colon I\to I$ is said to be *open* if it maps open subsets of the domain onto open subsets of the range, and a mapping $\nu\colon I\times I\to I$ is said to be *homogeneous* if for each constant $t\in[0,1]$ the equality

$$\nu(tx,ty)=t\nu(x,y) \tag{8}$$

holds for every $x,y\in I$. Observe that the means $\mu(x,y)$ on I defined by $(x+y)/2$, $\sqrt{xy}$, $\min(x,y)$ and $\max(x,y)$ are homogeneous, while $\mu(x,y) = \min(x,y)/(1+|x-y|)$ is not.

Two surjective mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ between topological spaces are said to be *equivalent* provided that there are homeomorphisms $h_X : X_1 \to X_2$ and $h_Y : Y_1 \to Y_2$ such that $f_2 \circ h_X = h_Y \circ f_1$. This concept generalizes the condition saying that $f_2 : [a,b] \to [a,b]$ is a *conjugate* of $f_1 : I \to I$ (considered in [6; p. 92]) in the sense that there is a homeomorphism $h : I \to [a,b]$ such that $f_1 = h^{-1} \circ f_2 \circ h$.

Now we formulate the main result of the paper.

9. Theorem. *Let mappings $\nu : I \times I \to I$ and $f : I \to I$ be such that*

$$\nu(x,x) = x \qquad \textit{for all} \quad x \in I \tag{2}$$

and

$$f(\nu(x,y)) = \nu(f(x), f(y)) \qquad \textit{for all} \quad x,y \in I. \tag{3}$$

If there are subintervals $[a,b]$ and $[c,d] = f([a,b])$ of I and homeomorphisms $h_1 : I \to [a,b]$ and $h_2 : [c,d] \to I$ which satisfy the conditions

$$g = h_2 \circ (f|[a,b]) \circ h_1, \tag{10}$$

$$h_1(\nu(x,y)) = \nu(h_1(x), h_1(y)) \qquad \textit{for all} \quad x,y \in I, \tag{11}$$

$$\nu([c,d] \times [c,d]) \subset [c,d], \tag{12}$$

$$h_2(\nu(x,y)) = \nu(h_2(x), h_2(y)) \qquad \textit{for all} \quad x,y \in [c,d], \tag{13}$$

then

$$\begin{aligned} &\textit{either} \quad \nu(x,y) = x \quad \textit{for all} \quad x,y \in I, \\ &\textit{or} \qquad \nu(x,y) = y \quad \textit{for all} \quad x,y \in I. \end{aligned} \tag{7}$$

14. Remarks.

1) The existence of homeomorphisms h_1 and h_2 satisfying (10) denotes that the restriction $f|[a,b] : [a,b] \to [c,d]$ and the mapping $g : I \to I$ defined by (4) are equivalent.

2) Condition (12) is assumed to make functional equation (13) possible; more precisely, to be sure that the composition $h_2 \circ \nu$ is well defined.

15. Proof of Theorem 9. We apply Theorem 5. To this end it is enough to show that the assumed conditions imply that the mapping ν under consideration satisfies functional equation (6). Put, for shortness, $f_0 = f|[a,b]$. Let $x,y \in I$, and observe the following sequence of equivalences.

$$\begin{aligned} g(\nu(x,y)) &= h_2\Big(f_0\Big(h_1(\nu(x,y))\Big)\Big) && \text{by (10)} \\ &= h_2\Big(f_0\Big(\nu(h_1(x), h_1(y))\Big)\Big) && \text{by (11)} \\ &= h_2\Big(\nu\Big(f_0(h_1(x)), f_0(h_1(y))\Big)\Big) && \text{by (3)} \\ &= \nu\Big(h_2\Big(f_0(h_1(x))\Big), h_2\Big(f_0(h_1(y))\Big)\Big) && \text{by (12) and (13)} \\ &= \nu(g(x), g(y)) && \text{by (10).} \end{aligned}$$

Thus ν fulfills (6), so Theorem 5 can be applied, from which (7) follows. The proof is complete. □

To present the above mentioned application of Theorem 9 we recall a countable family of open mappings of I onto itself. Let a positive integer k be given and let $m \in \{0, 1, \ldots, k\}$. Define a surjection $g_k \colon I \to I$ by the following conditions:

(a) if m is even, then $g_k(\frac{m}{k}) = 0$, and if m is odd, then $g_k(\frac{m}{k}) = 1$;
(b) for each m, the restriction $g_k|[\frac{m}{k}, \frac{m+1}{k}] : [\frac{m}{k}, \frac{m+1}{k}] \to I$ is defined as linear.

