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HOMOGENEOUS MEANS AND SOME FUNCTIONAL EQUATIONS

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ABSTRACT. Conditions are shown concerning a continuous open surjection \( f \) on the closed unit interval \([0, 1]\) under which the functional equation \( f(\mu(x, y)) = \mu(f(x), f(y)) \) has no solution \( \mu: [0, 1] \times [0, 1] \to [0, 1] \) among homogeneous means on \([0, 1]\).

A mean on a nonempty topological Hausdorff space \( X \) is a (continuous) mapping \( m: X \times X \to X \) such that \( m(x, y) = m(y, x) \) and \( m(x, x) = x \) whenever \( x, y \in X \).

Various kinds of means and their basic properties have been discussed in Aumann's habilitation thesis [2] and [3]. Aumann [4], Bacon [5], Eckmann [8], Eckmann, Ganea and Hilton [9], and Sigmon [11] have shown that there are wide classes of spaces which do not admit any mean.

And although the concept of a mean is defined for an arbitrary topological Hausdorff space, the most important space which admits a mean is the simplest one, viz. the closed unit interval \([0, 1]\) of reals. For some open questions concerning means on \([0, 1]\) see Bacon [5; p. 13] and Baker and Wilder [6; p. 103]. In the present paper just means \( m \) on \([0, 1]\) will be discussed.

Functional equations of the type

\[ f(\mu(x, y)) = \mu(f(x), f(y)), \]

with \( \mu \) given and \( f \) unknown, have been studied extensively (see e.g. Aczél [1]). In this paper, however, we will consider (1) with \( f \) given and \( \mu \) unknown, exactly as it is done in [6]. In this context, the equation is a functional equation in a single variable in the sense of the book by Kuczma [10]. But unlikely theorems concerning equation (1) in [10], we do not assume that \( f \) is injective.

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In what follows all mappings are assumed to be continuous. For shortness we let $I$ to denote the interval $[0,1]$. We will consider a mapping $\nu: I \times I \to I$ such that
\[ \nu(x,x) = x \quad \text{for all} \quad x \in I \] (2)
and that the functional equation
\[ f(\nu(x,y)) = \nu(f(x),f(y)) \] (3)
is satisfied for all $x,y \in I$, where $f: I \to I$ is a given mapping. Further, $g: I \to I$ will always denote a surjection defined by
\[ g(x) = \begin{cases} 2x & \text{if } x \in [0,1/2], \\ 2 - 2x & \text{if } x \in [1/2,1]. \end{cases} \] (4)

Some relations between the functional equation (1) (or (3)) and the mapping $f = g$ were studied by Wilder [13] and by Baker and Wilder [6]. In particular, it is shown in [6] that if $f = g$, then the functional equation (1) has no solution $\mu$ among the means on $[0,1]$. This is a corollary to a more general result (see [6; p. 92, Theorem 5]) which runs as follows.

5. THEOREM. (Baker and Wilder) If a mapping $\nu: I \times I \to I$ satisfies condition
\[ \nu(x,x) = x \quad \text{for all} \quad x \in I \] (2)
and the functional equation
\[ g(\nu(x,y)) = \nu(g(x),g(y)) \quad \text{for} \quad x,y \in I, \] (6)
then
\begin{align*}
\text{either} & \quad \nu(x,y) = x \quad \text{for all} \quad x,y \in I, \\
\text{or} & \quad \nu(x,y) = y \quad \text{for all} \quad x,y \in I. \quad \text{(7)}
\end{align*}

We apply the above theorem to show that the conclusion (7) holds true in the case when equation (6) is replaced by (3), where $f$ is a mapping from $I$ into $I$, a restriction of which is similar to $g$ in a sense that will be explained below, provided that the mapping $\nu$ satisfies some additional conditions. Next, the obtained result will be applied to get (7) in case when (3) holds not necessary for $f = g$ but for any member $f: I \to I$ of a countable family of open mappings, provided that $\nu$ is homogeneous. Recall that a mapping $f: I \to I$ is said to be open if it maps open subsets of the domain onto open subsets of the range, and a mapping $\nu: I \times I \to I$ is said to be homogeneous if for each constant $t \in [0,1]$ the equality
\[ \nu(tx,ty) = tv(x,y) \] (8)
holds for every $x,y \in I$. Observe that the means $\mu(x,y)$ on $I$ defined by $(x+y)/2$, $\sqrt{xy}$, $\min(x,y)$ and $\max(x,y)$ are homogeneous, while $\mu(x,y) - \min(x,y)/(1 + |x - y|)$ is not.
Two surjective mappings \( f_1: X_1 \to Y_1 \) and \( f_2: X_2 \to Y_2 \) between topological spaces are said to be equivalent provided that there are homeomorphisms \( h_X: X_1 \to X_2 \) and \( h_Y: Y_1 \to Y_2 \) such that \( f_2 \circ h_X = h_Y \circ f_1 \). This concept generalizes the condition saying that \( f_2: [a, b] \to [a, b] \) is a conjugate of \( f_1: I \to I \) (considered in [6; p. 92]) in the sense that there is a homeomorphism \( h: I \to [a, b] \) such that \( f_1 = h^{-1} \circ f_2 \circ h \).

