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Mathematica Slovaca, Vol. 35 (1985), No. 2, 175--184

Persistent URL: http://dml.cz/dmlcz/130597

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BASES IN r-SPASES

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In [5] a new class of spaces, called r-spaces, was introduced and studied as a generalization of topological spaces. In the present paper som other properties of r-space are investigated.

1. INTRODUCTION

Throughout this paper we shall use the notations from [2] and 2^X will denote the class of all subsets of X.

Let X be a nonempty set and $\mathcal{D}$ be a classes of subsets of X satisfying the following conditions:

Ω_1: $\emptyset, X \in \mathcal{D}$
Ω_2: for each $A \subset X$ and $B \in \mathcal{D}$ such that $B \subset A$ there is a maximal element C of $\{M \in \mathcal{D}: M \subset A\}$ such that $B \subset C \subset A$.

The pair $(X, \mathcal{D})$ is called an r-space and $\mathcal{D}$ the class of open subsets of this r-space.

The class $\mathcal{D}$ define uniquely a relation σ on 2^X by: $A\sigma B$ iff A is a maximal element of the class $\{M \in \mathcal{D}: M \subset B\}$. The relation σ is called a relation of the interior and if $A\sigma B$, then A is called an interior of B. A subset A of X is said to be closed if $(X - A)$ is open. We say that B is a closure of A if $(X - B)\sigma(X - A)$.

A subset of X of the form $\{x\} \cup A$, where $x \in X$ and A is open is said to be a preneighbourhood of x. By a neighbourhood of a point $x \in X$ we mean any open subset containing x.

In some sense the paper is a continuation of [5] and some results from [5] will be used essentially in the sequel.

In the second section we introduce the notion of an adherent point relative to an open set and the notions of semiopen and semiclosed sets. In terms of this notions closed sets, closures of the sets and some of their properties are characterized.

The third section is devoted to the questions of bases in r-spaces.

If $(X, \mathcal{D})$ denotes an r-space, then $\mathcal{D}$ always means the class of all open subsets of this r-space. To simplify the notation we often refer to the r-space X instead of the more proper form $(X, \mathcal{D})$.

2. Adherent points and semiopen sets

In the sequel X will be an r-space, $\mathscr{D}$ the class of all open subsets and $\mathscr{T}$ the class of all closed subsets of X.

Definition 1. *If $A \subset X$, $B \in \mathscr{D}$ and $x \in X$, then x is said to be an adherent point of A relative to B if $V \cap A \neq \emptyset$ for each neighbourhood V of x containing the preneighbourhood $V_1 = B \cup \{x\}$ of x.*

If $A \subset D \subset X$ and $D \in \mathscr{T}$, then we denote the set of all adherent points of A relative to $(X - D)$ by ${}^{D}A$.

As an immediate consequence of Definition 1 we have

Corollary 1. *Let $A \subset B \subset X$, where B is a closed set and let $x \in X$. Then $x \in {}^{B}A$ iff $V \cap A \neq \emptyset$ for each neighbourhood V of x that contains the preneighbourhood $(\{x\} \cup (X - B))$ of x.*

Now suppose that $A \subset B \subset X$ and B is closed. It is clear that if $x \in A$, then $V \cap A \neq \emptyset$ for each neighbourhood V of x. From this it follows $A \subset {}^{B}A$ by Corollary 1.

Next it is evident that if $x \in (x - B)$, the set $V = (X - B)$ is a neighbourhood of x containing the preneighbourhood $(\{x\} \cup (X - B))$ of x and such that $V \cap A = \emptyset$. Therefore if $x \in (X - B)$, then $x \notin {}^{B}A$ again by Corollary 1. This means that ${}^{B}A \subset B$.

Thus we may conclude

Theorem 1. *If $A \subset B \subset X$ and B is closed, then $A \subset {}^{B}A \subset B$.*

Definition 2. *A subset A of X such that $A \subset B$, where B is closed, is said to be a semiclosed set relative to B if ${}^{B}A = A$.*

From Corollary 1 it follows that if $A \subset B \subset X$ and B is closed, then $x \notin {}^{B}A$ iff there is a neighbourhood V of x containing the preneighbourhood $(\{x\} \cup (X - B))$ and such that $V \cap A = \emptyset$. From this and from Definition 2 we have the following characterization of semiclosed sets.

