Ján Haluška

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ON A SYMBOL OF OPERATORS GENERATING FINITE DIMENSIONAL ALGEBRAS

JÁN HALUŠKA

ABSTRACT. We investigate the so called symbol of operator generating finite dimensional algebras. In our consideration the kernel of the symbol need not be the subset of an ideal of compact operators.

In the paper [1] there were given examples showing that a finite dimensional operator algebra over a ring of continuous functions is a natural generalization of the structure of many important linear operators appering in the research of singular integral equations. In [1] the authors introduced the notion of the so called symbol of such an algebra. Moreover, they showed that if the kernel of the symbol is a subset of the ideal of compact operators then all algebraic and other properties of such an operator are characterized by the symbol alone. In this paper wee deal with the case when the kernel of the symbol need not be a subset of the ideal of compact operators.

Our approach has mainly an algebraic character. How ever, from the general algebraic point of view the solution of such problems is not known or need not exist in general. Further, these algebraic problems are specific for the theory of singular integral equations, e.g. such is the problem of existence of the symbols of operators themselves. Therefore we prefer to investigate a not general but suffuciently concrete situation and to use terms which the theory of singular integral equations deals with.

In the paper we use upper and lower indices. In the case of the exponent we exclusively use the brackets. The summing is executed always from 1 to K through the same upper and lower index.

1. Definitions and basic notions

1.1. Let $\mathscr{L}$ be a simple closed smooth curve in the complex plane. Let $C(\mathscr{L})$ be a space of all continuous functions $a: \mathscr{L} \to \mathbf{C}$ with the usual supremum norm

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$\| \,.\, \|_{C(\mathscr{L})}$. Let $X(\mathscr{L}) \supset C(\mathscr{L})$ be a Banach space of complex valued functions on $\mathscr{L}$ with the norm $\| \,.\, \|_{X(\mathscr{L})}$ such that the restriction of $\| \,.\, \|_{X(\mathscr{L})}$ to $C(\mathscr{L})$ equals the norm $\| \,.\, \|_{C(\mathscr{L})}$. Let $L(X(\mathscr{L}))$ be a Banach space of all continuous linear operators Y: $X(\mathscr{L}) \to X(\mathscr{L})$ with the supremum norm. Let $\mathscr{F}$ be a subring of the space $C(\mathscr{L})$ such that the operator $Y: f \to af, f \in X(\mathscr{L}), a \in \mathscr{F}$, is continuous in $X(\mathscr{L})$ and $\| Y \| \leq \| f \|_{X(\mathscr{L})} . \| a \|_{C(\mathscr{L})}$.

Let $\mathscr{D}$ be an ideal of compact operators in $X(\mathscr{L})$.

Let $S_k \in L(X(\mathscr{L}))$, $k = 1, 2, \ldots, K$, be operators such that

(a) $S_1 = \mathbf{I}$ is a unit operator,

(b) $a(S_k(f)) - S_k(af) = T(f)$, or shortly $aS_k - S_k a = T$, where $f \in X(\mathscr{L})$, $a \in \mathscr{F}$, $T \in \mathscr{D}$,

(c) S_k $(k = 1, 2, \ldots, K)$ are linearly independent over the ring $\mathscr{F}$,

(d) further we suppose that the composition of two operators S_i, S_j can be expressed as follows:

$$S_i \circ S_j = \sum_{k=1}^{K} \gamma_{i,j}^k S_k + T_{i,j}, \tag{1}$$

where $\gamma_{i,j}^k \in \mathscr{F}$, $T_{i,j} \in \mathscr{D}$, $i, j = 1, 2, \ldots, K$.

Let us consider the opearators $A \in L(X(\mathscr{L}))$ of the following form:

$$A = \sum_{k=1}^{K} a^k S_k + T,$$

where $T \in \mathscr{D}$, $a^k \in \mathscr{F}$, $k = 1, 2, \ldots, K$. The set of such operators clearly forms an algebra. Denote it by $\mathscr{R}$.

