Lubica Holá
An extension theorem for multifunctions and a characterization of complete metric spaces


Persistent URL: http://dml.cz/dmlcz/130635

Terms of use:
© Mathematical Institute of the Slovak Academy of Sciences, 1988

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
AN EXTENSION THEOREM FOR MULTIFUNCTIONS AND A CHARACTERIZATION OF COMPLETE METRIC SPACES

LUBICA HOLÁ

Let $X$ and $Y$ be topological spaces. If $Y$ is a metrizable space, then $Y$ is topologically complete (see [1], p276) iff each continuous mapping $f: A \rightarrow Y$ with $A$ dense in $X$ has a continuous extension to a $G_δ$-set containing $A$. To see this just consider the identity mapping $i: Y \rightarrow Y$ and view $Y$ as a dense subspace of its completion.

We show that if $Y$ is a metrizable space, then $Y$ is topologically complete if and only if each upper semicontinuous compact-valued multifunction $F: A \rightarrow Y$ with $A$ dense in $X$ has an upper semicontinuous compact-valued extension to a $G_δ$-set containing $A$.

We shall use the terminology from [1].

**Notation.** In what follows $X, Y$ denote topological spaces. The closure of a subset $M$ of a topological space $X$ will be denoted by $ar{M}$.

The intersection of a family $\mathcal{U}$ of sets will be denoted by $\cap \mathcal{U}$.

$\mathcal{P}(Y)$ denotes the collection of all subsets of $Y$, $C(Y)$ denotes the collection of all nonempty closed subsets of $Y$. If $(Y, d)$ is a metric space, $(C(Y), \delta)$ denotes the metric space equipped with the Hausdorff metric, i.e. $\delta(A, B) = \inf \{\varepsilon: A \subset S_{\varepsilon}[B], \quad B \subset S_{\varepsilon}[A]\}$, where $S_{\varepsilon}[A] = \bigcup \{S_{\varepsilon}[x]: x \in A\}$ and $S_{\varepsilon}[x] = \{y: d(x, y) < \varepsilon\}$.

$\mathcal{V}(x)$ denotes the set of all open neighbourhoods of $x$. $N$ denotes a set of all positive integers, $R$ denotes the set of all real numbers.

A family $\mathcal{U}$ of sets has the finite intersection property if the intersection of every finite subfamily is not empty. A centred family is a family of sets having the finite intersection property.

A multifunction $F$ from $X$ to $Y$ is a mapping $F: X \rightarrow \mathcal{P}(Y)$. We write $F: X \rightarrow Y$ for brevity. We suppose $F(x) \neq \emptyset$ for any $x \in X$.

A multifunction $F: X \rightarrow Y$ is upper semicontinuous at $x \in X$ if for every open set $V$ in $Y$ such that $F(x) \subset V$, there exists an open set $U$ in $X$ such that $x \in U$ and $F(U) \subset V$, where $F(U) = \bigcup_{x \in U} F(x)$. 


Let $A$ be a subset of $X$ and $F: A \to Y$ be a multifunction from $A$ to $Y$. A multifunction $F^*: X \to Y$ is an extension of $F$ if $F^*(x) = F(x)$ for every $x \in A$.

Let $(Y, d)$ be a metric space. Let $\chi$ be a functional defined on $\mathcal{P}(Y)$ as follows: $\chi(0) = 0$ and if $A$ is a nonempty subset of $Y$, then $\chi(A) = \inf \{\varepsilon: A \text{ has a finite } \varepsilon\text{-dense subset}\}$. In literature $\chi$ has been called the Hausdorff measure of noncompactness functional.

Remark. (See [2]) The Hausdorff measure of noncompactness functional has some good properties.

**Lemma 1.** (See [2]) The Hausdorff measure of noncompactness functional $\chi$ for a metric space $(Y, d)$ acts as follows:

(a) $\chi(A) = \infty$ if and only if $A$ is unbounded
(b) $\chi(A) = 0$ if and only if $A$ is totally bounded
(c) If $A \subseteq B$, then $\chi(A) \leq 2\chi(B)$
(d) If $A$ is totally bounded, then for each $\varepsilon > 0$, $\chi(S_\varepsilon[A]) \leq \varepsilon$
(e) $\chi(\bar{A}) = \chi(A)$
(f) If \{F_n\} is a sequence in $C(Y)$ convergent in the Hausdorff metric to $F \in C(Y)$, then $\lim_{n \to \infty} \chi(F_n) = \chi(F)$.

