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TOPOLOGICAL CONDITIONS FOR THE EXISTENCE OF FIXED POINTS

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ABSTRACT. Let X be an arbitrary topological space. Let $F\colon X \to X$ be a (multi)function with closed graph. A sufficient condition for the existence of a fixed point of F is given. Topological and quasi-uniform approaches to contractibility are compared.

1. Introduction

In this paper we give a topological generalization of the Banach contraction principle. We generalize the result published in [2]. We prove that if X is an arbitrary topological space and $F\colon X \to X$ is a feebly topologically contractive multifunction with a closed graph, then F has a fixed point. We compare our approach with the approach of Morales in [4].

The present result differs from the old one ([2]) in several points. Instead of Hausdorff spaces we consider arbitrary topological spaces. Instead of continuous functions we consider functions with closed graphs. The contractivity condition is weakened and the idea used in proof is different and more simple.

Before presenting results, we establish some terminology.

DEFINITION 1. ([2]) Let X be an arbitrary topological space. Let $f\colon X \to X$ be a function. Then f is said to be *topologically contractive* (in what follows *t-contractive*) if and only if

($*$) for every open cover $\mathfrak{c}$ of X and for every couple of points $a, b \in X$ there exists $n \in \mathbb{N}$ such that $\forall k \geq n$ $\exists P \in \mathfrak{c}$ such that $f^k(a) \in P$ and $f^k(b) \in P$ holds.

Now, we define a weaker contractivity condition. The topological contractivity means, that $f^k(a)$ and $f^k(b)$ are "near" for every k which is sufficiently large.

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Instead, we will show that it suffices to assume a weaker concept: always, when we choose what it means "near" (in terms of an open cover), there exists at least one index k such that $f^k(a)$ and $f^k(b)$ are near one to another.

DEFINITION 2. Let X be an arbitrary topological space. Let $f\colon X \to X$ be a function. Then f is said to be *feebly topologically contractive* (in what follows *f–t-contractive*) if and only if

(*) for every open cover $\mathfrak{c}$ of X and for every couple of points $a, b \in X$ there exists $k \in \mathbb{N}$ such that there exists $P \in \mathfrak{c}$ such that $f^k(a) \in P$ and $f^k(b) \in P$ holds.

2. Results

THEOREM 1. *Let X be an arbitrary topological space. Let $f\colon X \to X$ be an f–t-contractive function with a closed graph. Then f has a fixed point.*

Proof. This is a consequence of Theorem 2. □

We will prove our result in a more general setting, than announced in Theorem 1. First, we need the notion of a feeble contractivity for multifunctions. Let $F\colon X \to X$ be a multifunction, let $A \subset X$. Let us denote $F(A) = \bigcup_{a\in A} F(a)$. Let z be an element of X, we denote $F^2(z) = F\big(F(z)\big)$ and, by induction, $F^k(z) = F\big(F^{k-1}(z)\big)$ for every natural k, $k \geq 3$. We say that a point $z \in X$ is a *fixed point* of F if and only if $z \in F(z)$.

DEFINITION 3. Let X be an arbitrary topological space. Let $F\colon X \to X$ be a multifunction. Then F is said to be *feebly topologically contractive* (*f–t-contractive*) if and only if (*) for every open cover $\mathfrak{c}$ of X and for every couple of points $a, b \in X$ there exists $k \in \mathbb{N}$ such that $\exists P \in \mathfrak{c}$ such that $F^k(a) \subseteq P$ and $F^k(b) \cap P \neq \emptyset$ holds.

THEOREM 2. *Let X be an arbitrary topological space. Let $F\colon X \to X$ be an f–t-contractive multifunction with a closed graph. Then F has a fixed point. Moreover, if X is T_1 then the fixed point is unique and if z is the fixed point then $F(z) = \{z\}$.*

Proof.

