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NONLINEAR BOUNDARY VALUE PROBLEMS
AT RESONANCE
FOR DIFFERENTIAL EQUATIONS
IN BANACH SPACES

BOGDAN PRZERADZKI

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ABSTRACT. The perturbation method developed in [12]–[16] is applied to non-
linear BVP's \( x' - A(t)x = f(t, x) \), \( B_1x(0) + B_2x(1) = B_3(x) \), in a Banach space, 
where the linear homogeneous problem possesses nontrivial solutions and the 
nonlinearities \( f \), \( B_3 \) have at most linear growth. Examples of such problems are given.

1. Introduction

The question of the solvability of boundary value problems \( Lx = N(x) \), where \( L \) is a linear differential operator with nontrivial kernel and \( N \) is a superposition 
operator, has a long history. The first remarkable result was obtained in 1969 by 
Landesman and Lazer [10] for the zero-data Dirichlet BVP for a second 
order elliptic equation in a bounded domain \( \Omega \) with \( N(x)(t) = \lambda_0 x + f(t, x) \), 
where \( \lambda_0 \) is a simple eigenvalue of the elliptic operator and \( f: \Omega \times \mathbb{R} \to \mathbb{R} \) is 
bounded. The authors used the well-known alternative method (see[1]), which 
was also applied by Williams [17] to generalize this result for an arbitrary 
eigenvalue (this means that the dimension of the linear space of solutions to 
\( Lx = 0 \) may be greater than 1 but finite; we shall say that the resonance is multidimensional). This and other methods were then used to get existence for 
many similar problems such as:

\[
\begin{align*}
    x'' + m^2 x &= f(t, x), & x(0) &= x(\pi) = 0, \\
    x'' &= f(t, x, x'), & x(0) &= x(T), & x'(0) &= x'(T), \\
    x' &= f(t, x), & x(0) &= x(T), 
\end{align*}
\]

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(see [6], [8], [11], for example). The almost complete list of references can be found in [4].

The perturbation method (this name was proposed by Kannan [9]) is based on the observation that if one perturbs the linear operator $L$ by $\lambda I$ ($I$ is the identity map, and $\lambda$, a small parameter), then it becomes invertible, solutions can be found and the only problem is to prove a compactness of the set of solutions to perturbed equations. Obviously, the nonlinearity $N$ should be bounded or, at least, sublinear as it is in the paper by de Figueiredo [3]. He obtained an abstract result for the equation $Lx = N(x)$, using the perturbation method, but his proof does not involve a form of the inverse operators $(L-\lambda I)^{-1}$. The present author has studied the abstract problem, taking into account a family of equations $L(\lambda)x = N(x)$ with $L(\lambda)$ invertible for $\lambda \neq \lambda_0$ and $L(\lambda_0)$, a Fredholm linear operator. The inverse operators are supposed to have the special form

$$L(\lambda)^{-1} = G_0(\lambda) + \sum_{j=1}^{n} c_j(\lambda) \langle u_j(\lambda), \cdot \rangle w_j(\lambda),$$

where all terms except $c_j(\lambda)$ have continuous extensions to $\lambda_0$, $|c_j(\lambda)| \to \infty$, $w_j(\lambda_0)$, $j = 1, \ldots , n$, span the kernel of $L(\lambda_0)$, and the common part of $\ker u_j(\lambda_0)$, $j = 1, \ldots , n$, equals the range of $L(\lambda_0)$. This generalization of $L-\lambda I$ to $L(\lambda)$ enables us to study equations depending explicitly on a real parameter (for instance, the bifurcation problems). On the other hand, the form of $L(\lambda)^{-1}$ is natural from point of view of applications: Green operators for ordinary differential equations have this form, and if the Hilbert-Schmidt theory is applicable, then $L(\lambda)^{-1}$ is a sum of a series built of eigenvalues and eigenfunctions, $c_j(\lambda) = (\lambda_0 - \lambda)^{-1}$, and $G_0(\lambda)$ is the rest of this series in whose terms $\lambda_0$ does not occur. The method is useful not only for sublinear nonlinearities. They may have a linear growth at infinity or even be superlinear. A lot of theoretical results based on the topological degree theory and similar techniques are given in [12]–[16]. Below, we shall show that this method (with some improvements) can be applied to BVP’s in a Banach space $E$. All difficulties connected with a partition of a function space into a topological sum of its subspaces are reduced to the same (but easier) problem for underlying space $E$. We shall also consider problems with a nonlinear boundary condition, using the observation of Furi and Perera [7]. We refer the reader to [2] for information on differential equations in infinite dimensional spaces.