1.2. Let $\bar{\mathscr{R}}$ be a factor algebra $\mathscr{R} / (\mathscr{D} \cap \mathscr{R})$. Every $\bar{A} \in \bar{\mathscr{R}}$ can be unambigously expressed in the form

$$\bar{A} = \sum_{k-1}^{K} a^k \bar{S}_k,$$

where $\bar{S}_k$ is the image of the operator S_k, $k = 1, 2, \ldots, K$, in the natural homogorphism $\mathscr{R} \to \bar{\mathscr{R}}$. Obviously

$$\bar{S}_i \;\; \bar{S}_j = \sum_{k=1}^{K} \gamma_{i,j}^k \bar{S}_k,$$

where $\gamma_{i,j}^k$, $i, j, k = 1, 2, \ldots K$, are the same functions as in (1). Both $\mathscr{R}$ and $\bar{\mathscr{R}}$ are algebras with units. The algebra $\bar{\mathscr{R}}$ is a left and right free module over the ring $\mathscr{F}$, shortly $\mathscr{F}$-module. We shall say that the system

$$\{\bar{S}_k;\, k = 1, 2, \ldots, K\} = \bar{\mathbf{S}}$$

forms a basis of the $\mathscr{F}$-module $\bar{\mathscr{R}}$.

1.3. Lemma. *Let* $[b_j^i(t)]$, $i, j = 1, 2, \dots, K$, *be a regular matrix for every* $t \in \mathscr{L}$ *and let*

$$\bar{P}_k = \sum_{j=1}^{K} b_k^j \bar{S}_j \tag{2}$$

where $b_k^j \in \mathscr{F}$ *for* $j, k = 1, 2, \dots, K$, *is a linear transform of the basis* $\bar{\mathbf{S}}$ *onto the basis*

$$\bar{\mathbf{P}} = \{\bar{P}_k;\ k = 1, 2, \dots, K\}.$$

Let $[c_k^p]$, $p, k = 1, 2, \dots, K$, *be the matrix of the inverse transform to the* (2). *Then* (1) *implies the following rule of the operator composition*:

$$\bar{P}_m \circ \bar{P}_n = \sum_{p=1}^{K} \eta_{m,n}^p \bar{P}_p,$$

where

$$\eta_{m,n}^p = \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{k=1}^{K} b_m^j b_n^i \gamma_{i,j}^j c_k^p,$$

and $m, n, p = 1, 2, \dots K$.

1.4. For every $\bar{X} \in \bar{\mathscr{R}}$ denote

$$\hat{A} = \sum_{k=1}^{K} \gamma_{k,j}^i a^k,\ i, j = 1, 2, \dots, K,$$

the matrix of the operator $F_{\bar{A}}$: $\bar{X} \to \bar{A} \circ \bar{X}$ with respect to the basis $\bar{\mathbf{S}}$. Suppose further that there exists the following decomposition of det $\hat{A}$:

$$\det \hat{A} = \prod_{j=1}^{K} \sigma_A^j,$$

where

$$\sigma_A^j = \sum_{i=1}^{K} \lambda_i^j a^i,$$

where λ_i^j, $i, j = 1, 2, \dots, K$, are some functions from $\mathscr{F}$. Note that in general it is not always possible, cf. [1]. Now, we define

$$\operatorname{sym} A = \sigma_A = (\sigma_A^1, \sigma_A^2, \dots, \sigma_A^K) \in (\mathscr{F})^K,$$

cf. [1], 2°. So, we define the symbol of the operator $A \in \mathscr{R}$ as the map sym in the following way:

$$\begin{array}{ccccc} A & \longrightarrow & \bar{A} & \longrightarrow & F_{\bar{A}} \\ \text{sym} \downarrow & & & & \downarrow \\ \sigma_A & \longleftarrow & \det \hat{A} & \longleftarrow & \hat{A} \end{array}$$

The operation of addition and also of multiplication of symbols are defined coordinatewisely. Evidently the $\mathscr{F}$-module $(\mathscr{F})^K$ is an algebra.

