COVERING CONDITION
IN THE LATTICE OF RADICAL CLASSES
OF LINEARLY ORDERED GROUPS

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Existence of covers in lattices of varieties and quasivarieties was studied by A. N. Trachtman [10], V. A. Gorbunov [11] and G. Pollák [9]; for the case of varieties of lattice ordered groups cf. N. Ja. Medvedev [8]. Analogous questions concerning covers of torsion classes and radical classes of lattice ordered groups were studied by J. Jakubík [4], [3].

Radical classes of linearly ordered groups were investigated by C. G. Chehata and R. Wiegandt [1]. J. Jakubík [5] studied radical classes of abelian linearly ordered groups.

All linearly ordered groups dealt with in this note are assumed to be abelian; thus the words linearly ordered group will in the following always mean abelian linearly ordered group.

Let \( \mathcal{R}_a \) be the lattice of all radical classes of linearly ordered groups (for definitions, cf. § 1 below). For each \( X \in \mathcal{R}_a \) we denote by \( a(X) \) the collection of all \( Y \in \mathcal{R}_a \) such that \( Y \) covers \( X \) (i.e., the interval \([X, Y]\) of the lattice \( \mathcal{R}_a \) is prime).

Put

\[ \mathcal{R}_1 = \{ X \in \mathcal{R}_a : X \text{ is principal and } a(X) = \emptyset \}. \]

In [5] it was proved that whenever \( X \) is a principal radical class generated by an archimedean linearly ordered group, then \( X \) belongs to \( \mathcal{R}_1 \). From this it follows that the class \( \mathcal{R}_1 \) is infinite. In this note it will be shown that \( \mathcal{R}_1 \) is a large class (in the sense that there exists an injective mapping of the class of all cardinals into \( \mathcal{R}_1 \)).

1. Preliminaries

For the basic definitions concerning linearly ordered groups cf. Fuchs [2] and Kokorin and Kopytov [6]. The group operation in a linearly ordered group will be written additively. In this paragraph we recall for the sake of completeness some
definitions and results from [5] concerning radical classes of linearly ordered
groups.

Let $\mathcal{G}_a$ be the class of all linearly ordered groups, $G \in \mathcal{G}_a$. Let

$$\{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \quad (\alpha < \delta)$$

be an ascending chain of convex subgroups of $G$. Put

$$H = \cup_{\alpha < \delta} G_\alpha.$$ 

For each $\beta < \delta$ let $G_\beta$ be a linearly ordered group isomorphic to $G_\beta / \cup_{\gamma < \beta} G_\gamma$. Then $H$ is said to be a transfinite extension of linearly ordered groups $G_\beta \ (\beta < \delta)$. Let

$$\{G_i\}_{i \in I}$$

be the set of those $G_\beta$ which are distinct from $\{0\}$; if this set is nonempty, then we also say that $H$ is a transfinite extension of linearly ordered groups $G_i (i \in I)$.

1.1. Definition. A nonempty class $X$ of linearly ordered groups is called a radical
class if

(a) $X$ is closed under homomorphisms, and
(b) $X$ is closed with respect to transfinite extensions.

Let $X$ be a radical class and $G \in \mathcal{G}_a$. Further let $\{H_\alpha\}$ be the set of all convex subgroups of $G$ belonging to $X$. We put

$$X(G) = \cup_{\alpha \in X} H_\alpha.$$

1.2. Proposition. (Cf. [5].) Let $X$ be a radical class and $G \in \mathcal{G}_a$. Then $X(G)$

belongs to $X$.

Let us denote by $\mathcal{R}_a$ the collection of all radical classes of linearly ordered
groups. For $X, Y \in \mathcal{R}_a$ we put $X \leq Y$ if $X$ is a subclass of $Y$. Then $\mathcal{G}_a$ is the greatest element of $\mathcal{R}_a$ and $R_0 = \{\{0\}\}$ is the least element of $\mathcal{R}_a$. Moreover, under the partial order $\leq$, $\mathcal{R}_a$ is a complete lattice.

For each nonempty subclass $Y$ of $\mathcal{G}_a$ let us denote by

$\text{TY}$ — the intersection of all radical classes $R$ with $Y \subseteq R$;

$\text{Hom } Y$ — the class of all homomorphic images of linearly ordered groups

belonging to $Y$;

$\text{Ext } Y$ — the class of all transfinite extensions of linearly ordered groups

belonging to $Y$.