2. General problem

Let $E$ be a Banach space, $A: (0,1) \to L(E)$, a continuous function taking values in the space of bounded linear operators of $E$, $f: (0,1) \times E \to E$, a continuous function, $B_1, B_2 \in L(E)$, and let $B_3: C(\bar{(0,1)},E) \to E$ be a non-
linear continuous mapping defined on the Banach space of continuous functions \( (0,1) \rightarrow E \). We look for a solution of the first order differential equation

\[
x' - A(t)x = f(t,x)
\]  
(2.1)
satisfying the boundary condition

\[
B_1 x(0) + B_2 x(1) = B_3(x).
\]  
(2.2)

System (2.1)-(2.2) is at resonance, which means that the linear homogeneous problem

\[
x' - A(t)x = 0, \quad B_1 x(0) + B_2 x(1) = 0,
\]

has a nonzero solution. We shall assume that there exists an operator \( A_0 \in L(E) \) commuting with the resolvent \( U : (0,1) \rightarrow L(E) \) of the operator \( x' - A(t)x = 0 \), such that \( B_1 + B_2 \exp(A_0)U(1) \) is an automorphism of \( E \) for \( \lambda \) from a neighbourhood (nhbd) of \( 0 \in \mathbb{R} \). Usually, \( A_0 = I \) is the identity operator. Moreover, let \( B_1 + B_2 U(1) \) be a linear Fredholm operator (its index must be 0, by the above). Our assumptions mean that the problems

\[
x' - A(t)x - \lambda A_0 x = 0, \quad B_1 x(0) + B_2 x(1) = 0,
\]  
(2.3)

have only the zero-solution for \( \lambda \neq 0 \) belonging to the nhbd of 0, the subspace of initial points of solutions to (2.3), with \( \lambda = 0 \), is finite dimensional, and the range of the operator \( B_1 + B_2 U(1) \) has a finite codimension.

Take any basis \( x_1, \ldots, x_n \) in \( \ker(B_1+B_2 U(1)) \) and suppose that the following limits

\[
\lim_{\lambda \rightarrow 0} B(\lambda)x_j/\|B(\lambda)x_j\| =: h_j, \quad j = 1, \ldots, n,
\]  
(2.4)

where \( B(\lambda) = B_1 + B_2 U(1) \exp \lambda A_0 \), exist and constitute a linearly independent system such that

\[
\text{Lin}\{h_1, \ldots, h_n\} \oplus \text{Im} B(0) = E.
\]

Then, of course, this condition is satisfied for each basis.

Let \( E_1 = \ker B(0) \), and let \( E_0 \) be its topological complement:

\[ E_1 \oplus E_0 = E. \]

We have \( B(\lambda)E_1 \oplus B(\lambda)E_0 = E \) for \( \lambda \neq 0 \) sufficiently close to 0. Moreover,

\[
\text{Lin}\{h_1, \ldots, h_n\} \oplus B(0)E_0 = E.
\]

Define the system of linear bounded functionals on \( E \): \( v_j(\lambda), \ j = 1, \ldots, n, \) for \( \lambda \neq 0 \) by the formulae

\[
\langle v_j(\lambda), B(\lambda)x_i \rangle = \delta_{ij} \|B(\lambda)x_i\|, \quad i = 1, \ldots, n,
\]

\[
v_j(\lambda) | B(\lambda)E_0 = 0.
\]
Obviously, $v_j$ are continuous functions of $\lambda$ and have continuous extensions to $0$ such that

$$\langle v_j(0), h_i \rangle = \delta_{ij}, \quad v_j(0) \mid B(0)E_0 = 0.$$ 

If we denote by $P_1(\lambda)$ (resp. $P_0(\lambda)$) the projectors on $B(\lambda)E_1$ (resp. $B(\lambda)E_0$) along $B(\lambda)E_0$ (resp. $B(\lambda)E_1$) for $\lambda \neq 0$ and similarly for $\lambda = 0$ with natural changes, then we can find the representation of $B(\lambda)^{-1}$:

$$B(\lambda)^{-1} = B(\lambda)^{-1}P_0(\lambda) + \sum_{j=1}^n \|B(\lambda)x_j\|^{-1} \langle v_j(\lambda), \cdot \rangle x_j,$$

where the first summand has a continuous extension to $0$: $(B(0) \mid E_0)^{-1}P_0(0)$. We shall denote this summand by $R(\lambda)$, and

$$c_j(\lambda) := \|B(\lambda)x_j\|^{-1}, \quad j = 1, \ldots, n,$$

are the only parts which make $\lambda = 0$ a singular point of $B(\lambda)^{-1}$. We have

$$B(\lambda)^{-1} = R(\lambda) + \sum_{j=1}^n c_j(\lambda)\langle v_j(\lambda), \cdot \rangle x_j, \quad (2.5)$$

which is similar to the corresponding formula from the previous papers [12] – [16].

