František Neuman

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GLOBAL CANONICAL FORMS OF LINEAR DIFFERENTIAL EQUATIONS

FRANTIŠEK NEUMAN

I. Two special forms of linear differential equations have been introduced in literature, the Laguerre—Forsyth form, and the Halphen form. Each of them is sometimes called canonical. As G. D. Birkhoff [2] pointed out already in 1910, the Laguerre—Forsyth form is not global, in the sense that not every linear differential equation can be transformed, on its whole interval of definition, into an equation of that form. It can be shown that the Halphen form is not global either. Neither of these forms becomes global by restricting ourselves to the class of linear differential equations where some smoothness (or even analyticity) of coefficients is required.

In this paper we give a construction of global canonical forms. In particular it is shown that

$$y^{(n)}+y^{(n-2)}+r_{n-3}(x)y^{(n-3)}+\ldots+r_0(x)y=0 \quad \text{on} \quad I\subset \mathbf{R}$$

is one of the global forms.

II. Let $P_n(y, x; I)=y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots+p_0(x)y$ on I, and $Q_n(z, t; J)=z^{(n)}+q_{n-1}(t)z^{(n-1)}+\ldots+q_0(t)z$ on J denote linear differential operators of the n-th order, $n\geqq 2$, $p_i\in C^0(I)$, $q_i\in C^0(J)$, I and J be open (bounded or unbounded) intervals of reals. Let id_I denote the identity of I.

We say that the equation $P_n(y, x; I)=0$ is globally transformable into $Q_n(z, t; J)=0$, if functions f and h exist, f: $J\to\mathbf{R}$, $f\in C^n(J)$, $f(t)\neq 0$ in J, $h\in C^n(J)$, $\mathrm{d}h(t)/\mathrm{d}t\neq 0$ on J, $h(J)=I$, such that

$$z(t)=f(t)\cdot y(h(t)), \quad t\in J,$$

is a solution of $Q_n(z, t; J)=0$ whenever y is a solution of $P_n(y, x; I)=0$, see [6, 8].

For such a transformation $T=\langle f, h\rangle$ and for the corresponding differential operators we shall also write

$$T*P_n(y, x; I)=Q_n(z, t; J), \quad \text{or briefly} \quad T*P_n=Q_n.$$

Let us point out that all operators are considered with leading coefficient 1. We also say that the transformation T transforms the differential equation $P_n = 0$ into the equation $Q_n = 0$. If $T * P_n = Q_n$, the equations $P_n = 0$ and $Q_n = 0$ will be called *globally equivalent*, since the global transformability is an equivalence relation.

Let $\mathscr{L}_n$ denote the set of all linear differential equations of the order n, $n \geqq 2$.

We say that $\mathscr{S}$ is the *set of global canonical forms* for $\mathscr{L}_n^* \subset \mathscr{L}_n$, if each linear differential equation of the n-th order from $\mathscr{L}_n^*$ can be globally transformed into at least one equation from the set $\mathscr{S}$.

A set $\mathscr{S} \subset \mathscr{L}_n^*$ is called the *set of unique canonical forms* for $\mathscr{L}_n^*$, if each equation from $\mathscr{L}_n^*$ can be globally transformed into at most one equation from $\mathscr{S}$.

Remark 1. $\mathscr{S} \subset \mathscr{L}_n^*$ is a set of canonical forms (with respect to the equivalence relation of "global transformation" on $\mathscr{L}_n^*$) in the sense of S. Mac Lane and G. Birkhoff [5] exactly when $\mathscr{S}$ is a set of global and unique canonical forms. The reason for introducing our definitions follows from the fact that, in the case of linear differential equations, under a very weak reasonable condition, it is impossible to satisfy both requirements (i.e. globality and uniqueness) at the same time, [7]. On the other hand, it is useful to have the two types of canonical forms, because global canonical forms are suitable for description of global behaviour of solutions (i.e. on the whole interval of definition), whereas invariants of linear differential equations can be evaluated from unique (generally only local) canonical forms.

Consider a linear differential equation of the second order in the Jacobi form

$$u'' = q(t)u, \quad q \in C^{n-2}(J). \tag{1}$$

Let u_1, u_2 be two linearly independent solutions of (1). Let $z_i(t) := u_1^{i-1}(t) \cdot u_2^{n-i}(t)$, $i = 1, \ldots, n$. The n-tuple $z_1, \ldots, z_n$ has a non-vanishing Wronskian on J, each $z_i \in C^n(J)$, hence z_i are solutions of a linear differential equation of the n-th order. Denote by $I_n[q]$ the linear differential operator with the leading coefficient 1 corresponding to the equation. It can be shown that

$$I_n[q] = z^{(n)} + \sum_{i=1}^{n} f_i[q] \cdot z^{(n-i)},$$

where

$$f_1[q] = 0, \quad f_2[q] = -\binom{n+1}{3} q, \quad f_3[q] = -2\binom{n+1}{4} q',$$

etc., see, e.g. [4].