$TY$ is the least radical class containing $Y$ as a subclass; it is said to be the radical
class generated by $Y$. If $Y = \{G\}$ is a one-element set, then we also write $TY = T(G)$; the radical class $T(G)$ is called principal.

1.3. Proposition. (Cf. [5].) Let $Y$ be a nonempty subclass of $\mathcal{G}_a$. Then

$TX = \text{Ext } \text{Hom } X$.

1.4. Proposition. (Cf. [5].) Let $J$ be a nonempty class and for each $j \in J$ let $X_j$ be

a radical class. Then $\cup_{j \in J} X_j = \text{Ext } \cup_{j \in J} X_j$.  

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1.5. **Proposition.** (Cf. [5], 4.6.) Let \( H \) be an archimedean linearly ordered group, \( X = T(H) \). Then \( X \in \mathcal{R}_1 \).

1.6. **Corollary.** The class \( \mathcal{R}_1 \) is infinite.

**Proof.** It is easy to verify that there exists an infinite set \( S = \{ G_i \}_{i \in I} \) of archimedean linearly ordered groups such that whenever \( i, j \) are distinct elements of \( I \), then \( G_i \) fails to be isomorphic to \( G_j \). Put \( X_i = T(G_i) \) for each \( i \in I \). Let \( i, j \in I \), \( i \neq j \) and assume that \( G_i \in T(G_j) \). Hence in view of 1.3 we have \( G_i \in \text{Ext Hom} \{ G_j \} \). Because \( G_i \) is \( o \)-simple, \( G_i \) belongs to \( \text{Hom} \{ G_j \} \). Since \( G_i \) is \( o \)-simple, \( G_i \) is isomorphic either to \( G_j \) or to \( \{ 0 \} \); thus \( G_i = \{ 0 \} \). Therefore for each \( k \in I \) with \( k \neq i \) we have \( G_k \notin T(G_j) \). Hence the set \( \{ T(G_k) \}_{k \in I} \) is infinite. According to 1.5 each \( T(G_k) \) belongs to \( \mathcal{R}_1 \).

### 2. Lexicographic products

We recall the notion of the lexicographic product of linearly ordered groups. Let \( I \) be a linearly ordered set and for each \( i \in I \) let \( G_i \) be a linearly ordered group. The lexicographic product \( H = \Gamma_{i \in I} G_i \) of the system \( \{ G_i \}_{i \in I} \) is defined to be the set of all functions \( f : I \to \bigcup_{i \in I} G_i \) such that (i) \( f(i) \in G_i \) for each \( i \in I \), and (ii) the set \( \{ i \in I : f(i) \neq 0 \} \) is either empty or is dually well-ordered; the group operation in \( H \) is defined coordinate-wise and for \( 0 \neq f \in H \) we put \( f > 0 \) if \( f(j) > 0 \), where \( j \) is the greatest element of the set \( \{ i \in I : f(i) \neq 0 \} \). In the case \( I = \{ 1, 2, \ldots, n \} \) we write also \( H = G_1 \circ G_2 \circ \ldots \circ G_n \).

For each linearly ordered group \( G \) we have \( G = G \circ \{ 0 \} = \{ 0 \} \circ G \). If from \( G, G_1, G_2 \in \mathcal{R}_n \), \( G = G_1 \circ G_2 \) it follows that \( G_1 = \{ 0 \} \) or \( G_2 = \{ 0 \} \), then \( G \) is said to be lexicographically indecomposable.

Let \( H = \Gamma_{i \in I} G_i \). We shall apply the following denotations: For each \( i \in I \) we denote by \( \tilde{G}_i \) the set of all \( h \in H \) such that \( h(j) = 0 \) for each \( j \in I \) with \( j > i \); then \( \tilde{G}_i \) is a convex subgroup of \( H \). For \( K \subset H \) and \( i \in I \), \( K(G_i) \) is defined to be the set \( \{ k(i) : k \in K \} \); if \( K \subset H \), then \( K(G_i) \) is a subgroup of \( G_i \). Further, we identify \( G_i \) with the set \( \{ h \in H : h(j) = 0 \) for each \( j \in I \) with \( j \neq i \} \).

Let \( \tilde{G}_i \) be the set of all elements \( h \in \tilde{G}_i \) with \( h(i) = 0 \); \( \tilde{G}_i \) is a convex subgroup of \( \tilde{G}_i \) and the factor linearly ordered group \( \tilde{G}_i / \tilde{G}_i \) is isomorphic with \( G_i \). Thus if \( I \) is a well-ordered set, then \( H \) is a transfinite extension of linearly ordered groups \( G_i \) \( (i \in I) \).