1.5. Lemma. *The map* sym: $\mathscr{R} \to (\mathscr{F})^K$ *is a homomorphism if and only if*

$$\lambda_i^p \lambda_j^p = \sum_{k=1}^{K} \gamma_{i,j}^k \lambda_k^p \tag{3}$$

where $i, j, p = 1, 2, \ldots, K$.

Remark. Let us note that the condition (3) is only necessary for proving that the function sym preserves multiplication.

1.6. Carollary. *If* ker (sym) $\subseteq \mathscr{D}$, *then the function* sym: $\mathscr{R} \to (\mathscr{F})^K$ *is an isomorhism. To this end it is enough to have*

$$\det [\lambda_k^p(t)] \neq 0 \tag{4}$$

for every $t \in \mathscr{L}$, *where* $p, k = 1, 2, \ldots, K$, *which follows from* (3).

1.7. Remark. If (4) is valid, then some necessary and sufficient conditions of the noetherness of the operator $A \in \mathscr{R}$ with respect to the symbol sym (A) are known. In this case we are able to compute the index of operators with respect to the symbol of operators and to the homotopic invariants of the ring $\mathscr{F}$, cf. [1], 3°.

2. Case ker (sym) $\nsubseteq \mathscr{D}$

2.1. The consideration of the case ker (sym) $\nsubseteq \mathscr{D}$ is equivalent to determining such an ideal $\mathscr{H}$ which should be a kernel of a homomorphism of the algebra $\mathscr{R}$ into some module of functions over $\mathscr{F}$. In the paper [1] the authors considered only the case (4), but this situation is not typical for the algebra $\mathscr{R}$. Indeed, (3) implies:

$$\lambda_i^p \lambda_j^p = \sum_{k-1}^{K} \gamma_{i,j}^k \lambda_k^p$$

and

$$\gamma_j^p \lambda_i^p = \sum_{k=1}^{K} \gamma_{j,i}^k \lambda_k^p,$$

where $i, j, k, p = 1, 2, \dots, K$. After subtracting these two equations we get for every $t \in \mathscr{L}$:

$$0 = \sum_{k=1}^{K} (\gamma_{i,j}^{k} - \gamma_{j,i}^{k}) \lambda_k^p, \tag{5}$$

where $i, j, p = 1, 2, \dots, K$.

When (4) is true, then the system (5) of the linear equations with the coefficients λ_k^p, $p = 1, 2, \dots, K$, for every $t \in \mathscr{L}$, has only a trivial solution, and so the algebra $\mathscr{R}$ is commutative. In other words, if the algebra $\mathscr{R}$ is not commutative, then certainly there exists a point $t_0 \in \mathscr{L}$ such that

$$\det [\lambda_j^i(t_0)] = 0, \; i, j = 1, 2, \dots, K.$$

2.2. Theorem. *Let $A \in \mathscr{R}$ and the symbol σ_A exists. Then for a fixed m, $m = 1, 2, \dots, K$, the set*

$$\mathscr{H}_m = \{\bar{A} \in \bar{\mathscr{R}}; \; \sigma_A^m = 0\} \tag{6}$$

is a maximal ideal of the algebra $\bar{\mathscr{R}}$.

Proof. Let us have two operators $\bar{X}, \bar{A} \in \bar{\mathscr{R}}$. so

$$\bar{X} = \sum_{i=1}^{K} x^i \bar{S}_i \quad \text{and} \quad \bar{A} = \sum_{j=1}^{K} a^j \bar{S}_j.$$

Immediately from (b) and (d) we have:

$$\begin{aligned} \bar{A} \circ \bar{X} &= \Big(\sum_{i=1}^{K} a^i \bar{S}_i\Big) \circ \Big(\sum_{j=1}^{K} x^j \bar{S}_j\Big) = \\ &= \sum_{i=1}^{K} \sum_{j=1}^{K} a^i x^j \bar{S}_i \circ \bar{S}_j = \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{k=1}^{K} a^i x^j \gamma_{i,j}^k \bar{S}_k. \end{aligned} \tag{7}$$

From (a) and (3) it follows that $\lambda_1^m = 1$.