Thus this restriction, and hence the mapping g_k, is a surjection. Note that $g_k(0) = 0$ and that $g_k(1)$ is either 1 or 0 according to k is either odd or even. Observe that g_1 is the identity and $g_2 = g$. Further, note that each g_k is open.

Now we apply Theorem 9 to prove the next result which is just the previously mentioned extension of Baker and Wilder's Theorem 5 in which the condition demanding that ν satisfies functional equation (3) with $f = g = g_2$ (see (6)) is weakened to one saying that ν has to satisfy (3) with $f = g_k$ for an arbitrary $k \geq 2$ provided that ν satisfies a condition of homogeneity type.

16. THEOREM. *If a mapping $\nu \colon I \to I$ is such that*

$$\nu(x, x) = x \qquad \textit{for all} \quad x \in I, \tag{2}$$

and if for some integer $k \geq 2$ and for all $x, y \in I$ it satisfies the functional equation

$$g_k\big(\nu(x, y)\big) = \nu\big(g_k(x), g_k(y)\big) \tag{17}$$

and the condition

$$\nu\big((2/k)x, (2/k)y\big) = (2/k)\nu(x, y), \tag{18}$$

then

$$\begin{aligned} &\textit{either} \quad \nu(x, y) = x \quad \textit{for all} \quad x, y \in I, \\ &\textit{or} \qquad \nu(x, y) = y \quad \textit{for all} \quad x, y \in I. \end{aligned} \tag{7}$$

Proof. In Theorem 9 put $a = 0$, $b = 2/k$, $c = 0$ and $d = 1$. Define $h_1 \colon I \to [a, b] = [0, 2/k]$ by $h_1(x) = (2/k)x$ for all $x \in I$ and take $h_2 \colon I \to I$ as the identity, i.e., $h_2(x) = x$ for all $x \in I$. We have to verify that all the assumptions of Theorem 9 are fulfilled. Indeed, (2) is assumed, and (17) stands for (3) with $f = g_k$. It can easily be observed that $g_2 = \big(g_k|[0, 2/k]\big) \circ h_1$, whence (10) follows. Further, (11) is an immediate consequence of the definition of h_1 and of (18). Finally, since $[c, d] = [0, 1]$ and since h_2 is the identity, conditions (12) and (13) trivially hold. Thus Theorem 9 can be applied, so (7) follows as needed. □

19. Remark. Note that if $k = 2$, then the coefficient $2/k$ equals 1, so the needed equality (18) turns into the identity. It is so because for $k = 2$ the theorem is a particular case of Baker and Wilder's Theorem 5 of [6] which was proved without any homogeneity assumption. Thus the following question is natural.

20. Question. Is condition (18) on the mapping ν an essential assumption in Theorem 16 for $k > 2$?

Since condition (18) is a very particular case of the homogeneity condition (8) for the mapping ν, we get the following corollaries to Theorem 16.

21. Corollary. *If a homogeneous mapping $\nu\colon I \times I \to I$ is such that*

$$\nu(x,x) = x \qquad \textit{for all} \quad x \in I\,, \tag{2}$$

and if, for some integer $k \geq 2$ and for all $x, y \in I$ it satisfies the functional equation

$$g_k\big(\nu(x,y)\big) = \nu\big(g_k(x), g_k(y)\big)\,, \tag{17}$$

then

$$\begin{aligned} &\textit{either} \quad \nu(x,y) = x \quad \textit{for all} \quad x,y \in I\,,\\ &\textit{or} \qquad\ \ \nu(x,y) = y \quad \textit{for all} \quad x,y \in I\,. \end{aligned} \tag{7}$$

22. Corollary. *If $k \geq 2$, then the functional equation*

$$g_k\big(\mu(x,y)\big) = \mu\big(g_k(x), g_k(y)\big) \qquad \textit{for all} \quad x,y \in I \tag{23}$$

has no solution μ among means on I satisfying the condition

$$\mu\big((2/k)x, (2/k)y\big) = (2/k)\mu(x,y) \qquad \textit{for all} \quad x,y \in I\,, \tag{24}$$

thus among homogeneous means on I.

