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ON A USEFUL ECONOMY IN THE FORMATION OF RIEMANN SUMS

ALEXANDER ABIAN*

In this paper it is shown that in order to ascertain that a function f is Reimann integrable it is sufficient to follow Riemann's procedure, however, by restricting the formation of Riemann sums to the sums of the product of the length of a typical subdivision $[x_k, x_{k+1}]$ by the value of f evaluated *only* at $z_k \in [x_k, x_{k+1}]$ where z_k is selected from $[x_k, x_{k+1}]$ by a function S belonging to a class of functions called „*selection functions*". A function S from the set of all nonempty real closed intervals $[x, y]$ is called a selection function if $S([x, y]) = z$ with $x \leqslant z \leqslant y$ where x is a function of $y - x$ and z, continous in z. Functions picking up the left endpoint or the right endpoint or the middle point of a clased interval are examples of selection functions. Other examples of selection functions are $S([x, y]) = x + (y - x) \cdot \sin^2(y - x)$ and $S([x, y]) = (y + xy - x^2)/(1 + y - x)$. Some other examples are given below. Related results have been expounden in [2].

Definition. *Let I be the set of all the closed nonempty intervals $[x, y]$ of the set of all real numbers R. A function S from I into R is called a selection function if and only if*

$$S([x, y]) = z \quad \textit{with} \quad x \leqslant z \leqq y \tag{1}$$

where

$$x = h(y - x, z) \tag{2}$$

is such that h is a function of two variables $y - x$ and z, continuous in z.

If S is a selection function then, in view of the above Definition, we say that „*S picks up the point z from the closed interval* $[x, y]$".

Examples. Let q be a real number such that $0 \leqslant q \leqq 1$. Consider the function S given by:

$$S([x, y]) = x + q(y - x) \tag{3}$$

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It is readily verified that S is a selection function. Indeed S picks up the point z from the closed interval $[x, y]$ such that z is located at the ratio q with respect to the endpoints x and y of the interval. For the case of (3), the corresponding function h required by (2) is given as:

$$x = h(y-x, z) = -q(y-x) + z$$

The three special cases of (3) corresponding to $q=0$, $q=1$ and $q=0.5$ yield selection functions which pick up respectively the left endpoint, the right endpoint and the middle point of a closed interval.

From arithmetic-mean-geometriac-mean inequality if follows that

$$S([x, y]) = x + \frac{n(y-x)^{(n+1)/n}}{y-x+n-1} \quad \text{for} \quad n = 2, 3, \ldots$$

is a selection function.

From Cauchy and Triangle inequalities it follows that (4) and (5) respectively are examples of selection functions, where:

(4) $$S([x, y]) = x + \frac{(y-x)^2 + (y-x)(n-1)}{\sqrt{n(y-x)^2 + n(n-1)}} \quad \text{for} \quad n = 2, 3, \ldots$$

and

$$S([x, y]) = x + (y-x) \cdot \frac{\sqrt{(y-x+1)^2 + n - 1}}{y-x+\sqrt{n}} \quad \text{for} \quad n = 1, 2, \ldots$$

Let f be a bounded real-valued function defined on a nonempty closed interval $[a, b]$ of real numbers. We recall that f is Riemenn integrable with $\int_a^b f = r$ if and only if

(6) $$\lim_{\text{mesh } P \to 0} \Sigma(x_{k+1} - x_k) \cdot f(z_k) = r$$

where P stands for a partition of $[a, b]$ into finitely many contiguous closed intervals $[x_k, x_{k+1}]$ and for every selection of the point z_k from the closed interval $[x_k, x_{k+1}]$.

Let f be as in the above and let $D(x)$ denote the discontinuity (or saltus [1, p. 95]) of f at x. We recall [1, p. 209] that f is not Riemann integrable if and only if there exists a *positive* real number d such that

(7) $$\text{measure } \{x \mid x \in [a, b] \text{ and } D(x) \geq d\} = m > 0$$

As mentioned earlier, we prove below that f is Riemann integrable if (6) holds even if the choice of z_k from $[x_k, x_{k+1}]$ is restricted only to the value of a fixed selection function S at $[x_k, x_{k+1}]$. In other words, f is Riemann integrable if (6) holds provided "z_k is picked up from $[x_k, x_{k+1}]$ by a fixed selection function s". This is a rather significant result since it implies that to ascertain the Reimann integrability of f it is enough to verify the validity of (6) not necessarily for all the possible selections of z_k from $[x_k, x_{k+1}]$ but only for one particular selection of z_k from $[x_k, x_{k+1}]$ given by a fixed selection function S.