We show first that $\mathscr{H}_m$ is an ideal.

Consider the new basis $\bar{\mathbf{P}} = \{\bar{P}_i; i = 1, 2, \dots, K\}$ of the $\mathscr{F}$-module $\mathscr{R}$ with the following property:

$$(\bar{S}_1, \bar{S}_2, \dots, \bar{S}_K) = (\bar{P}_1, \bar{P}_2, \dots, \bar{P}_K) \begin{bmatrix} \lambda_1^m & \lambda_2^m & \lambda_3^m & \dots & \lambda_K^m \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{8}$$

Since $\lambda_1^m = 1$, the transform (8) is regular for every $t \in \mathscr{L}$. (8), (7) and (3) imply

$$\bar{A} \circ \bar{X} = \sigma_A^m \bar{P}_1 \sigma_X^m + \sum_{i=1}^{K} \gamma_{k,n}^i a^k x^n \bar{P}_i. \tag{9}$$

From (8) we have:

$$\bar{A} = \sum_{i=1}^{K} a^i \bar{S}_i = \sigma_A^m \bar{P}_1 + \sum_{i=2}^{K} a^i \bar{P}_i. \tag{10}$$

Take $\bar{A} \in \bar{\mathscr{H}}_m$ and $\bar{X} \in \bar{\mathscr{R}}$. (9) and (10) imply $\bar{A} \circ \bar{X} \in \bar{\mathscr{H}}_m$ and $\bar{X} \circ \bar{A} \in \bar{\mathscr{H}}_m$.

The maximality of the ideal $\bar{\mathscr{H}}_m$ follows from the dimensional reasons. ∎

2.3. Corollary. *It follows from* (9) *that the* $\mathscr{F}$*-factor module* $\bar{\mathscr{R}} / \bar{\mathscr{H}}_m$, $m = 1, 2, \ldots, K$, *is a comutative algebra.*

2.4. Corollary. *Again from* (9) *and the fact that* $\bar{\mathbf{P}}$ *is a basis of* $\bar{\mathscr{R}}$ *we obtain that the nember of the different maximal ideals of algebra* $\bar{\mathscr{R}}$ *is not greater than K.*

2.5. Corollary. *If* (4) *is true, then we may take a matrix* $[\lambda_j^i]$, $i, j = 1, 2, \ldots, K$, *istead of the matrix*

$$\begin{bmatrix} \lambda_1^m & \lambda_2^m & \lambda_3^m & \ldots & \lambda_K^m \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

in the proof of the theorem. Then (9) *and* (10) *can be rewritten in the following way*:

$$\bar{A} \circ \bar{X} = \sum_{i=1}^{K} \sigma_A^i \bar{P}_i \sigma_X^i \quad \text{and} \quad \bar{A} = \sum_{i=1}^{K} \sigma_A^i \bar{P}_i.$$

The fact, that the algebra $\bar{\mathscr{R}}$ is a direct sum of the ideals

$$\bar{\mathscr{H}}_m = \{\bar{A} \in \bar{\mathscr{R}};\ \bar{A} = a^m \bar{P}_m\},$$

$m = 1, 2, \ldots, K$, is implied by the diagonality of the matrix $\hat{A}$ with respect to the basis $\bar{\mathbf{P}}$.