The next result is also a consequence of Theorem 16.

25. Corollary. *Given a closed bounded interval J, let a mapping $\psi\colon J \times J \to J$ be such that*

$$\psi(x,x) = x \qquad \textit{for all} \quad x \in J\,. \tag{26}$$

If there are a mapping $f\colon J \to J$ with

$$f\big(\psi(x,y)\big) = \psi\big(f(x), f(y)\big) \qquad \textit{for all} \quad x,y \in J\,, \tag{27}$$

and a homeomorphism $h\colon I \to J$ such that, for some $k \geq 2$,

$$g_k = h^{-1} \circ f \circ h\,, \tag{28}$$

and if the mapping $\nu\colon I \times I \to I$ *defined by*

$$\nu(x,y) = h^{-1}\bigl(\psi\bigl(h(x),h(y)\bigr)\bigr) \qquad \textit{for} \quad x,y \in I \tag{29}$$

satisfies the condition

$$\nu\bigl((2/k)x,(2/k)y\bigr) = (2/k)\nu(x,y) \qquad \textit{for all} \quad x,y \in I\,, \tag{18}$$

then

$$\begin{array}{llll} \textit{either} & \psi(x,y) = x & \textit{for all} & x,y \in J\,, \\ \textit{or} & \psi(x,y) = y & \textit{for all} & x,y \in J\,. \end{array} \tag{30}$$

Proof. It is easy to verify that (26) and (29) imply (2). Further, we get (17) by the following sequence of arguments:

$$\begin{aligned} g_k\bigl(\nu(x,y)\bigr) &= h^{-1}\Bigl(f\Bigl(h\bigl(\nu(x,y)\bigr)\Bigr)\Bigr) && \text{by (28)} \\ &= h^{-1}\Bigl(f\Bigl(h\Bigl(h^{-1}\bigl(\psi\bigl(h(x),h(y)\bigr)\bigr)\Bigr)\Bigr)\Bigr) && \text{by (29)} \\ &= h^{-1}\Bigl(f\Bigl(\psi\bigl(h(x),h(y)\bigr)\Bigr)\Bigr) \\ &= h^{-1}\Bigl(\psi\Bigl(f\bigl(h(x),h(y)\bigr)\Bigr)\Bigr) && \text{by (27)} \\ &= h^{-1}\Bigl(\psi\Bigl(h\Bigl(h^{-1}\bigl(f\bigl(h(x)\bigr)\bigr)\Bigr),h\Bigl(h^{-1}\bigl(f\bigl(h(y)\bigr)\bigr)\Bigr)\Bigr)\Bigr) \\ &= \nu\Bigl(h^{-1}\Bigl(f\bigl(h(x)\bigr)\Bigr),h^{-1}\Bigl(f\bigl(h(y)\bigr)\Bigr)\Bigr) && \text{by (29)} \\ &= \nu\bigl(g_k(x),g_k(y)\bigr) && \text{by (28)}\,. \end{aligned}$$

Finally, condition (18) is assumed. Thus Theorem 16 can be applied, whence we conclude that alternative (7) holds. In the first case, if $\nu(x,y) = x$ for all $x,y \in I$, then for all $x,y \in J$ we have

$$\psi(x,y) = h\bigl(\nu\bigl(h^{-1}(x),h^{-1}(y)\bigr)\bigr) = h\bigl(h^{-1}(x)\bigr) = x\,.$$

Similarly, in the second case, we find $\psi(x,y) = y$ for all $x,y \in J$. Thus (30) follows and the proof is complete. □

31. Question. Does the conclusion (30) of Corollary 25 hold under a (more natural) assumption of a particular case of the homogeneity condition concerning the mapping ψ instead of (18) for ν?

Recall the following characterization of open mappings of closed bounded intervals, which is due to Whyburn (see [12; p. 184, (1.3)]).

32. Proposition. (Whyburn) *A surjective mapping* $f\colon J_1 \to J_2$ *between closed bounded intervals* J_1 *and* J_2 *is open if and only if* f *is equivalent to* $g_k\colon I \to I$ *for some positive integer* k.

33. Remark. In Corollary 25 the existence of the homeomorphism $h\colon I \to J$ satisfying (28) for some $k \geq 2$ means that f is a conjugate of g_k, so it is equivalent to g_k. Therefore f is open by Proposition 32.