Corollary 2. *Let $A \subset B \subset X$ and let B is closed. Then ${}^{B}A = A$ iff for each $x \in X - A$ and the preneighbourhood $V_1 = (\{x\} \cup (X - B))$ of x there exists a neighbourhood V of x such that $V_1 \subset V$ and $V \cap A = \emptyset$.*

Theorem 2. *A subset A of an r-space* X *is closed iff for each closed subset* B *of* X *such that $A \subset B$ the set A is semiclosed relative to B (i.e. ${}^{B}A = A$).*

Proof. Let first $A \subset B \subset X$, where A, B are closed sets. We shall prove that ${}^{B}A = A$. If $x \in X - A$, then $V = X - A$ is a neighbourhood of x, since A is closed. From $A \subset B$ it follows, that $(\{x\} \cup (X - B)) \subset V$ and $V \cap A = \emptyset$. But then by Corollary 2 ${}^{B}A = A$.

To prove the converse, suppose that $A \subset B \subset X$, B is closed and $V_1 = (V \cup \{x\})$

is a preneighbourhood of x such that $V_1 \cap A = \emptyset$. Since $V_1 \cap A = \emptyset$ and therefore $A \subset (X-V)$, where $(X-V)$ is closed, by our assumptions ${}^{(X-V)}A = A$. Apllying Corollary 2 we see that for $((X-(X-V)) \cup \{x\}) = V \cup \{x\} = V_1$ there exists a neighbourhood V_2 of x such that $V_1 \subset V_2$ and $V_2 \cap A = \emptyset$. But then by Corollary 3 of [5] it is clear that A is closed, which completes the proof.

In [5] the author proved that if X is an r-space, $A \subset B \subset X$ and B is a closed set, then B is a closure of A iff $V \cap A \neq \emptyset$ for each neighbourhood V of x including the preneighbourhood $V_1 = (\{x\} \cup (X-B))$ of x. By Corollary 1 this means that if B is a closure of A, then $B \subset {}^BA$. Since from Theorem 1 it follows ${}^BA \subset B$, we thus have

Theorem 3. *Let $A \subset B \subset X$, where B is a closed set. Then B is a closure of A iff ${}^BA = B$.*

It is natural to define the notion of a semiopen set as dual to the notion of semiclosed set.

Definition 3. *Let $B \subset A \subset X$ and let B be an open set. Then A is semiopen relative to B iff $(X-A)$ is semiclosed relative to $(X-B)$, i.e. ${}^{(X-B)}(X-A) = (X-A)$.*

From this, from Corollary 2 and Theorem 2 we have the following two statements.

Corollary 3. *Let $C \subset D \subset X$ and let C be an open set. Then D is semiopen relative to C iff for each $x \in D$ and the preneighbourhood $V_x = C \cup \{x\}$ of x there is a neighbourhood V of x satisfying $V_x \subset V \subset D$.*

Theorem 4. *A subset D of X is open iff D is semiopen relative to each $C \subset D$, where C is open.*

The operator ${}^BA$ assigns to each pair A, B of subsets of X, where $A \subset B$ and B is closed, the subset ${}^BA$ of X. It is clear that it ispossible to define an operator dual to ${}^BA$ as follows: Let A, B be subsets of X, where B is an open set and $B \subset A$. Let us denote

$$(1) \qquad {}_BA = X - {}^{(X-B)}(X-A).$$

As an immediate consequence of (1) and Corollary 1 it follows

(2) ${}_BA = \{x \in A$: for the preneighbourhood $V_x = B \cup \{x\}$ *there is an neighbourhood* V of x such that $V_x \subset V \subset A\}$.

As a consequence of (2) and Corollary 3 we have

Corollary 4. *Let $C \subset D \subset X$ and let C be an open set. Then D is semiopen relative to C iff ${}_CD = D$.*

From this and from Theorem 4 it follows

Corollary 5. *A subset D of X is open iff ${}_CD = D$ for each open subset C of X such that $C \subset D$.*

If $B \subset A \subset X$ and B is open, then by Theorem 11 of [5] B is an interior of A iff $V \cap (X - A) \neq \emptyset$ for each $x \in (A - B)$ and each neighbourhood V of x containing the preneighbourhood $V_x = \{x\} \cup B$. This means that B is an interior of A iff $\{x \in X$: there is a neighbourhood V containing the preneighbourhood $\{x\} \cup B$ and such that $V \subset A\} = B$. But from this and from (2) it follows

Corollary 6. *An open subset B of A, where $A \subset X$, is an interior of A iff ${}_BA = B$.*

The last theorem of this section shows the connection between the sets of the form ${}^BA$ and the closures of the set A.