2.6. Corollary. *Let us denote*

$$\bar{\mathscr{H}} = \bigcap_{m=1}^{K} \bar{\mathscr{H}}_m.$$

The ideal $\bar{\mathscr{H}}$ *is a kernel of the considered homorphism. If the matrix* $\hat{A}$ *has a constant rank*, rank $\hat{A} = r$, $1 \leq r \leq K$, *over the whole curve* $\mathscr{L}$, *then* $\bar{\mathscr{R}} / \bar{\mathscr{H}} \to \to (\mathscr{F})^r$ *is an isomorphism and the algebra* $\mathscr{R} / \mathscr{H}$ *is commutative, where the ideal* $\mathscr{H} = \bar{\mathscr{H}} + (\mathscr{D} \cap \mathscr{R})$.

To illustrate our result we give the following example.

2.7. Example. Let $X(\mathscr{L}) = L_p(\mathscr{L})$, $p > 1$, $\mathbf{I}$ denote the unit operator, $\mathbf{S}$ be a singular operator, and $\mathbf{N}$ be an operator with the polar-logarithm kernel. Recall that

$$(\mathbf{S}f)(t) = \frac{1}{\pi i} \int_{\mathscr{L}} \frac{f(v)}{v - t} \, \mathrm{d}v, \ t \in \mathscr{L}, f \in \mathscr{F},$$

$$(\mathbf{N}f)(t) = \int_{\mathscr{L}} \frac{f(v)}{(v-t)\cdot \ln(v-t)}\, dv, \; t \in \mathscr{L}, f \in \mathscr{F},$$

where the operators **S** and **N** are defined on a Hölder space $H_\mu(\mathscr{L})$, $0 < \mu < 1$, as Cauchy singular integrals and then they are extended to $L_p(\mathscr{L})$, $p > 1$.

The composition rules of operators in our algebra $\mathscr{R}$,

$$\mathscr{R} = \{A \in L(X(\mathscr{L}));\; A = a\mathbf{I} + b\mathbf{S} + c\mathbf{N} + T,\; T \in \mathscr{D}\},$$

are given by the properties of operators **N** and **S**, cf. [2], Corollary of Theorem 3. Namely:

$$\mathbf{S}^2 = \mathbf{I},$$

$$\mathbf{N}^2 = -\omega\mathbf{N} + T,$$

$$\mathbf{S}\circ\mathbf{N} = \omega\mathbf{I} + \mathbf{N} - \omega\mathbf{S} + T,$$

$$\mathbf{N}\circ\mathbf{S} = -\mathbf{N} + T,$$

where $\omega \in \mathscr{F}$, $T \in \mathscr{D}$. We have

$$\hat{A} = \begin{pmatrix} a & c\omega & c \\ b \; a & -b\omega + c & -b \\ c & -c\omega & a \end{pmatrix},$$

$$\det \hat{A} = (a+c)^2(a - b\omega - c).$$

So, over the whole curve $\mathscr{L}$ rank $\hat{A} = 2 = \text{const}$. We have

$$\bar{\mathscr{H}} = \{\bar{A} \in \bar{\mathscr{R}};\; a + c = 0 \quad \text{and} \quad a - b\omega - c = 0\},$$

$$(\bar{\mathbf{I}}, \bar{\mathbf{N}}, \bar{\mathbf{S}}) = (\bar{P}_1, \bar{P}_2, \bar{P}_3)\begin{pmatrix} 1 & -\omega & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\bar{\mathscr{H}} = \{d(\omega\mathbf{I} + 2\mathbf{N} - \omega\mathbf{S}) \in \bar{\mathscr{R}}\},$$

$$\bar{\mathbf{S}} \circ \bar{\mathbf{N}} - \bar{\mathbf{N}} \circ \bar{\mathbf{S}} = \omega\mathbf{I} + 2\mathbf{N} - \omega\mathbf{S} \in \bar{\mathscr{H}},$$

where $d, \omega \in \mathscr{F}$. We see that $\hat{\mathscr{R}} = \bar{\mathscr{R}} / \bar{\mathscr{H}}$ is a commutative algebra

2.8. Problem. Does there exist a symbol of the operator $A \in \mathscr{R}$ considered in [3]?

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Matematický ústav SAV
Dislokované pracovisko
Grešákova 6
040 01 Košice