It is easy to see that $V_\lambda(t) = \exp(\lambda tA_0)U(t)$ is the resolvent for the operator $x' - A(t)x - \lambda A_0x$. This implies that the unique solution to the BVP

$$x' - A(t)x - \lambda A_0x = b(t), \quad B_1x(0) + B_2x(1) = 0,$$

is the function

$$x(t) = V_\lambda(t)x_0 + V_\lambda(t) \int_0^t V_\lambda^{-1}(s)b(s) \, ds \quad (2.6)$$

with the initial vector $x_0$ for which

$$B(\lambda)x_0 = -B_2 \exp(\lambda A_0)U(1) \int_0^1 \exp(-\lambda sA_0)U^{-1}(s)b(s) \, ds. \quad (2.7)$$

We shall denote the right-hand side of the last equality by $C(\lambda, b)$, where $b \in C([0,1), E)$. Applying (2.5), we get, for $\lambda \neq 0$,

$$x_0 = R(\lambda)C(\lambda, b) + \sum_{j=1}^n c_j(\lambda)\langle v_j(\lambda), C(\lambda, b) \rangle x_j$$
and
\[ x(t) = \exp(\lambda t A^0) U(t) R(\lambda) C(\lambda, b) + \exp(\lambda t A^0) U(t) \int_0^t \exp(\lambda s A^0) U^{-1}(s) b(s) \, ds \]
\[ + \sum_{j=1}^n c_j(\lambda) \langle v_j(\lambda), C(\lambda, b) \rangle \exp(\lambda t A^0) U(t) x_j. \]

Now, we are able to write down the system equivalent to the BVP
\[ x' - A(t)x - \lambda A^0 x = f(t, x), \quad B_1 x(0) + B_2 x(1) = B_3 x, \]
for \( \lambda \neq 0 \):
\[ x_0 = R(\lambda) \left( C(\lambda, N(x)) + B_3(x) \right) + \sum_{j=1}^n c_j(\lambda) \langle v_j(\lambda), C(\lambda, N(x)) + B_3(x) \rangle x_j, \]  
\[ (2.8) \]
\[ x(t) = V_\lambda(t) R(\lambda) \left( C(\lambda, N(x)) + B_3(x) \right) + V_\lambda(t) \int_0^t V_\lambda^{-1}(s) N(x)(s) \, ds \]
\[ + \sum_{j=1}^n c_j(\lambda) \langle v_j(\lambda), C(\lambda, N(x)) + B_3(x) \rangle V_\lambda(t) x_j, \]  
\[ (2.9) \]
where \( N(x)(t) = f(t, x(t)) \). The scheme of our considerations is the following. First, we shall show that the operator defined by the right-hand sides of (2.8), (2.9) on \( E \times C([0,1], E) \) is completely continuous (under some assumptions on \( f \) and \( B_3 \)). Then we can find solutions to \( (2.8) - (2.9) \) for \( \lambda \neq 0 \) if \( f \) and \( B_3 \) are sublinear, by the Rothe fixed point theorem [5], and prove that the existence of a bounded sequence of solutions for \( \lambda_m \to 0 \) implies the solvability of the studied resonance problem (2.1)–(2.2). Next, we should find conditions excluding the existence of unbounded sequence of solutions (they correspond to the well-known Landesman-Lazer condition). The case of nonlinearities with a linear growth will be examined separately by homotopy arguments.

3. Compactness

We shall assume that \( f \) is more than continuous: \( f(t, \cdot) \) is completely continuous, i.e. it maps bounded sets into compact ones, for any \( t \in (0,1) \), and functions \( f_x \in C([0,1], E) \), \( f_x(t) = f(t, x) \), are equicontinuous for \( x \) belonging to every bounded set. Moreover, let \( B_3 \) be completely continuous. Fix \( \lambda \neq 0 \). The following lemma is essential for the proof of the complete continuity of (2.8)–(2.9).
Lemma 1. If \( f \) is as above and \( K : (0,1) \to L(E) \) is continuous, then \( F : C((0,1),E) \to C((0,1),E) \), given by

\[
F(x)(t) = \int_0^t K(s)f(s,x(s)) \, ds,
\]
is completely continuous.

Proof. The continuity of \( F \) is obvious. If we take all continuous functions \( x \in C((0,1),E), \|x\|_\infty := \sup_t \|x(t)\| \leq M \), then

\[
\|F(x)(t)\| \leq \sup_s \|K(s)\| \cdot \sup_{s,\|x\| \leq M} \|f(s,x)\| < \infty,
\]
so, the functions \( F(x) \) are equibounded. Similarly, they are equicontinuous. In order to apply the Generalized Ascoli-Arzela theorem, we should show that the sets

\[
\left\{ \int_0^t K(s)f(s,x(s)) \, ds : \|x\|_\infty \leq M \right\}
\]
are relatively compact in \( E \), \( t \in (0,1) \). Take \( \varepsilon > 0 \) and \( \delta > 0 \), such that \( |t_2 - t_1| < \delta \) implies \( \|f(t_2,x) - f(t_1,x)\| < \frac{\varepsilon}{3} \sup \|K(s)\| \) if \( \|x\| \leq M \) and \( \|K(t_2) - K(t_1)\| < \frac{\varepsilon}{3} \sup \|f(t,x)\| \). Divide the interval \( (0,1) \) into subintervals of length less than \( \delta: 0 < t_1 < t_2 < \cdots < t_k = 1 \), and choose finite \( \frac{\varepsilon}{3} \)-nets for \( K(t_j)f(t_j,\bar{B}(0,M)) : K(t_j)f(t_j,x_i), \ j = 1,\ldots,k, \ i = 1,\ldots,l(j) \). We get