Theorem 5. *Let X be an r-space. Let $A \subset B \subset X$, where B is closed and let $\{A_s\}_S$ be the class of all closures of A with $A_s \subset B$ for each $s \in S$. Then ${}^BA = \bigcap_S A_s$.*

Proof. By R_5 of the definition of an r-space in [5], S is a nonempty set. Since for each $s \in S$ $A \subset A_s \subset B$ and B is closed, then from Corollary 1 it is easy to see that ${}^BA \subset {}^{A_s}A$ for each $s \in S$. Aplying Theorem 3 and using the fact that A_s is a closure of A for each $s \in S$ it follows that ${}^{A_s}A = A_s$. But this implies ${}^BA \subset {}^{A_s}A = A_s$ for each $s \in S$ and so ${}^BA \subset \bigcap_S A_s$. To show that ${}^BA = \bigcap_S A_s$ it satisfies to prove $\bigcap_S A_s \subset {}^BA$. Suppose $\bigcap_S A_s \not\subset {}^BA$; then there is $x \in \bigcap_S A_s$ with $x \notin {}^BA$. Since $x \notin {}^BA$ then there is a neighbourhood V of x that contains the preneighbourhood $\{x\} \cup (X - B)$ of x with $A \cap V = \emptyset$. From this it follows $A \subset (X - V) \subset B$. But since $X - V$ is closed, then by R_5 of a definition of r-space given in [5] there exists a closure A_{s_1}, $s_1 \in S$ of A such that $A \subset A_{s_1} \subset (X - V)$. This shows that $x \in X - V$ which contradicts the fact that $x \in V$. Now is the proof complete.

3. Base and r-base

Let X be an r-space and $\mathscr{D}$ the class of all open subsets of X. In [5] the following characterization of an open subset of X was given:

A subset A of X is open iff

(3) for each $x \in A$ and each $V \in \mathscr{D}$ such that $V \subset A$ there exists $V_1 \in \mathscr{D}$ satisfying $(\{x\} \cup V) \subset V_1 \subset A$.

Now we shall show that if $\mathscr{D}_0$ is a subclass of $\mathscr{D}$ and $\mathscr{D}_0$ has some properties, then the set V in (3) can be considered to be from $\mathscr{D}_0$ only.

Definition 4. *A class $\mathscr{D}_0 \subset \mathscr{D}$ is said to be a base for $\mathscr{D}$ if $\mathscr{D}_0$ has the following properties*

I. *if $A \in \mathscr{D}$ and $x \in A$, then there exists $V \in \mathscr{D}_0$ such that $x \in V \subset A$.*

II. *if $A \subset B \subset X$, $A \in \mathscr{D}$, $x \in B - A$ and for each $V \in \mathscr{D}_0$, where $V \subset A$, there is $V_1 \in \mathscr{D}$ satisfying $(\{x\} \cup V) \subset V_1 \subset B$, then there exists $V_2 \in \mathscr{D}$ such that $(\{x\} \cup A) \subset V_2 \subset B$.*

Example 1. If $(X, \mathscr{D})$ is an r-space and $\mathscr{D}_0 = \mathscr{D}$, then it is clear that $\mathscr{D}_0$ is a base for $\mathscr{D}$.

Example 2. In [5] it was shown that if X is a connected topological space and $\mathscr{D}$ is the class of all connected subsets of X, then $(X, \mathscr{D})$ is an r-space. Let

$$\mathscr{D}_0 = \{A = \{x\} : x \in X\}. \tag{4}$$

It is well known (see e.g. [1] p. 170) that each one-point subset of a connected topological space is a connected subset of X. This means that $\mathscr{D}_0$ satisfies I of Definition 4. From Theorem 2 of [1] it follows that $\mathscr{D}_0$ satisfies the condition II of Definition 4 too. Thus, $\mathscr{D}_0$ is a base for $\mathscr{D}$.

Example 3. Let X be an universal algebra and let $\mathscr{D}$ consist of all subalgebras of X and of the empty set. As it was shown in [5], $(X, \mathscr{D})$ is an r-space.