The set of equations of either of the forms

$$
\begin{array}{llr}
I_n[q]+z^{(n-3)}+d_{n-4}(t)z^{(n-4)}+d_{n-5}(t)z^{(n-5)}+\dots+d_0(t)z=0, & & (2_1)\\
I_n[q]+\qquad z^{(n-4)}+d_{n-5}(t)z^{(n-5)}+\dots+d_0(t)z=0, & & (2_2)\\
\dots & & \\
I_n[q]+\qquad z=0, & & (2_{n-2})\\
I_n[q]\qquad =0, & & (2_{n-1})
\end{array}
$$

for all n, $n\geqq 3$, all $J\subset \mathbf{R}$ and all $q\in C^{n-2}(J)$, $d_i\in C^0(J)$, is called the set of the Halphen canonical forms, see [3] and [9].

III. Theorem 1. *The Halphen canonical forms do not form a set of global canonical forms for linear differential equations of any order n, $n\geqq 3$.*

Proof. For $n\geqq 3$, consider the equation

$$I_n[p]+x\cdot y^{(n-3)}=0 \quad \text{on} \quad I=(a,b)\ni 0, \tag{3}$$

$p\in C^{\omega}(I)$. Suppose there exists a global transformation $T=\langle f, h\rangle$ that transforms (3) into (2_i) on J for some i, $1\leqq i\leqq n-1$. Then $h\in C^n(J)$, $h'(t)\neq 0$ on J, $h(J)=(a,b)$. Since the coefficients of $y^{(n-1)}$ and $z^{(n-1)}$, in both (3) and (2_i) are zero, we get $f(t)=c\cdot|h'(t)|^{-(n-1)/2}$, c being a nonzero constant. Then the transformation $\langle |h'|^{-1/2}, h\rangle$ globally transforms $y''=p(x)y$ on I into $z''=q(t)z$ on J, that gives

$$T*(I_n[p]+x\cdot y^{(n-3)})=I_n[q]+h(t)\cdot(h'(t))^3\cdot z^{(n-3)}+d_{n-4}(t)z^{(n-4)}+\dots+d_0(t)z.$$

If $i=1$, then $h(t)\cdot(h'(t))^3=1$. Since $h'\neq 0$ on J, we have $h(t)\neq 0$ on J that contradicts $h(J)=(a,b)\ni 0$. If $i>1$, then $h(t)\cdot(h'(t))^3\equiv 0$, that is again a contradiction to $h'(t)\neq 0$ on J. Q.E.D.

Remark 2. The coefficients of (3) were analytic, hence the Halphen canonical forms are not even global for the class of linear differential equations with analytic coefficients.

For $n\in\mathbf{N}$, $n\geqq 2$, let $\mathscr{L}_n^\circ$ denote the set of all linear differential equations $P_n(y,x;I)=0$, with $p_n\equiv 1$, $p_{n-1}\in C^{n-1}(I)$ and $p_{n-2}\in C^{n-2}(I)$.

Theorem 2. *Let the equation*

$$z''=\bar{q}(t)z \quad on \quad J, \quad \bar{q}\in C^{n-2}(J) \tag{4}$$

have solutions oscillatory to both sides of J.

For each n, $n\geqq 2$, the set of the differential equations

$$z^{(n)}-\binom{n+1}{3}\bar{q}(t)z^{(n-2)}+\sum_{i=3}^{n}r_{n-i}(t)z^{(n-i)}=0 \quad on \quad J^*, \tag{5}$$

for (a fixed function $\bar{q}$ on J), arbitrary $J^*\subset J$, arbitrary $r_{n-i}\in C^0(J^*)$, is a set of global canonical forms for $\mathscr{L}_n^\circ$.

Proof. Let $P_n(y, x; I)=0$ from $\mathscr{L}_n^\circ$ be given. The transformation $\left\langle \exp\left\{\frac{1}{n}\int p_{n-1}\right\}, \mathrm{id}_I \right\rangle$ transforms it into

$$v^{(n)}+s_{n-2}(x)v^{(n-2)}+\sum_{i=3}^{n}s_{n-i}(x)v^{(n-i)}=0 \text{ on } I, \tag{6}$$

where $s_{n-2}\in C^{n-2}(I)$, $s_{n-i}\in C^0(I)$ for $i=3, \ldots, n$. For $\bar{p}(x):=-s_{n-2}(x)\Big/\binom{n+1}{3}$ and $\bar{r}_{n-i}(x):=s_{n-i}(x)-f_i[\bar{p}], i=3, \ldots, n$, we may write (6) as

$$I_n[\bar{p}]+\sum_{i=3}^{n}\bar{r}_{n-i}(x)v^{(n-i)}=0 \text{ on } I, \tag{7}$$

$\bar{p}\in C^{n-2}(I)$.