\[
\|K(s)f(s,x(s)) - K(t_j)f(t_j,x_i)\| < \varepsilon
\]
for any \( \|x\|_\infty \leq M \) and \( s \in (0,1) \), where we take \( t_j \) such that \( |t_j - s| < \delta \), and \( x_i \) such that \( \|K(t_j)f(t_j,x(s)) - K(t_j)f(t_j,x_i)\| < \frac{\varepsilon}{3} \). Hence the set \( \{K(s)f(s,x(s)) : s \in (0,1), \|x\|_\infty \leq M \} \) is relatively compact, and so is its closed convex hull (the Mazur theorem). But (3.1) are contained in this hull, which ends the proof.

Notice that \( C(\lambda,N(\cdot)) \) and the second summand in (2.9) are completely continuous by Lemma 1, and that the remaining terms involve \( B_3 \) or are finite dimensional. Therefore the right-hand sides of (2.8), (2.9) define a completely continuous operator on \( E \times C((0,1),E) \).

Suppose that

\[
\lim_{\|x\|_\infty \to \infty} \|f(t,x)\|/\|x\| = \lim_{\|x\|_\infty \to \infty} \|B_3(x)\|/\|x\|_\infty = 0 \quad (3.2)
\]
(the nonlinearity is sublinear). It is easy to see that the boundary of a sufficiently large ball centred at 0 is mapped by the above mentioned operator into this ball. Due to the Rothe fixed point theorem [5], we have a solution \((x_0^\lambda, x^\lambda) \in E \times C([0,1], E)\) to system (2.8)-(2.9) for any \(\lambda \neq 0\). However, the radius of the ball tends to infinity as \(\lambda \to 0\), and the assumption of the following lemma is not unconditionally satisfied.

**Lemma 2.** If \(\lambda_m \to 0\) and \((x_0^m, x^m)_m\) is a bounded sequence of solutions to (2.8)-(2.9) for \(\lambda = \lambda_m\), \(m \in \mathbb{N}\), then problem (2.1)-(2.2) is solvable.

**Proof.** Passing to convergent subsequences, we may assume without loss of generality that

\[
C(\lambda_m, N(x^m)) \to y_1, \quad B_3(x^m) \to y_2,
\]

\[
V_{\lambda_m}(t) \int_0^t V_{\lambda_m}^{-1} N(x^m)(s) \, ds \to y(t).
\]

By the linear independence of \(x_j, j = 1, \ldots, n\), and \(V_{\lambda_m}(t)x_j, j = 1, \ldots, n\), the scalar sequences contain the convergent subsequences

\[
c_j(\lambda_m)\langle v_j(\lambda_m), C(\lambda_m, N(x^m)) + B_3(x^m) \rangle \to d_j
\]

for \(j = 1, \ldots, n\); thus

\[
x_0^m \to R(0)(y_1 + y_2) + \sum d_j x_j =: x_0,
\]

\[
x^m(t) \Rightarrow U(t)\left(R(0)(y_1 + y_2) + \sum d_j x_j\right) + y(t) =: x(t).
\]

Therefore

\[
y(t) = U(t) \int_0^t U^{-1}(s)f(s, x(s)) \, ds,
\]

which implies that the function \(x\) satisfies equation (2.1). Since \(c_j(\lambda_m) \to \infty\), we have

\[
\langle v_j(0), C(0, N(x)) + B_3(x) \rangle = 0, \quad j = 1, \ldots, n,
\]

which means that

\[
y_1 + y_2 = C(0, N(x)) + B_3(x) \in B(0)E_0 = B(0)E;
\]

so

\[
B(0)x_0 = C(0, N(x)) + B_3(x).
\]

The last equality is equivalent to boundary condition (2.2). \(\square\)
4. The Landesman-Lazer condition

Suppose that the sequence \((x_m^0, x_m^1)\) from Lemma 2 is unbounded. Then \((x_m^0)\) is unbounded and one may assume that \(\|x_m^0\|_\infty \to \infty\). Dividing both sides of (2.9), for \(\lambda = \lambda_m\), by \(\|x_m^0\|_\infty\), we find that the first and second summands on the right tend to 0, hence the sequence

\[
\|x_m^0\|_\infty^{-1} \sum_{j=1}^n c_j(\lambda_m) \langle v_j(\lambda_m), C(\lambda_m, N(x_m^0)) + B_3(x_m^0) \rangle V_{\lambda_m}(t)x_j
\]

is bounded and, as in the proof of Lemma 2, one can choose convergent scalar subsequences

\[
\|x_m^0\|_\infty^{-1} c_j(\lambda_m) \langle v_j(\lambda_m), C(\lambda_m, N(x_m^0)) + B_3(x_m^0) \rangle \to d_j,
\]