Let for each $A \subset X$

$$J(A) = \bigcap \{B \in \mathscr{D} : A \subset B\} \tag{5}$$

be the algebraic closure operator on X (see e.g. [2]). Define the following class of subsets of X:

(6) $\mathscr{D}_0 = \{A \subset X$: there exists a finite subset B of X *such that* $J(B) = A\}$.

Algebraic operator J of closure on a set X has the following properties

(7) if $A \subset X$ and $a \in X$, then from $a \in J(A)$ it follows that $a \in J(B)$, where B is a finite subset of A

(8) if $A \subset B \subset X$, then $J(A) \subset J(B)$

(9) $A \subset J(A)$, for each $A \subset X$

(10) $A \in \mathscr{D}$ iff $J(A) = A$

We now show that $\mathscr{D}_0$ satisfies the conditions I and II of Definition 4.

First let $x \in A \in \mathscr{D}$. Since $A \in D$, then by (10) $J(A) = A$ and by (8) $J(\{x\}) \subset J(A) = A$. Since by (6) $J(\{x\}) \in \mathscr{D}_0$, then I. holds. To show II, suppose that $A \subset B \subset X$, $A \in \mathscr{D}$, $x \in B - A$ and for each $V \in \mathscr{D}_0$ such that $V \subset A$ there is $V_1 \in \mathscr{D}$ satisfying $(\{x\} \cup V) \subset V_1 \subset B$. We want to show that there is V_2 such that $(\{x\} \cup A) \subset V_2 \subset B$. Consider $J(A \cup \{x\})$. We prove that $J(A \cup \{x\}) \subset B$. By (7), it suffices to prove that for each $C \subset A$, where C is a finite set, it holds $J(C \cup \{x\}) \subset$

B. Let C be a finite subset of A. By our assumptions $J(J(C) \cup \{x\}) \subset B$. But by (9) $C \subset J(C)$ and therefore $(C \cup \{x\}) \subset (J(C) \cup \{x\})$. From this it follows $J(C \cup \{x\}) \subset J(J(C) \cup \{x\}) \subset B$. Since $J(A \cup \{x\}) \in \mathscr{D}$, it suffices to put $V_2 = J(A \cup \{x\})$.

Theorem 6. *Let $(X, \mathscr{D})$ be an r-space and $\mathscr{D}_0$ be a base for $\mathscr{D}$. Then a subset A of X is open iff*

(11) *for each $x \in A$ and each $V \in \mathscr{D}_0$ such that $V \subset A$ there exists $V_1 \in \mathscr{D}$ satisfying $(\{x\} \cup V) \subset V_1 \subset A$.*

Proof. If a subset A of X is open, then (11) follows immediatelly from (3). To prove the converse suppose that a subset A of X satisfies (11). By (3), it suffices to prove that if $x \in A$, $V \subset A$, where $V \in \mathscr{D}$, there is $V_1 \in \mathscr{D}$ satisfying $(\{x\} \cup V) \subset V_1 \subset A$. But by (11) for each $V_0 \subset V$ and x, where $V_0 \in \mathscr{D}_0$, there exists $V_2 \in \mathscr{D}$ such that $(V_0 \cup \{x\}) \subset V_2 \subset A$. But then by II of Definition 4 there is $V_1 \in \mathscr{D}$ satisfying $(\{x\} \cup V) \subset V_1 \subset A$, which completes the proof of the theorem.

Remark 1. If X is a topological space and $\mathscr{D}$ the class of all open subsets of X, then a class $\mathscr{D}_0 \subset \mathscr{D}$ satisfying I of Definition 4, is a base for $\mathscr{D}$ (in the sense of the terminology used in the theory of topological spaces). In topological spaces the condition II of Definition 4 follows from I.

Now we give an example illustrating that if $(X, \mathscr{D})$ is an r-space, $\mathscr{D}_0 \subset \mathscr{D}$ and $\mathscr{D}_0$ satisfies I of Definition 4, then an open subset A of X cannot be described by (11).

Example 4. Let X be an infinite set and $\mathscr{D}$ be the class of subsets of X consisting of

— all subsets of X consisting of $10 \cdot k$ elements, where $k = 1, 2, 3, \dots$
— all infinite subsets of X.