**Lemma 2.** (See [2]) Let $\{A_n\}$ be a decreasing sequence of nonempty closed sets in a complete metric space $(Y, d)$. The following are equivalent: (1) $\bigcap_{n=1}^{\infty} A_n$ is a nonempty compact set, and $\{A_n\}$ is a sequence convergent in the Hausdorff metric to $\bigcap_{n=1}^{\infty} A_n$. (2) $\lim_{n \to \infty} \chi(A_n) = 0$.

Remark 1. Let $(Y, d)$ be a metric space and $F: A \to Y$ be a multifunction with $A$ dense in $X$. Put $G = \{x \in X: \text{the net } \{\chi(F(V \cap A)): V \in \mathcal{V}^-(x)\} \text{ converges to zero}\}$, where $\mathcal{V}^-(x)$ denotes the set of all open neighbourhoods of $x$.

It is easy to verify that $G = \{x \in X: \text{for any } n \in \mathbb{N} \text{ there exists } V \in \mathcal{V}^-(x) \text{ such that } \chi(F(V \cap A)) \leq \frac{1}{n}\}$, which mens that $G$ is a $G_\sigma$-set in $X$.

If $F: A \to Y$ is a compact-valued upper semicontinuous multifunction, then $A \subseteq G$. Let $x \in A$ and $n \in \mathbb{N}$. Since $F(x)$ is compact by Lemma 1 (d) $\chi\left(S_{\frac{1}{2n}}[F(x)]\right) \leq \frac{1}{2n}$. The upper semicontinuity of $F$ at $x$ implies there exists a set $V \in \mathcal{V}^-(x)$ such that $F(V \cap A) \subseteq S_{\frac{1}{2n}}[F(x)]$. Then by Lemma 1 (c) $\chi(F(V \cap A)) \leq \frac{1}{n}$. The inclusion $A \subseteq G$ is proved.
**Theorem 1.** Let $Y$ be a complete metric space. Let $F: A \to Y$ be an upper semicontinuous closed-valued multifunction, where $A$ is dense in $X$. Let the net $\{\chi(F(V \cap A)): V \in \mathcal{V}(x)\}$ converge to zero for any $x \in X \setminus A$. There exists an upper semicontinuous extension $F^*$ of $F$ defined on $X$.

**Proof.** Put $F^*(x) = F(x)$ for $x \in A$. Now let $x \in X \setminus A$ and $\mathcal{V}(x)$ be the set of all open neighbourhoods of $x$. First we show that $\cap \{F(V \cap A): V \in \mathcal{V}(x)\} \neq \emptyset$. There exists a decreasing sequence $\{V_n\}$ of open sets from $\mathcal{V}(x)$ such that $\chi(F(V_n \cap A)) \leq 1/n$. By Lemma 1 (e) and Lemma 2 $\bigcap_{n=1}^{\infty} F(V_n \cap A)$ is a nonempty compact set and $\{F(V_n \cap A)\}_{n=1}^{\infty}$ is a sequence convergent in the Hausdorff metric to $\bigcap_{n=1}^{\infty} F(V_n \cap A)$.

Let $V \in \mathcal{V}(x)$. Then $\{F(V \cap V_n \cap A)\}_{n=1}^{\infty}$ is a decreasing sequence of closed sets such that the sequence $\{\chi(F(V \cap V_n \cap A))\}_{n=1}^{\infty}$ converges to zero and thus by Lemma 2 $\bigcap_{n=1}^{\infty} F(V \cap V_n \cap A)$ is a nonempty compact set and $\{F(V \cap V_n \cap A)\}_{n=1}^{\infty}$ is a sequence convergent in the Hausdorff metric to $\bigcap_{n=1}^{\infty} F(V \cap V_n \cap A)$.