\(j = 1, \ldots, n\), and obtain

\[
\|x_m^0\|_\infty^{-1} x_m^0(t) \Rightarrow \sum_{j=1}^n d_j U(t)x_j.
\]

Thus \(\langle v_j(\lambda_m), C(\lambda_m, N(x_m^0)) + B_3(x_m^0) \rangle\) has the same sign as \(d_j\) for large \(m\) and each \(j\). Introduce the following condition:

for any \((x_m^0) \in C(\langle 0, 1 \rangle, E)\) with the properties \(\|x_m^0\|_\infty \to \infty\), \(\|x_m^0\|_\infty^{-1} x_m^0 \to \sum d_j U(\cdot)x_j\) for some \((d_1, \ldots, d_n) \in \mathbb{R}^n\), there exists \(j \in \{1, \ldots, n\}\) such that

\[
\limsup_{m \to \infty} d_j \langle v_j(0), D(x_m^0) \rangle < 0,
\]

where

\[
D(x) = -B_2 U(1) \int_0^1 U^{-1}(s)f(s, x(s)) \, ds + B_3(x).
\]

From the above arguments, this condition (referred to as the L-L condition) implies that the assumption of Lemma 2 holds. We have proved

**Theorem 1.** Under the assumptions of Sections 2,3, if the L-L condition is satisfied, then boundary value problem (2.1)–(2.2) has a solution.

If there exist limits \(D(d_1, \ldots, d_n) = \lim_{m \to \infty} D(x_m^0)\) independently of \((x_m^0)\) such that \(\|x_m^0\|_\infty \to \infty\), \(\|x_m^0\|_\infty^{-1} x_m^0 \to \sum d_j U(\cdot)x_j\), then the L-L condition has the form: for each \((d_1, \ldots, d_n) \in \mathbb{R}^n \setminus \{0\}\), there exists \(j\) such that

\[
d_j \langle v_j(0), D(d_1, \ldots, d_n) \rangle < 0.
\]
5. The nonlinearity with linear growth

Keep all the assumptions and notations of Sections 2 and 3 in mind, except (3.2), which we replace by

\[ \beta(s) := \limsup_{\|x\| \to \infty} \frac{\|f(s, x)\|}{\|x\|} < \infty, \quad (5.1) \]
\[ \gamma_2 := \limsup_{\|x\| \to \infty} \frac{\|B_3(x)\|}{\|x\|} < \infty. \quad (5.2) \]

Let

\[ \begin{cases} \hat{\beta} := \frac{1}{\sigma_2} \int_0^1 \beta(s) \|U^{-1}(s)\| \, ds, \\ \gamma_1 := \|B_2 U(1)\| \hat{\beta}, \\ \gamma := \gamma_1 + \gamma_2. \end{cases} \quad (5.3) \]

and suppose that

\[ \hat{\beta} \sup_t \|U(t)\| < 1. \quad (5.4) \]

**Theorem 2.** Assume that there exist \( \sigma_1 > 0 \) and \( r > 0 \) such that, for any \( j \in \{1, \ldots, n\}, \)

\[ \sup_{d_j} \langle v_j(0), C(0, N(x)) + B_3(x) \rangle < 0 \quad (5.5) \]

over the set of all \( x(t) = U(t)(x_0 + \sum d_i x_i + y(t)) \) with \( |d_j| \geq r, \hat{x}_0 \in E_0, \)
\( |d_i| \leq |d_j|, \|\hat{x}_0\| \leq \sigma_1 \|\sum d_i x_i\|, \|y\|_\infty \leq \hat{\beta} \sigma_2 \|\hat{x}_0 + \sum d_i x_i\|, \) \( y(0) = 0, \) where

\[ \sigma_2 = \sup_t \|U(t)\| \left( 1 - \hat{\beta} \sup_t \|U(t)\| \right)^{-1}. \quad (5.6) \]

If

\[ \gamma \|R(0)\| \sigma_2 (1 + \sigma_1) \sigma_1^{-1} < 1, \quad (5.7) \]

then BVP (2.1)–(2.2) has a solution.

**Proof.** Define a homotopy \( H = (H_0, H_1): E \times C(\langle 0, 1 \rangle, E) \times (0, 1) \to E \times C(\langle 0, 1 \rangle, E) \) by the formulae

\[ H_0(x_0, x, \alpha) = (1 - \alpha) R(\alpha \lambda_1) \left( C(\alpha \lambda_1, N(x)) + B_3(x) \right) + \sum_j c_j(\alpha \lambda_1) \langle v_j(\alpha \lambda_1), C(\alpha \lambda_1, N(x)) + B_3(x) \rangle x_j, \]

\[ H_1(x_0, x, \alpha) = V_{\alpha \lambda_1}(t) H_0(x_0, x, \alpha) + V_{\alpha \lambda_1}(t) \int_0^t V_{\alpha \lambda_1}^{-1}(s) N(x)(s) \, ds, \]