Now we formulate the main result of the paper.

9. THEOREM. Let mappings \( \nu: I \times I \to I \) and \( f: I \to I \) be such that

\[
\nu(x, x) = x \quad \text{for all } x \in I \quad (2)
\]

and

\[
f(\nu(x, y)) = \nu(f(x), f(y)) \quad \text{for all } x, y \in I. \quad (3)
\]

If there are subintervals \([a, b]\) and \([c, d] = f([a, b])\) of \( I \) and homeomorphisms \( h_1: I \to [a, b] \) and \( h_2: [c, d] \to I \) which satisfy the conditions

\[
g = h_2 \circ (f|_{[a, b]}) \circ h_1, \quad (10)
\]

\[
h_1(\nu(x, y)) = \nu(h_1(x), h_1(y)) \quad \text{for all } x, y \in I, \quad (11)
\]

\[
\nu([c, d] \times [c, d]) \subset [c, d], \quad (12)
\]

\[
h_2(\nu(x, y)) = \nu(h_2(x), h_2(y)) \quad \text{for all } x, y \in [c, d], \quad (13)
\]

then

either \( \nu(x, y) = x \) for all \( x, y \in I \),

or \( \nu(x, y) = y \) for all \( x, y \in I \). \quad (7)


1) The existence of homeomorphisms \( h_1 \) and \( h_2 \) satisfying (10) denotes that the restriction \( f|_{[a, b]}: [a, b] \to [c, d] \) and the mapping \( g: I \to I \) defined by (4) are equivalent.

2) Condition (12) is assumed to make functional equation (13) possible; more precisely, to be sure that the composition \( h_2 \circ \nu \) is well defined.

15. Proof of Theorem 9. We apply Theorem 5. To this end it is enough to show that the assumed conditions imply that the mapping \( \nu \) under consideration satisfies functional equation (6). Put, for shortness, \( f_0 = f|_{[a, b]} \). Let \( x, y \in I \), and observe the following sequence of equivalences.

\[
g(\nu(x, y)) = h_2\left(f_0\left(h_1(\nu(x, y))\right)\right) \quad \text{by (10)}
\]

\[
= h_2\left(f_0\left(\nu(h_1(x), h_1(y))\right)\right) \quad \text{by (11)}
\]

\[
= h_2\left(\nu\left(f_0(h_1(x)), f_0(h_1(y))\right)\right) \quad \text{by (3)}
\]

\[
= \nu\left(h_2\left(f_0(h_1(x))\right), h_2\left(f_0(h_1(y))\right)\right) \quad \text{by (12) and (13)}
\]

\[
= \nu(g(x), g(y)) \quad \text{by (10)}.
\]
Thus $\nu$ fulfills (6), so Theorem 5 can be applied, from which (7) follows. The proof is complete.

To present the above mentioned application of Theorem 9 we recall a countable family of open mappings of $I$ onto itself. Let a positive integer $k$ be given and let $m \in \{0, 1, \ldots, k\}$. Define a surjection $g_k: I \rightarrow I$ by the following conditions:

(a) if $m$ is even, then $g_k \left( \frac{m}{k} \right) = 0$, and if $m$ is odd, then $g_k \left( \frac{m}{k} \right) = 1$;

(b) for each $m$, the restriction $g_k\left[ \frac{m}{k}, \frac{m+1}{k} \right]: \left[ \frac{m}{k}, \frac{m+1}{k} \right] \rightarrow I$

is defined as linear.

Thus this restriction, and hence the mapping $g_k$, is a surjection. Note that $g_k(0) = 0$ and that $g_k(1)$ is either 1 or 0 according to $k$ is either odd or even. Observe that $g_1$ is the identity and $g_2 = g$. Further, note that each $g_k$ is open.