A family $\emptyset = \{F(V \cap A) \cap \left( \bigcap_{n=1}^{\infty} F(V_n \cap A) \right): V \in \mathcal{V}(x)\}$ is a compact family of nonempty compact sets. Then $\emptyset \neq \cap \emptyset \subseteq \cap \{F(V \cap A): V \in \mathcal{V}(x)\}$. It is easy to verify that $\cap \{F(V \cap A): V \in \mathcal{V}(x)\} = \cap \emptyset$.

Since $\emptyset$ is a compact set, the set $\cap \{F(V \cap A): V \in \mathcal{V}(x)\}$ is also a compact set. Put $F^*(x) = \cap \{F(V \cap A): V \in \mathcal{V}(x)\}$ for $x \in X \setminus A$.

We show that $F^*$ is upper semicontinuous. Let $x \in A$. Let $U$ be an open set in $Y$ such that $F^*(x) \subseteq U$. Since $F^*(x) = F(x)$ is a closed set in $Y$ and $Y$ is a normal space, there exists an open set $U_1$ such that $F^*(x) \subseteq U_1 \subseteq U$. The upper semicontinuity of $F$ at $x$ implies, there is an open neighbourhood $V$ of $x$ such that $F(V \cap A) \subseteq U_1$. Let $z \in V \setminus A$. Then $F^*(z) = \cap \{F(G \cap A): G \in \mathcal{V}(x)\} \subseteq F(V \cap A) \subseteq U_1 \subseteq U$. The upper semicontinuity of $F^*$ at $x \in A$ is proved.

Now let $x \in X \setminus A$. It is sufficient to prove that for any $\varepsilon > 0$ there exists a neighbourhood $V$ of $x$ such that $F^*(V) \subseteq S_{\varepsilon}(F^*(x))$ ($F^*(x)$ is a compact set for any $x \in X \setminus A$).

Let $\varepsilon > 0$. $F^*(x) = \cap \left\{F(U \cap A) \cap \left( \bigcap_{n=1}^{\infty} F(V_n \cap A) \right): U \in \mathcal{V}(x) \right\} \subseteq S_{\varepsilon/2}(F^*(x))$, where $\{V_n\}$ is a decreasing sequence of neighbourhoods of $x$ such that the sequence $\{\chi(F(V_n \cap A))\}_{n=1}^{\infty}$ converges to zero. Put $B = \bigcap_{n=1}^{\infty} F(V_n \cap A)$. The com-
pactness of $B$ implies that $\overline{F(U \cap A)} \cap B$ is a compact set for any $U \in \mathcal{F}(x)$. Since $\bigcap \{F(U \cap A) \cap B : U \in \mathcal{F}(x)\}$ is a subset of the open set $S_{\varepsilon 2}[F^*(x)]$ by [1] there exist sets $U_1, U_2, \ldots, U_n \in \mathcal{F}(x)$ such that $\bigcap_{i=1}^{n} F(U_i \cap A) \cap B \subset S_{\varepsilon 2}[F^*(x)]$.

Put $G = \bigcap_{i=1}^{n} U_i$. Then $G \in \mathcal{F}(x)$ and $F(G \cap A) \cap B = F(G \cap A) \cap \left( \bigcap_{i=1}^{n} F(U_i \cap A) \right) \subset S_{\varepsilon 2}[F^*(x)]$.

By Lemma 2 $\{F(G \cap V_n \cap A)\}_{n=1}^{\infty}$ is a sequence convergent in the Hausdorff metric to $\bigcap_{n=1}^{\infty} F(G \cap V_n \cap A)$. Thus there exists $M$ such that for any $m \geq M$ $F(G \cap V_m \cap A) \subset S_{\varepsilon 2} \left[ \bigcap_{n=1}^{\infty} F(G \cap V_n \cap A) \right] \subset S_{\varepsilon 2} \left[ \bigcap_{i=1}^{n} F(G \cap A) \cap \bigcap F(V_i \cap A) \right] \subset S_{\varepsilon 2}[S_{\varepsilon 2}[F^*(x)]] \subset S_{\varepsilon 2}[F^*(x)_]$. That implies $F^*(z) \subset S_{\varepsilon 2}[F^*(x)]$ for any $z \in G \cap V_M$. The upper semicontinuity of $F^*$ is proved.