where \( \lambda_1 \) is a positive number sufficiently close to 0. We shall show that the homotopy \( H \) has fixed points (if they exist) in a bounded set.
First of all, notice that (5.5) is satisfied for 0 replaced by \( A \in (0, \alpha_i) \), and 
\[ x(t) = \frac{1}{V(t)}(x_0 + \sum d_i x_i + y(t)) \]
with \( x_0, d_i, y \) as above (in definition (5.6) of \( \sigma_2 \), \( U \) is replaced by \( V \)), but \( \sigma_1 \) and \( \sigma_2 \) satisfying 
\[ \gamma ||R(\lambda)||\sigma_2(1 + \sigma_1)\sigma_1^{-1} < 1. \]

If \( x = H_1(x_0, x, \alpha), x_0 = H_0(x_0, x, \alpha) \), then 
\[ ||x||_\infty \leq (1 - \beta \sup ||V_{\alpha_1}(t)||)^{-1} \sup ||V_{\alpha_1}(t)|| ||x_0|| \]  
and 
\[ ||\tilde{x}_0|| = \left( (1 - \alpha)R(\alpha \lambda_1) \left( C(\alpha \lambda_1, N(x)) + B_3(x) \right) \right) \]
\[ \leq \|R(\alpha \lambda_1)\| \left( \|B_2 V_{\alpha_1}(1)\| \int_0^1 \|V_{\alpha_1}^{-1}(t)\|N(x)(s)\| \, ds + \|B_3(x)\| \right). \]

Enlarging \( \gamma_1, \gamma_2 \) with (5.7) kept, we can estimate this norm for large \( ||x||_\infty \) : 
\[ ||\tilde{x}_0|| \leq \gamma ||R(0)|| ||x||_\infty \leq \gamma ||R(0)|| \sigma_2 ||x_0|| \leq \gamma ||R(0)|| \sigma_2 \left( ||\tilde{x}_0|| + \left\| \sum d_i x_i \right\| \right), \]
thus 
\[ ||\tilde{x}_0|| \leq \gamma ||R(0)|| \sigma_2 (1 - \gamma ||R(0)|| \sigma_2)^{-1} \left\| \sum d_i x_i \right\| < \sigma \left\| \sum d_i x_i \right\|. \]

For such fixed points, we have 
\[ ||y||_\infty = \sup_t \left\| \int_0^t V_{\alpha_1}^{-1}(s) N(x)(s) \, ds \right\| \leq \beta ||x||_\infty \leq \beta \sigma_2 ||x_0||, \]
which is needed to apply (5.5).

Take any solution \( x_0 = \tilde{x}_0 + \sum d_i x_i, d \in \mathbb{R}^n \) and \( d_j \) with the maximal modulus. Obviously, \( |d_j| < r \) by (5.5) (with \( \alpha \lambda_1 \) instead of 0). So, \( \left\| \sum d_i x_i \right\| \) is bounded, which gives an estimate on \( ||\tilde{x}_0|| \), then on \( ||x_0|| \) and, at last, on \( ||y||_\infty \). Due to (5.8), we have an upper bound for the norms of solutions \( x \). Denote by \( \Omega_0 \) (resp. \( \Omega_1 \)) a ball containing fixed points of \( H_0 \) (resp. \( H_1 \)). The Leray-Schauder degree 
\[ \deg_{LS}((I-H_0) \times (I-H_1), \Omega_0 \times \Omega_1, 0) \]  
(5.9)

does not depend on \( \alpha \in (0,1) \). We can deform \( H(\cdot, 1) \) by \( H(x_0, x, \mu) = \mu H(x_0, x, 1) \) which is fixed point free on the boundary of \( \Omega_0 \times \Omega_1 \) by similar (but simpler) arguments. For \( \mu = 1 \), we have degree (5.9) and, for \( \mu = 0, \deg_{LS}(I, \Omega_0 \times \Omega_1, 0) \). Therefore, \( H \) has a fixed point in \( \Omega_0 \times \Omega_1 \) for any \( \alpha > 0 \). Repeating the arguments from the proof of Lemma 2 with a slight change, we get the assertion. \( \square \)
6. Examples

Let us consider the BVP:

\[ x' = f(t, x), \]  
\[ x(1) = Bx(0) \]  

in a Hilbert space \( E \) with \( B \) being a linear self-adjoint completely continuous operator in \( E \) and \( f \) satisfying the continuity assumptions from Section 3. Since we are interested in resonance problems, \( 1 \in \text{Sp} B \), and we can take \( A_0 = I \), \( B(\lambda) = B - e^{\lambda}I \). Obviously, \( E_1 = \ker(B - I) \) is finite dimensional. Take any orthonormal set \( \{x_1, \ldots, x_n\} \) spanning \( \ker(B - I) \). We have \( h_j = -x_j \) for \( \lambda \to 0^+ \), \( j = 1, \ldots, n \), and

\[ \langle v_j(0), x \rangle = -(x_j, x). \]