Theorem. *Let f be a bounded real valued function defined on a nonempty closed interval $[a, b]$ of real numbers. Let S be a selection function such that*

$$\lim_{\text{mesh } P\to 0} \sum_P (x_{k+1}-x_k)\cdot f(S([x_k, x_{k+1}]))=r \tag{8}$$

where P stands for a partition of $[a, b]$ into finitely many contiguous closed intervals $[x_k, x_{k+1}]$.

Then f is Riemann integrable. Moreover,

$$\int_a^b f=r \tag{9}$$

Proof. We prove the Theorem by showing that the assumption that f is not Riemann integrable implies the negation of (8). To this end we prove that if f is not Riemann integrable then there exists an $\varepsilon>0$ such that for every $e>0$ there exist two partitions P_1 and P_2 of $[a, b]$ into finitely many contiguous closed intervals such that

$$\text{mesh } P_1=\text{mesh } P_2=e \tag{10}$$

whereas

$$\left|\sum_{P_1}(x_{k+1}-x_k)\cdot f(S([x_k, x_{k+1}]))-\sum_{P_2}(x_{k+1}-x_k)\cdot f(S([x_k, x_{k+1}]))\right|\geqslant\varepsilon \tag{11}$$

Since we assume that f is not Riemann integrable, in view of (7), there exists a real number $d>0$ such that the measure of the set K of the points of $[a, b]$ at each of which the discontinuity of f is greter than or equal to d, is a positive real number m, i.e.,

$$\text{measure } K=m \quad \textit{with} \quad m>0 \tag{12}$$

where K is the set appearing in (7).

We choose ε as

$$\varepsilon=md/6 \tag{13}$$

Let $e>0$ be given. For the sake of simplicity, we extend the closed interval $[a, b]$ from the left by the segment $a'a$ of length e and from the right by a segment bb' of length at least e such that the length of the newly obtained closed interval $[a', b']$ is an integral multiple of $3e$. Again, without loss of generality and for the sake of simplicity we extend the function f to $[a', b']$ by defining it to be identically zero on $[a', a)$ and on $(b, b']$.

Next, we partition $[a', b']$ into $3n$ equal segments each of length e as follows:

(14) $$a' \ a \ t_1 \ t_2 \ t_3 \ t_4 \ \dots \ t_{3n-1} = b'$$

The closed interval $[a', b']$ is evidently the union of three pairwise disjoint subsets E_1, E_2, E_3 where

(15) $$\begin{aligned} E_1 &= [a', a) \cup [t_2, t_3) \cup [t_5, t_6) \cup \dots \cup [t_{3n-4}, t_{3n-3}) \\ E_2 &= [a, t_1) \cup [t_3, t_4) \cup [t_6, t_7) \cup \dots \cup [t_{3n-3}, t_{3n-2}) \\ E_3 &= [t_1, t_2) \cup [t_4, t_5) \cup [t_7, t_8) \cup \dots \cup [t_{3n-2}, t_{3n-1}] \end{aligned}$$

i.e., E_i is the union of every fourth interval in partition (14).

Since $K \subseteqq [a', b'] = E_1 \cup E_2 \cup E_3$, obviously,

$$\text{measure } (K \cap E_i) \geqslant m/3 \quad \textit{for some} \quad i = 1, 2, 3$$

Without loss of generality, let

(16) $$\text{measure } (K \cap E_3) \geqslant m/3$$

For the sake of simplicity, and without loss of generality, we assume that there are two intervals of the form $[t_{3k-2}, t_{3k-1})$, say, $[t_1, t_2)$ and $[t_4, t_5)$ such that:

(17) $$m_1 = \text{measure } (K \cap (t_1, t_2)) > 0 \textit{ and } m_4 = \text{measure } (K \cap (t_4, t_5)) > 0$$

with

(18) $$(m_1 + m_4) \geqslant m/3$$

Let

(19) $$g_1 = \text{glb } (K \cap (t_1, t_2)) \quad \textit{and} \quad g_4 = \text{glb } (K \cap (t_4, t_5))$$

But then, in view of (12) and (17), for every real number $p_1 > 0$ there exist (even to the right of g_1) two distict points z_1 and z_1' in the oper interval (t_1, t_2) such that:

(20) $$|f(z_1) - f(z_1')| \geqslant d \quad \textit{with} \quad |z_1 - z_1'| < p_1$$

Also, for every $p_4 > 0$ there exist (even to the right of g_4) two distinct points z_4 and z_4' in the open interval (t_4, t_5) such that:

(21) $$|f(z_4) - f(z_4')| \geqslant d \quad \textit{with} \quad |z_4 - z_4'| < p_4$$

Let

(22) $$M = \text{lub } |f(x)| \quad \textit{for} \quad a \leqslant x \leqq b$$

Consider the real number (without loss of generality $nM > 0$)

(23) $$md/36nM$$

where d is mentioned in (7) and m in (7) and (12), and where $[a', b']$ is partitioned into $3n$ equal segments each of length e as mentioned in (10) and (14).