It is not difficult to verify that $\mathscr{D}$ satisfies Ω_1 and Ω_2 of the definition of an r-space and therefore $(X, \mathscr{D})$ is an r-space. Consider the class

$$\mathscr{D}_1 = \{A \subset X\colon A \text{ has precisely 10 elements}\}.$$

It is clear that $\mathscr{D}_1 \subset \mathscr{D}$ and $\mathscr{D}_1$ satisfies I of Definition 4. Now let A be a subset of X having exactly 25 elements. The set A has the following property:

For each $x \in A$ and $V \in \mathscr{D}_1$ there exists $V_1 \in \mathscr{D}$ such that $(V \cup \{x\}) \subset V_1 \subset A$.

We see that A is described by (11), but from the definition of the class $\mathscr{D}$ it follows that A is not open.

On the other hand, let $V \subset A$ and let V be a 20-point set. It is clear that $V \in \mathscr{D}$ and for each $V_1 \subset V$, where $V_1 \in \mathscr{D}_1$ and for eeach $x \in A - V$ there is $V_2 \in \mathscr{D}$ such that $(V_1 \cup \{x\}) \subset V_2 \subset A$. But there does not exist an element $V_3 \in \mathscr{D}$ satisfying $(V \cup \{x\}) \subset V_3 \subset A$. This means that $\mathscr{D}_1$ does not sytisfy II of Definition 4 and therefore.

As an immediate consequence of Theorem 6 and Theorem 7 of [6] we have the following assertion.

Theorem 7. *Let $(X, \mathscr{D}_1)$ and $(Y, \mathscr{D}_2)$ be r-spaces, $\mathscr{D}_0$ be a base for $\mathscr{D}_1$ and f be a mapping from X into Y. Then f is continuous mapping iff for each $x \in X$ and each $U \in \mathscr{D}_2$ such that $f(x) \in U$ and $V \in \mathscr{D}_0$, where $f(V) \subset U$, there exists $V_1 \in \mathscr{D}_1$ satisfying $(\{x\} \cup V) \subset V_1$ and $f(V_1) \subset U$.*

Theorem 8. *Let $(X, \mathscr{D})$ be an r-*space, *$\mathscr{D}_0$ be a base for $\mathscr{D}$ and let $A \subset B \subset X$, where A is open. Then A is an interior of B iff*

(12) *for each $x \in B - A$ there exists $V_x \in \mathscr{D}_0$, where $V_x \subset A$ such that $V \cap (X - B) \neq \emptyset$ for each $V \in \mathscr{D}$ satisfying $(V_x \cup \{x\}) \subset V$.*

Proof. If (12) holds, then it is clear that for each $V_1 \in \mathscr{D}$ such that $(A \cup \{x\}) \subset V_1$ we have $V_x \subset V_1$ and therefore $V_1 \cap (X - A) \neq \emptyset$. But then from Theorem 11 of [5] it follows that A is an interior of B.

To prove the converse, suppose that A is an interior of B, but (12) is not true. It is clear that (12) is not true iff there exists a point $a \in (B - A)$ such that for each $V_0 \in \mathscr{D}_0$, where $V_0 \subset A$, there is $V_1 \in \mathscr{D}$ satisfying $(V_0 \cup \{a\}) \subset V_1 \subset B$. Since $\mathscr{D}_0$ is a base for D, then by II. of Definition 4 there is $V_2 \in D$ such that $(A \cup \{a\}) \subset V_2 \subset B$. But then again by Theorem 11 of [5] A is not an interior of B, which is a contradiction.

Theorem 9. *Let $(X, \mathscr{D})$ be an r-space, $\mathscr{D}_0$ be a base for $\mathscr{D}$ and let $A \subset B \subset X$, where $A \in \mathscr{D}$. Then $x \in {}_AB$ iff $x \in {}_CB$ for each $C \in \mathscr{D}_0$ such that $C \subset A$.*

Proof. Let $x \in {}_AB$ and let $C \in \mathscr{D}_0$ such that $C \subset A$. Since $x \in {}_AB$, then there exists $V \in \mathscr{D}$ such that $(\{x\} \cup A) \subset V \subset B$. But $C \subset A$ and therefore also $(\{x\} \cup C) \subset V \subset B$. This means that $x \in {}_CB$. To prove the converse, suppose that $x \in {}_CB$ for each $C \in \mathscr{D}_0$ such that $C \subset A$. If $x \in A$, then from Definition 1 it follows immediately that $x \in {}_AB$. Noe it suffices to consider the case $x \notin A$. But by definition of the set ${}_C$B for each $C \subset A$, where $C \in \mathscr{D}_0$, there exists $V \in \mathscr{D}$ satisfying $(C \cup \{x\}) \subset V \subset B$. From this and from II of Definition 4 it follows that there is $V_1 \in \mathscr{D}$ such that $(\{x\} \cup A) \subset V_1 \subset B$. So we have $x \in {}_AB$.