The L-L condition, in the sublinear case, has the form:

for any sequence \( (x^m) \) such that \( \|x^m\|_{\infty} \to \infty \), \( \|x^m\|_{\infty}^{-1} x^m \Rightarrow \sum d_i x_i \), there exists \( j \in \{1, \ldots, n\} \) such that

\[ \liminf_{m \to \infty} d_j \left( x_j, \int_0^1 f(s, x^m(s)) \, ds \right) > 0. \]  

(6.3)

One can use the weaker condition summing up (6.3) over the numbers \( j \):

\[ \liminf_{m \to \infty} \int_0^1 (x^m(s), f(s, x^m(s))) \, ds > 0 \]  

(6.4)

or even

\[ \liminf_{\|x\| \to \infty, x \in G} \int_0^1 (x, f(s, x)) \, ds > 0, \]

where \( G = \{\lambda x : \lambda \in \mathbb{R}, x \in W\} \) and \( W \) is a nbhd of \( \{x \in \ker(B - I) : \|x\| = 1\} \).

One can examine a more general boundary condition

\[ x(1) = Bx(0) + B_3(x) \]  

(6.5)

with \( B_3 \) sublinear. Assumption (6.4) should be replaced by

\[ \liminf_{m \to \infty} \int_0^1 \left( x^m(s), f(s, x^m(s)) + B_3(x^m) \right) \, ds > 0 \]
or
\[
\liminf_{\|x\| \to \infty} \int_0^1 (x, f(s, x) + B_3(x)) \, ds > 0 ,
\]
where \( B_3(x) \) means the value of \( B_3 \) on the constant function equal to \( x \). If
\[
B_3(x) = \int_0^1 \|x(t)\|^\rho \, dt \cdot x_0 ,
\]
where \( \rho \in (0, 1) \) and \( x_0 \) is a fixed vector orthogonal to \( \ker(B - I) \), then (6.4) is still a sufficient condition for the solvability of (6.1), (6.5).

Now, we consider BVP (6.1), (6.2) with the nonlinearity \( f \) of a linear growth. In the notations of Section 5,
\[
\gamma = \gamma_1 = \hat{\beta} = \int_0^1 \beta(s) \, ds .
\]
Let \( f_j(t, x) = (f(t, x), x_j) \), \( j = 1, \ldots, n \), and let \( f_0 \) be the orthogonal projection of \( f \) onto \( E_0 = \text{Im}(B - I) \). We shall assume that
\[
\limsup_{d_j \to -\infty} \int_0^1 f_j(s, x_0 + \sum d_i x_i) \, ds < 0 < \liminf_{d_j \to +\infty} \int_0^1 f_j(s, x_0 + \sum d_i x_i) \, ds (6.6)
\]
for any \( j \in \{1, \ldots, n\} \), \( d = (d_1, \ldots, d_n) \in \mathbb{R}^n \) and \( x_0 \in E_0 \), and that the limits are separated from 0 uniformly on bounded sets. Moreover, let
\[
\hat{\beta} < (\sqrt{n} + 1)^{-1} (6.7)
\]
and
\[
\frac{\hat{\beta}}{1 - (\sqrt{n} + 1)\hat{\beta}} \max_{\lambda \in \text{Sp} B, \lambda \neq 1} |1 - \lambda|^{-1} < 1 (6.8)
\]
(the maximum in this inequality equals \( \|R(0)\| \)). It is easy to calculate that there exists \( \sigma_1 > 0 \) satisfying inequality (5.7) \( (\sigma_2 = (1 - \hat{\beta})^{-1}) \) and such that
\[
\sigma_1 < \frac{1 - (\sqrt{n} + 1)\hat{\beta}}{\sqrt{n}\hat{\beta}} .
\]
Hence, if we take \( \tilde{x}_0 \in E_0 \), \( d \in \mathbb{R}^n \), \( y \in C((0,1), E) \), such that \( \|\tilde{x}_0\| \leq \sigma_1 \|\sum d_i x_i\| \), \( \|y\|_\infty \leq \hat{\beta}\sigma_2 \|\tilde{x}_0 + \sum d_i x_i\| \), then if \( |d_j| = \max_i |d_i| \) and we have the worst case: “\( y \) takes values in \( \text{Lin}\{x_1, \ldots, x_n\}\)” ; then we obtain
\[
\|y\|_\infty < (1 - \epsilon)|d_j|.
\]
for some positive \( \varepsilon \). Thus, we can make the coefficient standing with \( x_j \) in the projection of \( \tilde{x}_0 + \sum d_i x_i + y(t) \) onto \( \text{Lin}\{x_1, \ldots, x_n\} \) arbitrarily large if \( |d_j| \to \infty \). By (6.6), this means that the assumption (5.5) of Theorem 2 holds and, therefore, BVP (6.1), (6.2) has a solution provided that the projections of \( f \) onto \( \text{Lin}\{x_1, \ldots, x_n\} \) satisfy (6.6), the constant \( \beta \) describing a linear growth of \( f \) satisfies (6.7), and \( \beta(1 - (\sqrt{n} + 1)\beta)^{-1} \) is less than the distance between 1 and the nearest eigenvalue of \( B \).