**Theorem 2.** A metric space $Y$ is complete if and only if each upper semicontinuous closed-valued multifunction $F: A \to Y$ with $A$ dense in $X$ and such that for any $x \in X \setminus A$ the net $\{\chi(F(V \cap A)) : V \in \mathcal{F}(x)\}$ converges to zero, has an upper semicontinuous extension to $X$.

**Proof.** The necessity is obvious from Theorem 1.

Suppose that a metric space $Y$ is not complete. Then there exists a Cauchy sequence $\{y_n\}$ such that no point in $Y$ is a cluster point of $\{y_n\}_{n=1}^{\infty}$. We can suppose that $y_i \neq y_j, i \neq j$. Put $X = \{y_1, \ldots, y_n, \ldots\}$. Let $\mathcal{T}$ consist of 0 and of the sets $\{y_1, y_n, y_{n+1}, \ldots\}, n = 1, 2, \ldots$, $\mathcal{T}$ is a topology on $X$. Put $A = \{y_2, y_3, \ldots\}$. It is easy to verify that $A$ is dense in $X$. Define a multifunction $F: A \to Y$ as follows: $F(y_n) = \{y_n, y_{n+1}, \ldots\} \subset A$. Then $F$ is a closed-valued upper semicontinuous multifunction on $A$.

Since $\{y_n\}$ is a cauchy sequence in $Y$, the net $\{\chi(F(V \cap A)) : V \in \mathcal{F}(x)\}$ converges to zero. There exists no upper semicontinuous extension $F^*$ of $F$ defined on $X$.

Suppose that $F^*$ is an upper semicontinuous extension of $F$ defined on $X$. The upper semicontinuity of $F^*$ at $y_n$ for $n = 2, 3$, implies $F^*(y_1)$ contains no point from the set $\{y_2, y_3, \ldots\}$. Let $i > 1$ be such that $y_i \in F^*(y_1)$. Let $n > 1$. There exists an open set $U$ in $X$ such that $F^*(y_n) = F(y_n) \subset U$ and $y_i \notin U$. Thus $F^*(y_1) \cap (X \setminus U) \neq 0$, which is a contradiction with the upper semicontinuity of $F^*$ at $y_1$, $F^*(y_1) \cap \{y_2, y_3, \ldots\} = 0$, that means $V = Y \setminus \{y_2, \ldots, y_n\}$ is open in $Y$ such that $F^*(y_1) \subset V$ and $F^*(y_n) \cap V = 0$ for any $n = 2, 3, \ldots$. However, that is a contradiction with the upper semicontinuity of $F^*$ at $y_1$.

**Theorem 3.** Let $Y$ be a metric space. $Y$ is topologically complete if and only if
each upper semicontinuous compact-valued multifunction \( F: A \to Y \) with \( A \) dense in \( X \) has upper semicontinuous compact-valued extension to a \( G_\sigma \)-set containing \( A \).

**Proof.** Suppose that a metric space \((Y,d)\) is topologically complete. That means, there exists a complete metric \( \varrho \) in \( Y \) topologically equivalent to \( d \).

Let \( G = \{ x \in X : \text{the net } \{ x(F(V_n A) : V_n \in \mathcal{V}(x) \} \text{ converges to zero} \} \). By Remark 1 \( G \) is a \( G_\sigma \)-set and \( A \subseteq G \).

Define \( F^* \) as in the proof of Theorem 1., that means \( F^*(x) = \bigcap \{ F(V_n A) : V_n \in \mathcal{V}(x) \} \) for \( x \in G \setminus A \) and \( F^*(x) = F(x) \) for \( x \in A \). Then \( F^* \) is an upper semicontinuous compact-valued multifunction. (see the proof of Theorem 1.)

Suppose that the metric space \((Y,d)\) is not topologically complete. We show that there exist a topological space \( X \) and an upper semicontinuous compact-valued multifunction \( F \) from a dense set in \( X \) to \( Y \), which has no upper semicontinuous compact-valued extension to a \( G_\sigma \)-set in \( X \).