I) Let X be an arbitrary topological space. Let us suppose, to obtain a contradiction, that F has no fixed point. Let us consider the space $X \times X$ with the product topology. In this case the set $O = X \times X - \operatorname{Gr} F$ is an open neighborhood of the diagonal $\{[x,x]\,;\ x \in X\}$. So we can define an open cover $\mathfrak{c}$ of X as follows: $\mathfrak{c} = \{V\,;\ V$ is open in X and $V \times V \subseteq O\}$. Let us take a point

$a \in X$ and a point $b \in F(a)$. From f–t-contractivity of F we obtain: there exists $k \in \mathbb{N}$ such that $\exists P \in \mathfrak{c}$ such that $F^k(a) \subseteq P$ and $F^k(b) \cap P \neq \emptyset$ holds. Since $b \in F(a)$, $F^k(b) \subseteq F^{k+1}(a)$ holds so $F^{k+1}(a) \cap P \neq \emptyset$ is true or, differently said, $F\big(F^k(a)\big) \cap P \neq \emptyset$. But we know that $F^k(a) \subseteq P$, so $F(P) \cap P \neq \emptyset$ holds, which gives $\operatorname{Gr} F \cap P \times P \neq \emptyset$ and $\operatorname{Gr} F \cap O \neq \emptyset$. But this is a contradiction.

II) Let X be a T_1 space. Let z be a fixed point of F. Let there exist $b \in F(z)$ which is different from z. Let us define an open cover $\mathfrak{c}$ of X by $\mathfrak{c} = \big\{X - \{z\}, X - \{b\}\big\}$. Since F is f–t-contractive, there exist $P \in \mathfrak{c}$ and $k \in \mathbb{N}$ such that $F^k(z) \subseteq P$. From $z \in F(z)$ we obtain $z \in F^{k-1}(z)$, so $\{z, b\} \subseteq F(z) \subseteq F^k(z) \subseteq P$. But this is impossible.

Now, let us suppose that F has at least two distinct fixed points a and b. Now we know that $F(a) = \{a\}$ and $F(b) = \{b\}$. Defining an open cover $\mathfrak{c}$ of X by $\mathfrak{c} = \big\{X - \{a\}, X - \{b\}\big\}$ and using the f–t-contractivity of F we obtain the same type of contradiction as above. □

EXAMPLE 1. The condition $F(a) \subseteq P$ in $(*)$ in Definition 3 of f–t-contractivity cannot be replaced by $F(a) \cap P \neq \emptyset$.

Let $X = \{1, 2, 3\}$ with the discrete topology. Let $F \colon X \to X$ be defined by $F(1) = \{2, 3\}$, $F(2) = \{1, 3\}$, $F(3) = \{1, 2\}$.

Then the graph of F is closed and

$(**)$ for every open cover $\mathfrak{c}$ of X and for every couple of points $a, b \in X$ there exists $k \in \mathbb{N}$ such that $\exists P \in \mathfrak{c}$ such that $F^k(a) \cap P \neq \emptyset$ and $F^k(b) \cap P \neq \emptyset$ holds,

since $F^2(z) = \{1, 2, 3\}$ for every $z \in X$. But F has no fixed point.

The function $G \colon \mathbb{R} \to \mathbb{R}$ defined by $G(x) = (-\infty, x-1] \cup [x+1, +\infty)$ $\forall x \in \mathbb{R}$ is an another example of a multifunction with closed graph which satisfies $(**)$ and has no fixed point. The condition $(**)$ is satisfied for $k = 1$ while $\forall a, b \in \mathbb{R}$ $G(a) \cap G(b) \neq \emptyset$.

EXAMPLE 2. Let $X = \mathbb{R}$ with the usual topology. Let $G \colon X \to X$ be defined by $G(x) = \{0, n\}$ if $x = \frac{1}{n}$ for some integer $n > 1$, $G(x) = 0$ otherwise.

Let us define $F \colon X \to X$ by $F(x) = \big\{-\frac{1}{2(n+1)}\big\}$ for $x \in \big(\frac{1}{n+1}, \frac{1}{n}\big)$ and n even, $F(x) = \big\{\frac{x}{2}\big\}$ for $x \in \big(\frac{1}{m+1}, \frac{1}{m}\big)$ and m odd, $F\big(\frac{1}{k}\big) = \big\{-\frac{1}{2(k+1)}, -\frac{1}{2k}, \frac{1}{2k}\big\}$ for every integer $k \geq 1$, $F(x) = \frac{1}{2}$ for $x > 1$ and $F(x) = \{0\}$ for $x \leq 0$.

Both G and F have closed graphs and they are f–t-contractive. Their fixed point is 0.