Remark. Since, for \( \lambda \to 0^- \), we have \( h_j = x_j \) instead of \( -x_j \), we can reverse inequalities (6.3) and (6.4) replacing “\( \liminf \)” by “\( \limsup \)”. Moreover, we can change mutually \( \pm \infty \) in (6.6) and the solvability does not fail.

Now, we shall study a BVP for second order differential equations in the Banach space \( l^\infty \) of bounded sequences:

\[
x'' + m^2 x = f(t, x, x'),
\]
where \( x = (x_j)_{j \in \mathbb{N}}, f = (f_j)_{j \in \mathbb{N}} \) and \( m \) is an odd integer. The boundary condition is partially periodic and partially antiperiodic:

\[
\begin{align*}
x_j(0) &= x_j(\pi), & x_j'(0) &= -x_j'(\pi), & j \leq n, \\
x_j(0) &= x_j(\pi), & x_j'(0) &= x_j'(\pi), & j > n.
\end{align*}
\]
(6.10)

It is easily seen that the corresponding homogeneous linear problem has non-trivial solutions \( \sin mt \sum_{i=1}^n c_i e_i \), where \( e_i \) are elements of the standard basis in \( l^\infty \). Consider the equivalent first order system and perturb it by \( \lambda I \):

\[
x' = y + \lambda x, \quad y' = -m^2 x + \lambda y.
\]

We can put \( E = l^\infty \oplus l^\infty, U(t) = \cos mt I + \frac{1}{m} \sin mt A, \) where

\[
A = \begin{pmatrix} 0 & 1 \\ -m^2 & 0 \end{pmatrix},
\]

\( x_j = (0, e_j), \ j = 1, \ldots, n. \) Then \( h_j = -x_j \) if \( \lambda \to 0^+ \), and \( -\langle v_j(0), z \rangle \) is the \( j \)th coordinate in the second summand of \( z \in E = l^\infty \oplus l^\infty \).

The nonlinearity has the form \( (0, f) \), where \( f \) should satisfy the following conditions (see Section 3): \( f_j \) are equi-uniformly continuous on bounded sets and, for any \( \varepsilon > 0 \) and \( M > 0 \), there exists \( k \in \mathbb{N} \) such that

\[
|f_j(t, x, y)| < \varepsilon, \quad \text{for} \quad \|x\|, \|y\| \leq M, \ t \in \{0, \pi\}, \ j > k.
\]

There are conditions less restrictive than the last one, but very complicated guaranteeing the compactness of \( f(t, \cdot) \). Moreover, let \( f \) be sublinear. It is easy to calculate the L-L condition for BVP (6.9), (6.10):

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for \((x^k) \subset X\), \(a_k = \max(||x^k||_\infty, ||x^k||_\infty) \rightarrow \infty\), \(a_k^{-1}x^k(t) \Rightarrow \sin mt(d_1, \ldots, d_n, 0)\), \(a_k^{-1}x^k(t) \Rightarrow m\cos mt(d_1, \ldots, d_n, 0)\), there exists \(j \in \{1, \ldots, n\}\) such that
\[
\liminf_{k \to \infty} d_j \int_0^\pi \cos ms f_j(s, x^k(s), x^k'(s)) \, ds > 0,
\]
where 0's stand for the \(j\)th coordinates of \(x\) and \(y\) with \(j > n\). The inequality can be reversed \((\lambda \to 0^-)\) with replacing “\(\lim \inf\)” by “\(\lim \sup\)” . If \(f\) does not depend on derivative \(x'\), \(n = 1\), and if there exist uniform limits
\[
\lim_{d \to \pm \infty} f_1(s, d, x) = f_1^\pm(s)
\]
independent of \(x = (x_2, x_3, \ldots)\), we can simplify this condition, as the numbers
\[
\int_{\sin mt > 0} f_1^+(s) \cos ms \, ds + \int_{\sin mt < 0} f_1^-(s) \cos ms \, ds,
\]
\[
\int_{\sin mt < 0} f_1^+(s) \cos ms \, ds + \int_{\sin mt > 0} f_1^-(s) \cos ms \, ds
\]
have opposite signs.

It is interesting that the last condition differs from the classical Landesman-Lazer condition only by the kernel function \(\cos ms\). This is a consequence of the fact that the BVP is not self-adjoint as
\[
x'' + m^2x = 0, \quad x(0) = x(\pi) = 0
\]
is. Similarly, one can introduce a nonlinearity \(B_3\) to boundary condition (6.10) and study BVP (6.9), (6.10) with nonlinearities having linear growth.

REFERENCES


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