Let \((\tilde{Y}, \tilde{d})\) be a completion of \((Y,d)\). Put \( X = (\tilde{Y}, \tilde{d}) \). Then \( Y \) is a dense subset of \( X \), which is not a \( G_\sigma \)-set in \( X \). (Suppose that \( Y \) is a \( G_\sigma \)-set in \( X \). Then by [4] p. 49 \( Y \) is topologically complete.)

Consider the identity mapping \( i: Y \to Y \). There exists no upper semicontinuous compact-valued extension of \( i \) to a \( G_\sigma \)-set in \( X \) containing \( Y \).

Suppose that there exists an upper semicontinuous compact-valued extension \( i^* \) of \( i \) to a \( G_\sigma \)-set \( L \) containing \( Y \). Let \( y \in L \setminus Y \). There exists a sequence \( \{ y_n \} \) of points of \( Y \) which is convergent to \( y \). The sequence \( \{ y_n \} \) has no cluster point in \( Y \), that means every subsequence of \( \{ y_n \} \) is a closed set in \( Y \).

There exists \( N_1 \in \mathbb{N} \) such that for any \( n \geq N_1 \), \( y_n \in i^*(y) \). Otherwise there exists a subsequence \( \{ y_{n_k} \} \) of \( \{ y_n \} \) such that \( y_{n_k} \notin i^*(y) \) for any \( k \in \mathbb{N} \). The upper semicontinuity of \( i^* \) at \( y \) implies there exists an open set \( V \) such that \( y \in V \) and \( i^*(z) \subseteq Y \setminus \{ y_n, y_{n+1}, \ldots \} \) for any \( z \in V \). But there exists \( N_2 \in \mathbb{N} \) such that for any \( k \geq N_2 \), \( y_{n_k} \in V \), which is a contradiction.

Thus \( i^*(y) \supseteq \{ y_n, y_{n+1}, \ldots \} \) where \( n \geq N_1 \), that means \( i^*(y) \) is not compact, which is a contradiction.

The following example shows that the assumption on values of the multifunction in Theorems 1 and 3 is essential.

**Example 1.** Put \( Y = R \) with the usual topology. Put \( X = \{ 1, 1/2, \ldots, 1/n, \ldots, 0 \} \). Let \( \mathcal{G} \) be a family consisting of \( \emptyset \) and of the sets of the form \( \{ 0, 1/n, 1/n + 1, \ldots \} \) for \( n = 1, 2, \ldots \). Then \( \mathcal{G} \) is a topology on \( X \). Put \( A = \{ 1, 1/2, 1/3, \ldots, 1/n, \ldots \} \). Then \( A \) is dense in \( X \) in the topology \( \mathcal{G} \) and only the \( G_\sigma \)-set containing \( A \) is the set \( X \). Define \( F: A \to Y \) in this way: \( F(1/n) = (−1/n, 0) \). Then \( F \) is upper semicontinuous on \( A \). It is easy to verify that the net \( \{ x(F(V \cap A)) : V \in \mathcal{V}(0) \} \) converges to zero, where \( \mathcal{V}(0) \) is the set of all open neighbourhoods of \( 0 \). There is no upper semicontinuous extension of \( F \) defined on \( X \). (Suppose that there exists an upper semicontinuous extension \( F^* \) of \( F \) defined on \( X \). The
upper semicontinuity of $F^*$ at points of $A$ implies $F^*(0) \subset \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 0 \right) = 0.
That is a contradiction.)

REFERENCES


Received May 7, 1986

Katedra teorie pravdepodobnosti
a matematickej štatistiky
Matematicko-fyzikálna fakulta UK
Mlynská dolina
842 15 Bratislava

ТЕОРЕМА О ПРОДОЛЖЕНИИ ДЛЯ МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ
И ХАРАКТЕРИЗАЦИЯ ПОЛНЫХ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

Lubica Holá

Резюме

Пусть $Y$ — метризуемое пространство. Доказывается: пространство $Y$ топологически полно, тогда и только тогда, когда каждое сверху наперывное многозначное отображение $\Phi: A \to Y$ с бикомпактными значениями, где $A$ плотное множество в $X$, имеет сверху непрерывное продолжение на $\Gamma_{\varphi}$-множество $\Gamma$, причем $\Gamma \supset A$. 

182