If we change slightly the definition of G by redefining only the value in the point 1 by $G(1) = \{0, 1\}$ (the old value being $\{0\}$) the situation changes. Firstly, 0 and 1 are the fixed points of G and, secondly, G is no more f–t-contractive. To see that it suffices to consider $a = 1$, $b = 0$ and $\mathfrak{c} = \big\{\mathbb{R} - \{0\}, \mathbb{R} - \{1\}\big\}$. So, it can happen that Theorem 2 does not fit for quite a nice multifunction and it works only for some of its selections. That is why we consider Theorem 1 as the main result of the paper.

EXAMPLE 3. ([2]) Let $X = \{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\} \cup \{0\}$ with the natural metric inherited from $\mathbb{R}$. Let us define $f\colon X \to X$ as follows:

$f(0) = 1$ and $f\big(\frac{1}{k}\big) = \frac{1}{k+1}$ for $k = 1, 2, \dots$. Then X is a complete metric space, f is t-contractive, but f has no fixed point. So, the condition of closedness of the graph of f in Theorem 1 is essential.

3. Topology versus quasi-uniformity

In this section we compare two approaches which permit to obtain fixed-point theorems in non uniformizable spaces. The first one is the approach of Morales [4]. He formulates his results in the quasi-uniform language. The second one is the purely topological approach of Theorem 2 of this paper.

A quasi-uniformity on a set X is a filter $\mathscr{U}$ on $X \times X$ satisfying the axioms of a uniformity, with the possible exception of the symmetry axiom.

Every topological space is quasi-uniformizable (see [5]).

Let (X, T) be a topological space. For each set $G \in T$, we put $S_G = (G \times G) \cup \big((X - G) \times X\big)$.

Then the family $S = \{S_G;\ G \in T\}$ is a subbase for a quasiuniformity $\mathscr{U}(T)$ which in turn generates T. So T is the family of all subsets G of X such that for each $x \in G$ there exists $U \in \mathscr{U}$ such that $U[x] \subset G$ ([5]). In what follows we denote by $\mathscr{U}(T)$ this quasi-uniformity introduced by Pervin.

DEFINITION 4. ([4]) Let $(X, \mathscr{U})$ be a quasi-uniform space. Let f be a function on X into itself. We say that f is *occasionally small* if, for every ordered pair $(x, y) \in X \times X$ and every $U \in \mathscr{U}$, there exists a positive integer $n = n\big((x, y), U\big)$ such that $\big(f^n(x), f^n(y)\big) \in U$.

DEFINITION 5. ([4]) We say that a topological space is *US* if every convergent sequence has a unique limit.

DEFINITION 6. Let $(X, \mathscr{U})$ be a quasi-uniform space. According to Davis [1], a filter F on X is "*Cauchy*" if, for every $U \in \mathscr{U}$, there exists $x = x(U) \in X$ such that $U[x] \in F$. Morales in [4] defines a *Cauchy sequence* in X to be a sequence $\{x_n\}_{n=1}^{\infty}$ in X whose corresponding Frechet filter is Cauchy, that is, for every $U \in \mathscr{U}$, there exist $x = x(U) \in X$ and a positive integer $n = n(U)$ such that $x_m \in U[x]$ for all $m \geq n$. If every Cauchy sequence in X converges, we say that X is *sequentially complete.*

The following result was proved by Morales in [4].

THEOREM 3. ([4; Theorem 1.1]) *Let $X = (X, \mathscr{U})$ be a sequentially complete US quasi-uniform space and let f be a function on X into itself. If at least one iterate f^k is an occasionally small contraction, then f has a unique fixed point u. Moreover, for arbitrary $x_0 \in X$ $\lim_{n\to\infty} (f^k)^n(x_0) = u$.*

Let us compare Theorem 1 and Theorem 3. Is one of these theorems more general than the other one? We will give a partial answer to this question.

If we restrict ourselves to Haussdorf spaces, the situation is clear. The assumptions of Theorem 3 imply the assumptions of Theorem 1. So for Hausdorff spaces Theorem 1 is more general. Let us show it.

First of all, since $h = f^k$ is a contraction and since being a contraction in [4] is more than being continuous, since X is Hausdorff, the graph of h is closed. One of the quickest ways to prove the f–t-contractivity of h (which works even for X non Hausdorff) could be the following one:

Let a, b be two arbitrary elements of X, let $\mathfrak{c}$ be an open cover of X. Since the assumptions of Theorem 3 hold, the function h has a fixed point x_0 such that each of the following two sequences $\{h^n(a)\}_{n=1}^{\infty}$ and $\{h^n(b)\}_{n=1}^{\infty}$ converges to x_0. There exists an open set $U \in \mathfrak{c}$ such that $x_0 \in U$. Then there exists a positive integer m such that the points $h^m(a)$, $h^m(b)$ are elements of U. So h is f–t-contractive.