Taking $J = I$ in Corollary 25 and replacing ψ by ν and ν by ν_0 we get a stronger version of Theorem 16.

34. Proposition. *Let mappings* $\nu\colon I \times I \to I$ *and* $f\colon I \to I$ *be such that*

$$\nu(x,x) = x \qquad \text{for all} \quad x \in I\,, \tag{2}$$

and

$$f\bigl(\nu(x,y)\bigr) = \nu\bigl(f(x), f(y)\bigr) \qquad \text{for all} \quad x,y \in I\,. \tag{3}$$

If f *is a conjugate of* g_k *for some* $k \geq 2$ *and if for a homeomorphism* $h\colon I \to I$ *with*

$$g_k = h^{-1} \circ f \circ h\,, \tag{28}$$

the mapping $\nu_0\colon I \times I \to I$ *defined by*

$$\nu_0(x,y) = h^{-1}\bigl(\nu\bigl(h(x), h(y)\bigr)\bigr) \qquad \text{for all} \quad x,y \in I \tag{35}$$

satisfies the condition

$$\nu_0\bigl((2/k)x, (2/k)y\bigr) = (2/k)\nu_0(x,y) \qquad \text{for all} \quad x,y \in I\,, \tag{36}$$

then

$$\begin{aligned} &\text{either} \quad \nu(x,y) = x \quad \text{for all} \quad x,y \in I\,, \\ &\text{or} \quad\quad\; \nu(x,y) = y \quad \text{for all} \quad x,y \in I\,. \end{aligned} \tag{7}$$

In the light of Proposition 32 and Remark 33 the following questions seem to be interesting.

37. Question. Is the conclusion (7) true if ν satisfies, besides (2), functional equation (3) for a fixed open mapping $f\colon I \to I$ distinct from a homeomorphism (and, perhaps, a kind of homogeneity condition of the form (18) or (36))?

38. Question. Is the result formulated in Corollary 22 true for all (not necessary homogeneous) means μ on I?

39. Remark. The methods presented in this paper were also successfully exploited to produce corresponding versions of Baker and Wilder's Theorem 4 of [6; p. 92] and its extension due to Wilder [13] concerning inverse limit means. For details see [7].

REFERENCES

[1] ACZÉL, J.: *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.

[2] AUMANN, G.: *Aufbau von Mittelwerten mehrerer Argumente I*, Math. Ann. **109** (1933), 235–253.

[3] AUMANN, G.: *Aufbau von Mittelwerten mehrerer Argumente II (Analytische Mittelwerte)*, Math. Ann. **111** (1935), 713–730.

[4] AUMANN, G.: *Über Räume mit Mittelbildungen*, Math. Ann. **119** (1943), 210 215.

[5] BACON, P.: *An acyclic continuum that admits no mean*, Fund. Math. **67** (1970), 11 13.

[6] BAKER, J. A.—WILDER, B. E.: *Constructing means on certain continua and functional equations*, Aequationes Math. **26** (1983), 89–103.

[7] CHARATONIK, J. J.: *Inverse limit means and some functional equations*, Rocky Mountain J. Math. **23** (1993), 41–48.

[8] ECKMANN, B.: *Räume mit Mittelbildungen*, Comment. Math. Helv. **28** (1954), 329 340.

[9] ECKMANN, B.—GANEA, T.—HILTON, P. J.: *Generalized means.* In: Studies in Mathematical Analysis and Related Topics, Stanford University Press, Stanford, Calif., 1962, pp. 82–92.

[10] KUCZMA, M.: *Functional Equations in a Single Variable.* Monografie Matematyczne Vol. 46, PWN, Warszawa, 1968.

[11] SIGMON, K.: *On the existence of a mean on certain continua*, Fund. Math. **63** (1968), 311–319.

[12] WHYBURN, G. T.: *Analytic Topology.* Amer. Math. Soc. Colloq. Publ. 28, Amer. Math. Soc., Providence, RI, 1942 (Reprinted with corrections 1971).

[13] WILDER, B. E.: *Inverse limit means are not preserved under homeomorphisms*, Rocky Mountain J. Math. **19** (1989), 549–551.

Received March 10, 1995

Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
PL-50-384 Wrocław
POLAND

E-mail: jjc@hera.math.uni.wroc.pl

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior
Ciudad Universitaria
04510 México, D. F.
MÉXICO

E-mail: jjc@gauss.matem.unam.mx