From this, from Corollary 4, Theorem 8 and Corollary 5 we have the following three corollaries.

Corollary 7. *Let $(X, \mathscr{D})$ be an r-space, $\mathscr{D}_0$ be a base for $\mathscr{D}$ and $A \subset B \subset X$, where A is open. Then B is semiopen relative to A iff B is semiopen relative to each C of C_0 such that $C \subset A$.*

Corollary 8. *Let $(X, \mathscr{D})$ be an r-*space and *$\mathscr{D}_0$ be a base for $\mathscr{D}$. A subset A of X is open iff for each $C \in \mathscr{D}_0$ such that $C \subset A$ it holds ${}_CA = A$ (e.g. A is open iff A is semiopen relative to each $C \in \mathscr{D}_0$ such that $C \subset A$).*

Corollary 9. *Let* $(X, \mathscr{D})$ *be an* r-space, $\mathscr{D}_0$ *be a base for* $\mathscr{D}$ *and let* $A \subset B \subset X$, *where* A *is open. Then* A *is an interior of* B *iff for each* $x \in B - A$ *there exists* $C \in \mathscr{D}_0$, *where* $C \subset A$ *such that* $x \notin {}_C B$.

Now we give dual statements to the Theorems 6 and 7.

Theorem 10. *Let* $(X, \mathscr{D})$ *be an* r*-space,* $A \subset B \subset X$, *where* B *is closed and let* $\mathscr{D}_0$ *be a base for* $\mathscr{D}$. *Then* B *is a closure of* A *iff for each* $x \in B - A$ *there exists* $V_x \in \mathscr{D}_0$ *such that* $(V_x \cap B) = \emptyset$ and $V \cap A \neq \emptyset$ *for each* $V \in D$ *such that* $(V_x \cup \{x\}) \subset V$.

Theorem 11. *Let* $(X, \mathscr{D})$ *be an* r*-space,* $\mathscr{D}_0$ *be a base for* $\mathscr{D}$. *Then a subset* B *of* X *is closed iff for each* $x \notin B$ *and each* $C \in \mathscr{D}_0$ *such that* $C \cap B = \emptyset$ *there is* $V \in \mathscr{D}$ *satisfying* $(\{x\} \cup C) \subset V$ *and* $V \cap B = \emptyset$.

In the rest of this section we shall show that if a base $\mathscr{D}_0 \subset \mathscr{D}$, where $\mathscr{D}$ is the class of all open subsets of an r-space (X, D) satisfies the following condition

III. if $A \in \mathscr{D}$, $x \in A$ and $V \in \mathscr{D}_0$ such that $V \subset A$, then there is $V_1 \in \mathscr{D}_0$ satisfying $(\{x\} \cup V) \subset V_1 \subset A$,

then each open subset of X can be described by the sets of the class $\mathscr{D}_0$ only.

Definition 5. *Let* $(X, \mathscr{D})$ *be an* r*-space. A base* $\mathscr{D}_0$ *for* $\mathscr{D}$ *is said to be an* r*-base for* $\mathscr{D}$ *if* $\mathscr{D}_0$ *satisfies the condition III.*

Theorem 12. *Let* $(X, \mathscr{D})$ *be an* r*-space and* $\mathscr{D}_0$ *be an* r*-base for* $\mathscr{D}$. *A subset* A *of* X *is open, iff*

(13) *for each* $x \in A$ *and* $V \in \mathscr{D}_0$ *such that* $V \subset A$, *there exists* $V_1 \in \mathscr{D}_0$ *satisfying* $(\{x\} \cup V) \subset V_1 \subset A$.

Proof. If A is open, then half of the proof follows from III of Definition 5.