Now we apply Theorem 9 to prove the next result which is just the previously mentioned extension of Baker and Wilder's Theorem 5 in which the condition demanding that $\nu$ satisfies functional equation (3) with $f = g = g_k$ (see (6)) is weakened to one saying that $\nu$ has to satisfy (3) with $f = g_k$ for an arbitrary $k \geq 2$ provided that $\nu$ satisfies a condition of homogeneity type.

16. THEOREM. If a mapping $\nu: I \rightarrow I$ is such that

$$\nu(x, x) = x \quad \text{for all} \quad x \in I, \quad (2)$$

and if for some integer $k \geq 2$ and for all $x, y \in I$ it satisfies the functional equation

$$g_k(\nu(x, y)) = \nu(g_k(x), g_k(y)) \quad (17)$$

and the condition

$$\nu((2/k)x, (2/k)y) = (2/k)\nu(x, y), \quad (18)$$

then

either $\nu(x, y) = x$ for all $x, y \in I$, \hspace{1cm} (7)

or $\nu(x, y) = y$ for all $x, y \in I$.

Proof. In Theorem 9 put $a = 0$, $b = 2/k$, $c = 0$ and $d = 1$. Define $h_1: I \rightarrow [a, b] = [0, 2/k]$ by $h_1(x) = (2/k)x$ for all $x \in I$ and take $h_2: I \rightarrow I$ as the identity, i.e., $h_2(x) = x$ for all $x \in I$. We have to verify that all the assumptions of Theorem 9 are fulfilled. Indeed, (2) is assumed, and (17) stands for (3) with $f = g_k$. It can easily be observed that $g_2 = (g_k\left[0, 2/k\right]) \circ h_1$, whence (10) follows. Further, (11) is an immediate consequence of the definition of $h_1$ and of (18). Finally, since $[c, d] = [0, 1]$ and since $h_2$ is the identity, conditions (12) and (13) trivially hold. Thus Theorem 9 can be applied, so (7) follows as needed. □
19. Remark. Note that if \( k = 2 \), then the coefficient \( 2/k \) equals 1, so the needed equality (18) turns into the identity. It is so because for \( k = 2 \) the theorem is a particular case of Baker and Wilder's Theorem 5 of [6] which was proved without any homogeneity assumption. Thus the following question is natural.

20. Question. Is condition (18) on the mapping \( \nu \) an essential assumption in Theorem 16 for \( k > 2 \)?

Since condition (18) is a very particular case of the homogeneity condition (8) for the mapping \( \nu \), we get the following corollaries to Theorem 16.

21. COROLLARY. If a homogeneous mapping \( \nu: I \times I \to I \) is such that
\[
\nu(x,x) = x \quad \text{for all } \ x \in I ,
\]
and if, for some integer \( k \geq 2 \) and for all \( x, y \in I \) it satisfies the functional equation
\[
g_k(\nu(x,y)) = \nu(g_k(x), g_k(y)),
\]
then
\[
\text{either } \ \nu(x,y) = x \quad \text{for all } \ x, y \in I , \quad \text{or} \quad \nu(x,y) = y \quad \text{for all } \ x, y \in I .
\]

22. COROLLARY. If \( k \geq 2 \), then the functional equation
\[
g_k(\mu(x,y)) = \mu(g_k(x), g_k(y)) \quad \text{for all } \ x, y \in I
\]
has no solution \( \mu \) among means on \( I \) satisfying the condition
\[
\mu((2/k)x, (2/k)y) = (2/k)\mu(x, y) \quad \text{for all } \ x, y \in I ,
\]
thus among homogeneous means on \( I \).

The next result is also a consequence of Theorem 16.

25. COROLLARY. Given a closed bounded interval \( J \), let a mapping \( \psi: J \times J \to J \) be such that
\[
\psi(x,x) = x \quad \text{for all } \ x \in J .
\]
If there are a mapping \( f: J \to J \) with
\[
f(\psi(x,y)) = \psi(f(x), f(y)) \quad \text{for all } \ x, y \in J ,
\]
and a homeomorphism \( h: I \to J \) such that, for some \( k \geq 2 \),
\[
g_k = h^{-1} \circ f \circ h ,
\]
and if the mapping $\nu: I \times I \to I$ defined by

$$\nu(x, y) = h^{-1}(\psi(h(x), h(y))) \quad \text{for } x, y \in I$$

(29)

satisfies the condition

$$\nu((2/k)x, (2/k)y) = (2/k)\nu(x, y) \quad \text{for all } x, y \in I,$$

(18)

then

either $\psi(x, y) = x \quad \text{for all } x, y \in J,$

or $\psi(x, y) = y \quad \text{for all } x, y \in J.$

(30)