The solutions of the equation (4) are oscillatory to both ends of J. Hence, in accordance with [1], there exists a $J^*\subset J$ such that (4) restricted to J^* is globally equivalent to $v''=\bar{p}(x)v$ on I. (The case $J^*=J$ is not excluded.) Let $\langle f, h\rangle$ transform $v''=\bar{p}(x)v$ on I into (4) on J^*. Then $f=c\cdot|h'|^{-1/2}$, $c=\text{const.}\neq 0$, $h(J^*)=I$. The function h is a composition of antiderivatives of $(v_1^2+v_2^2)^{-1/2}$ and $(z_1^2+z_2^2)^{-1/2}$, and their inverses, where v_1, v_2, and z_1, z_2 are linearly independent solutions of $v''=\bar{p}(x)v$ and $z''=\bar{q}(t)z$, resp., see [1].

Since $\bar{p}$, $\bar{q}\in C^{n-2}$, solutions v_1, v_2, z_1, $z_2\in C^n$, hence our $h\in C^{n+1}(J^*)$. We see that the transformation $\langle |h'|^{-(n-1)/2}, h\rangle$ globally transforms $I_n[\bar{p}]$ into $I_n[\bar{q}]$ restricted to $J^*\subset J$. At the same time it transforms globally (7) into $I_n[\bar{q}]+\sum_{i=3}^{n}r^*_{n-i}(t)z^{(n-i)}=0$ on J^* with continuous r^*_{n-i}, $i=3, \ldots, n$, that can be written in the form (5). Hence the composition of two transformations,

$$\left\langle \exp\left\{\frac{1}{n}\int p_{n-1}\right\}, \mathrm{id}_I \right\rangle \quad \text{and} \quad \langle |h'|^{-(n-1)/2}, h\rangle$$

globally transforms $P_n(y, x; I)=0$ from $\mathscr{L}_n^\circ$ into the form (5). Q.E.D.

Since every z satisfying $z''=-z\Big/\binom{n+1}{3}$ is oscillatory to both sides of $(-\infty, \infty)$ for any integer n, $n\geqq 2$, we get the following consequence of Theorem 2.

Corollary. *The set of equations*

$$z^{(n)}+z^{(n-2)}+\sum_{i=3}^{n}r_{n-i}(t)z^{(n-i)}=0$$

on arbitrary intervals $J^*\subset(-\infty, \infty)$, $n\geqq 2$, with arbitrary $r_{n-i}\in C^0(J^*)$, is a set of global canonical forms for $\mathscr{L}_n^\circ$.

Remark 3. If Laguerre and Forsyth had required the first three coefficients 1, 0, 1 instead of their 1, 0, 0, they would have got our global forms in the Corollary instead of their local ones.

Remark 4. Using a geometrical approach to global problems in the theory of linear differential equations, another type of global canonical forms for $\mathscr{L}_n$ was described in [6].

Remark 5. Let us note that our n-th order global canonical forms (5) depend on $n-2$ functions only, i.e. on coefficients $r_0, r_1, \ldots, r_{n-3}$ ($\bar{q}$ is fixed).

IV. Let us give a survey of canonical forms for $n=2, 3$; I being arbitrary intervals.

$n=2$:

$$y''=0 \quad \text{on} \quad I$$

is the Laguerre—Forsyth canonical form (it is not global);

$$y''+y=0 \quad \text{on} \quad I$$

is both the form described in [6], and our form for $\mathscr{L}_2$ (it is global).

$n=3$:

$$y'''+r(x)y=0 \quad \text{on} \quad I$$

is the Laguerre—Forsyth form (it is not global);

$$y'''+p(x)y'+\left(\frac{1}{2}p'(x)+1\right)y=0,$$

and

$$y'''+p(x)y'+\frac{1}{2}p'(x)y=0$$

on I are the Halphen forms (they are not global);

$$y'''-\frac{\alpha'(x)}{\alpha(x)}y''+(1+(\alpha(x))^2)y'-\frac{\alpha'(x)}{\alpha(x)}y=0 \quad \text{on} \quad I,$$

$\alpha \in C^1(I)$, $\alpha>0$ on I, is a global canonical form for $\mathscr{L}_3$ described in [6].

$$y'''-4\bar{q}(x)y'+r(x)y=0 \quad \text{on} \quad I\subset J,$$

(where the fixed function $\bar{q}$ is such that the solutions of $y''=\bar{q}(x)y$ are oscillatory to both ends of J) is a global canonical form for $\mathscr{L}_3^0$.

In particular,

$$y'''+y'+r(x)y=0 \quad \text{on} \quad I\subset(-\infty, \infty)$$

is a global canonical form for $\mathscr{L}_3^0$.

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Matematický ústav ČSAV
pobočka v Brně
Janáčkovo nám. 2a
662 95 Brno

ГЛОБАЛЬНЫЕ КАНОНИЧЕСКИЕ ФОРМЫ ДЛЯ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

František Neuman

Резюме

В этой работе в отличие от классических локальных форм Лагерь–Форсайта или Гальфена дана конструкция глобальных канонических форм для линейных дифференциальных уравнений n-го порядка, $n \geqq 2$. В частности показано, что

$$y^{(n)} + y^{(n-2)} + r_{n-3}(x)y^{(n-3)} + \ldots + r_0(x)y = 0$$

одна из этих глобальных форм.