Since S is a selection function, in view of (2), we can assert that

(24) $$|h(e, z_1) - h(e, z'_1)| < md/36nM \quad \textit{with} \quad z_1, z'_1 \in (t_1, t_2)$$

and

(25) $$|h(e, z_4) - h(e, z'_4)| < md/36nM \quad \textit{with} \quad z_4, z'_4 \in (t_4, t_5)$$

hold together with the first parts of (20) *and* (21).

Let

(26) $$x_1 = h(e, z_1) \quad \text{and} \quad x_2 = x_1 + e$$

Moreover, let

(27) $$x_4 = h(e, z_4) \quad \text{and} \quad x_5 = x_4 + e$$

In partition P_1 of $[a', b']$ we include the segments

$$x_1 x_2 \quad \text{and} \quad x_4 x_5$$

each of length e, as (26) and (27) show.

Clearly, in view of (1), (26), (27), we have

(28) $$S([x_1, x_2]) = z_1 \quad \text{and} \quad S([x_4, x_5]) = z_4$$

i.e., selection function S picks up z_1 from $[x_1, x_2]$ and picks up z_4 from $[x_4, x_5]$.

Similarly, let

(29) $$x'_1 = h(e, z'_1) \quad \text{and} \quad x'_2 = x'_1 + e$$

and

(30) $$x'_4 = h(e, z'_4) \quad \text{and} \quad x'_5 = x'_4 + e$$

In partition P_2 of $[a', b']$ we include the segments

$$x'_1 x'_2 \quad \text{and} \quad x'_4 x'_5$$

each of length e, as (29) and (30) show.

Again, clearly, in view of (1), (29), (30), we have

(31) $$S([x'_1, x'_2]) = z'_1 \quad \text{and} \quad S([x'_4, x'_5]) = z'_4$$

i.e., selection function S picks up z'_1 from $[x'_1, x'_2]$ and picks up z'_4 from $[x'_4, x'_5]$.

Let us observe that from (10), (14), (17), (18) it follows that

(32) $$2e \geqq m/3$$

But then, in view of (26), (27), (29), (30), (20), (21) and (32), we have:

(33)
$$\sum_{\substack{k=1\\k=4}} (x_{k+1}-x_k)\cdot f(z_k) - \sum_{\substack{k=1\\k=4}} (x'_{k+1}-x'_k)\cdot f(z'_k) =$$
$$= \left(\sum_{\substack{k=1\\k=4}} 2ef(z_k - f(z'_k)\right) \geqslant md/3$$

Consequently, from (33), (28), (31) we derive:

(34)
$$\sum_{\substack{k=1\\k=4}} (x_{k+1}-x_k) f(S([x_k, x_{k+1}])) -$$
$$- \sum_{\substack{k=1\\k=4}} (x'_{k+1}-x'_k) f(S([x'_k, x'_{k+1}])) \geqslant md/3$$

It can be readily verified that there is a partition P_1 of $[a', b']$ into finitely many contigous closed intervals including $[x_1, x_2]$ and $[x_4, x_5]$ represented as:

$$a' \; a \; t_1 \; x_1 \; x_2 \; t_3 \; t_4 \; x_4 \; x_5 \; t_6 \; t_7 \; \dots \; t_{3n-1} = b'$$

and there is a partition P_2 of $[a', b'\}$ into finitely many contigous closed intervals including $[x'_1, x'_2]$ and $[x'_4, x'_5]$ represented as:

$$a' \; a \; t_1 \; x'_1 \; x'_2 \; t_3 \; t_4 x'_4 \; x'_5 \; t_6 \; t_7 \; \dots \; t_{3n-1} = b'$$

such that both partitions P_1 and P_2 satisfy (10) and where P_1 and P_2 are identically partitioned on

$$[a', b'] - ([x_1, x_2) \cup [x_4, x_5)) \quad \text{and} \quad [a', b'] - ([x'_1, x'_2) \cup [x'_4, x'_5))$$

perhaps with the exception of a set of pairwise disjoint intervals whose length, in view of (23), is at most

$$2(3n)md/36nM = md/6M$$

But then, in view of (34), (22), (23) and (24), the difference of the sums in (11) is greater than or equal to

$$md/3 - M(md/6M) = md/6$$

which, in view of (13), establishes (11), as desired. Hence f is Riemann integrable. But then (8) implies (9) trivially since the sum appearing in (8) is a Riemann sum.

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Department of Mathematics
Iowa State University
Ames, Iowa 50011