Proof. It is easy to verify that (26) and (29) imply (2). Further, we get

(17) by the following sequence of arguments:

$$g_k(\nu(x, y)) = h^{-1}\left( f(h(\nu(x, y))) \right)$$

by (28)

$$= h^{-1}\left( f\left(h\left(h^{-1}(\psi(h(x), h(y)))\right)\right) \right)$$

by (29)

$$= h^{-1}\left( f\left(\psi(h(x), h(y))\right) \right)$$

by (27)

$$= h^{-1}\left(\psi\left(f(h(x)), h(y)\right)\right)$$

by (27)

$$= h^{-1}\left(\psi\left(h\left(h^{-1}(f(h(x)))\right), h\left(h^{-1}(f(h(y)))\right)\right)\right)$$

$$= \nu\left(h^{-1}\left( f(h(x))\right), h^{-1}\left( f(h(y))\right)\right)$$

by (29)

$$= \nu(g_k(x), g_k(y))$$

by (28).

Finally, condition (18) is assumed. Thus Theorem 16 can be applied, whence we conclude that alternative (7) holds. In the first case, if $\nu(x, y) = x$ for all $x, y \in I$, then for all $x, y \in J$ we have

$$\psi(x, y) = h(\nu(h^{-1}(x), h^{-1}(y))) = h(h^{-1}(x)) = x.$$ 

Similarly, in the second case, we find $\psi(x, y) = y$ for all $x, y \in J$. Thus (30) follows and the proof is complete.

31. Question. Does the conclusion (30) of Corollary 25 hold under a (more natural) assumption of a particular case of the homogeneity condition concerning the mapping $\psi$ instead of (18) for $\nu$?

Recall the following characterization of open mappings of closed bounded intervals, which is due to Whyburn (see [12; p. 184, (1.3)]).
32. **Proposition.** (Whyburn) A surjective mapping \( f: J_1 \to J_2 \) between closed bounded intervals \( J_1 \) and \( J_2 \) is open if and only if \( f \) is equivalent to \( g_k: I \to I \) for some positive integer \( k \).

33. **Remark.** In Corollary 25 the existence of the homeomorphism \( h: I \to J \) satisfying (28) for some \( k \geq 2 \) means that \( f \) is a conjugate of \( g_k \), so it is equivalent to \( g_k \). Therefore \( f \) is open by Proposition 32.

Taking \( J = I \) in Corollary 25 and replacing \( \psi \) by \( \nu \) and \( \nu \) by \( \nu_0 \) we get a stronger version of Theorem 16.

34. **Proposition.** Let mappings \( \nu: I \times I \to I \) and \( f: I \to I \) be such that

\[
\nu(x, x) = x \quad \text{for all } x \in I, \tag{2}
\]

and

\[
f(\nu(x, y)) = \nu(f(x), f(y)) \quad \text{for all } x, y \in I. \tag{3}
\]

If \( f \) is a conjugate of \( g_k \) for some \( k \geq 2 \) and if for a homeomorphism \( h: I \to I \) with

\[
g_k = h^{-1} \circ f \circ h, \tag{28}
\]

the mapping \( \nu_0: I \times I \to I \) defined by

\[
\nu_0(x, y) = h^{-1}(\nu(h(x), h(y))) \quad \text{for all } x, y \in I \tag{35}
\]

satisfies the condition

\[
\nu_0((2/k)x, (2/k)y) = (2/k)\nu_0(x, y) \quad \text{for all } x, y \in I, \tag{36}
\]

then

\[
either \nu(x, y) = x \quad \text{for all } x, y \in I, \tag{7}
or \nu(x, y) = y \quad \text{for all } x, y \in I.
\]

In the light of Proposition 32 and Remark 33 the following questions seem to be interesting.

37. **Question.** Is the conclusion (7) true if \( \nu \) satisfies, besides (2), functional equation (3) for a fixed open mapping \( f: I \to I \) distinct from a homeomorphism (and, perhaps, a kind of homogeneity condition of the form (18) or (36))?  

38. **Question.** Is the result formulated in Corollary 22 true for all (not necessary homogeneous) means \( \mu \) on \( I \)?

39. **Remark.** The methods presented in this paper were also successfully exploited to produce corresponding versions of B a k e r and W i l d e r 's Theorem 4 of [6; p. 92] and its extension due to W i l d e r [13] concerning inverse limit means. For details see [7].
REFERENCES


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