To prove the converse, suppose that $A \subset X$ and A satisfies (13). To prove that A is open, by Theorem 6 it suffices to show that if $V_0 \subset A$, where $V_0 \in \mathscr{D}_0$ and $x \in A$, then there is $V_2 \in \mathscr{D}$ such that $(\{x\} \cup V_0) \subset V_2 \subset A$. If $V_0 = \emptyset$, then according to I there exists a $V_2 \in \mathscr{D}_0 \subset \mathscr{D}$ such that $(\{x\} \cup V_0) \subset V_2 \subset A$. Let $V_0 \neq \emptyset$. If $x \in V_0$, then we can put $V_2 = V_0$. Let $x \neq V_0$. But then by (13) there exists $V_1 \in \mathscr{D}_0 \subset \mathscr{D}$ such that $(V_0 \cup \{x\} \subset V_1 \subset A$ which completes the proof of the theorem.

Remark 2. The base $\mathscr{D}_0$ defined in Example 3 satisfies also the condition III. and is therefore an r-base for $\mathscr{D}$. To show this, suppose that A is an subalgebra of X, $x \in A$, $V \subset A$ and $V \in \mathscr{D}_0$. Since $V \in \mathscr{D}_0$ then there is a finite subset B of A such that $J(B) = V$. Consider now $J(B \cup \{x\})$. Since $B \subset (B \cup \{x\})$, then $J(B) \subset J(B \cup \{x\})$ by (8) and $(J(B) \cup \{x\}) \subset J(B \cup \{x\}) \subset J(A) = A$ by (9). From this again by (8) it follows $J(J(B) \cup \{x\}) \subset J(A) = A$. To prove (13) it suffices to put $V_1 = J(J(B) \cup \{x\})$, since then $(V \cup \{x\}) \subset V_1 \subset A$.

Remark 3. It is easy to see that the base $\mathcal{D}_0$ defined in Example 2 and consisting of all one-pointed subsets of X does not satisfy III of Definition 5 and therefore is not an r-base for $\mathcal{D}$.

Finally we give an example showing that if a class $\mathcal{D}_0 \subset \mathcal{D}$, where $\mathcal{D}$ is the class of all open subsets of an r-space X, satisfies the conditions I and III, but does not satisfy II, then the open subsets of X cannot be described as the subsets of X satisfying (13).

Example 5. Let X be an infinite set and let the class $\mathcal{D}$ of subsets of X consists of:

— the set X

— all subsets of X containing exactly $10 \cdot k$ elements, where $k = 1, 2, 3, \ldots$

— all infinite subsets of X of the form $B = X - A$, where $A \subset X$ and A has 10 or more elements.

It is easy to see that $\mathcal{D}$ satisfies the conditions Ω_1 and Ω_2 of Definition of r-space given in the first section of this paper, which means that $(X, \mathcal{D})$ is an r-space.

If $\mathcal{D}_0 = \{M \subset X\colon M$ consists of $10 \cdot k$ elements, where $k = 1, 2, 3, \ldots\}$, then $\mathcal{D}_0$ has properties I and III of Definitions 4 and 5, respectively. Let $B = X - N$, where $N \subset X$ and has precisely 5 elements. It is not difficult to verify that for each $x \in B$ and $V \in \mathcal{D}_0$ such that $V \subset B$, there exists $V_1 \in \mathcal{D}_0$ satisfying $(\{x\} \cup V) \subset V_1 \subset B$. This means that B satisfies the condition (13), but on the other hand from the definition of $\mathcal{D}$ it follows $B \notin \mathcal{D}$.

Now let $A \subset B$, where A has the form $X - M$ and M consists of 10 elements. If $x \in B - A$ and $V \in \mathcal{D}_0$ such that $V \subset A$, then it is clear that there is $V_1 \in \mathcal{D}$ satisfying $(\{x\} \cup V) \subset V_1 \subset B$. But there is no $V_2 \in \mathcal{D}$ such that $(\{x\} \cup A) \subset V_2 \subset B$, which means that $\mathcal{D}$ does not satisfy II of Definition 4.

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Received January 10, 1983

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БАЗИСЫ В *r*-ПРОСТРАНСТВАХ

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Резюме

r-пространства являются обобщениями топологических пространств. В настоящей работе вводится понятие точки, близкости, полуоткрытого множества и понятия базисов и изучаются некоторые свойства *r*-пространств, связанные с этими понятиями.