In our opinion the basic merit of the topological approach is that it does not require X to be sequentially complete. Any topological space is sufficient. Of course, the closedness of the graph of f reveals that X does have some qualities.

The following example shows that in general the assumptions of Theorem 1 do not imply the assumptions of Theorem 3, even when we restrict ourselves on domain spaces X which are compact metric.

EXAMPLE 4. Let $X = \{\frac{1}{n}\,;\ n \text{ is a positive integer}\} \cup \{\frac{1}{n} + \frac{1}{n^2}\,;\ n \text{ is a positive integer}\} \cup \{0\}$. We consider on X the the usual metric inherited from $\mathbb{R}$.

Let $f\colon X \to X$ be defined as follows:

for every positive integer n put

$$f\left(\frac{1}{n} + \frac{1}{n^2}\right) = \frac{1}{n} \quad \text{and} \quad f\left(\frac{1}{n}\right) = \frac{1}{n+1}$$
$$f(0) = 0\,.$$

The graph of f is closed since f is continuous, and it is easy to check the f–t-contractibility of f since for every $x \in X$ the sequence $\{f^n(x)\}_{n=1}^{\infty}$ converges to 0. But f is not contractive with respect to the classic metric (nor to the uniformity $\mathscr{U} = \{U_\varepsilon\,;\ \varepsilon > 0\}$ where $U_\varepsilon = \{[a,b] \in X \times X\,;\ |a-b| < \varepsilon\}$).

4. Occasionally small or f–t-contractive?

The following examples try to give a clearer picture of the interdependence of the two notions mentioned above.

EXAMPLE 5. If $f\colon X \to X$ is f–t-contractive it need not be occasionally small even when X is a compact topological space. Let (X,T) be a topological space defined by $X = \langle 1,4\rangle$, $T = \{\langle 1,4\rangle, (1,3), (2,4), (2,3), (1,4)\}$. Let us define $f\colon X \to X$ as follows: $f(2) = 3$, $f(x) = 2$ for each $x \neq 2$.

Then f is f–t-contractive while for every open cover $\mathfrak{c}$ of X the set X is an element of $\mathfrak{c}$.

But f is not occasionally small with respect to the uniformity $\mathscr{U}(T)$.

To see that, put

$$A = S_{(1,3)} = \big((1,3)\times(1,3)\big) \cup \big((\{1\}\cup\langle 3,4\rangle)\times\langle 1,4\rangle\big),$$
$$B = S_{(2,4)} = \big((2,4)\times(2,4)\big) \cup \big((\langle 1,2\rangle\cup\{4\})\times\langle 1,4\rangle\big),$$
$$U = A\cap B, \qquad a=2,\ b=3.$$

Of course $U \in \mathscr{U}(T)$. Let n be a positive integer. Since for every positive integer k we have $\{f^k(a), f^k(b)\} = \{2,3\}$, the couples $[f^n(a), f^n(b)]$ and $[f^n(b), f^n(a)]$ are not elements of U.

EXAMPLE 6. If $f\colon X\to X$ is occasionally small then it need not be f–t-contractive. Let $(X,\mathscr{U})$ be a uniform space defined as follows: $X = \{1, \frac12, \frac13, \frac14, \dots\}$, $\mathscr{U} = \{U_\varepsilon;\ \varepsilon > 0\}$ where $U_\varepsilon = \{[a,b]\in X\times X;\ |a-b|<\varepsilon\}$.

Let us define $f\colon X\to X$ as follows: $f\left(\frac1n\right) = \frac{1}{n+1}$ for every positive integer n.

The graph of f is closed. It is easy to see that f is occasionally small. But f is not f–t-contractive. To show this, let us put $\mathfrak{c} = \bigcup\limits_{k=1}^{\infty}\left\{\frac1k\right\}$. Then $\mathfrak{c}$ is an open cover of X. Let $a=1$, $b=\frac12$. Let n be an arbitrary positive integer. Since $f^n(a)\neq f^n(b)$, then there exists no set $P\in\mathfrak{c}$ such that $\{f^n(a), f^n(b)\}\subset P$ holds.

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