Let us consider the case when there are given two lexicographic decompositions of \( H \):

\[
H = \Gamma_{i \in I} G_i, \quad H = \Gamma_{j \in J} K_j.
\]

The following lemma is a corollary of the Malcev Theorem ([7]; cf. also [2]) on the existence of isomorphic refinements of two lexicographic decompositions.
2.1. Lemma. Let \( H, G, \) and \( K_j \) be as above. Assume that all \( G_j \) are non-zero and lexicographically indecomposable. Then there is a (unique) partition \( I = \cup_{i \in I^a} I_i \) such that

(i) if \( j_1, j_2 \in J, j_1 < j_2, i_1 \in I_{j_1}, i_2 \in I_{j_2}, \) then \( i_1 < i_2; \)

(ii) for each \( j \in J \) there exists a lexicographic decomposition \( K_j = \Gamma_{i \in I_j} G_i \) such that \( G_i \) is isomorphic with \( G_j \) for each \( i \in I_j. \)

2.2. Lemma. Let \( H, G, \) and \( K_j \) be as in 2.1. Assume that

(i) there exists a partition \( I = I' \cup I'' \) with \( I' \neq \emptyset \) \( \neq I'' \) such that if \( i_1, i_2 \in I', i_3, i_4 \in I'', \) then \( G_{i_1} \) and \( G_{i_2} \) are isomorphic, \( G_{i_3} \) and \( G_{i_4} \) are isomorphic, \( G_{i_1} \) fails to be isomorphic with \( G_{i_2} \) and \( i_1 < i_2; \)

(ii) all \( K_j \) are isomorphic.

Then \( J \) is a one-element set.

Proof. From 2.1 it follows that there are \( j', j'' \in J \) with \( I' \cap I_j \neq \emptyset, I'' \cap I_j \neq \emptyset. \) If there exists \( j \in J \) such that either (a) \( I_j \subset I', \) or (b) \( i_1 \subset I'', \) then we have a contradiction with 2.1 (since \( K_j \) is isomorphic to \( K_{i_1} \) and to \( K_{i_2} \)). Hence for each \( j \in J \) both \( I_j \cap I' \) and \( I_j \cap I'' \) are nonempty. Assume that \( \text{card } J > 1. \) Let \( j_1 \) and \( j_2 \) be distinct elements of \( J \) with \( j_1 < j_2. \) Then there are elements \( i'' \in I_{j_1} \cap I'', i' \in I_{j_2} \cap I'. \)

In view of \( i' \in I_{j_2}, i'' \in I_{j_1} \) we have \( i'' < i'; \) on the other hand, from \( i' \in I', i'' \in I'' \) we obtain \( i' < i'' \), which is a contradiction. Hence \( \text{card } J = 1. \)

3. The class \( \mathcal{R}_1 \)

In the following lemmas 3.1—3.3 we assume that \( G \in \mathcal{G}_a, B = T(G), A \in \mathcal{R}_a \) and that \( A \) covers \( B \) in the lattice \( \mathcal{R}_a. \)

3.1. Lemma. There exists \( H \in A \setminus B \) such that \( B(H) = \{0\} \) and \( B \vee T(H) = A. \)

Proof. Since \( B \) is covered by \( A, \) there is \( H_1 \in A \setminus B. \) Hence (cf. 1.2) \( B(H_1) \neq H_1 \) and \( B(H_1) \in B. \) Denote \( H = H_1 / B(H_1). \) Then \( H \in A. \) From \( H \in B \) it would follow \( H_1 \in B \) (with regard to 1.1, (b)), which is a contradiction; therefore \( H \in A \setminus B. \) In view of 1.4 we have \( H_1 \in T(H) \vee B, \) hence \( B < T(H) \vee B \subseteq A \) and thus \( T(H) \vee B = A. \) Moreover, from 1.2 we easily obtain that \( B(H) = \{0\} \) is valid.

Let \( \alpha \) be an infinite cardinal. We denote by \( \omega(\alpha) \) the least ordinal having the property that the power of the set of all ordinals less than \( \omega(\alpha) \) is \( \alpha. \) For each \( G \in \mathcal{G}_a \) we put

\[ G_a = \Gamma_{i \in I(\omega)} G_i, \]

where \( I(\omega) \) is a linearly ordered set isomorphic with \( \omega(\alpha) \) and \( G_i \) is a linearly ordered group isomorphic with \( G \) for each \( i \in I(\omega). \)

3.2. Lemma. Let \( H \) be as in 3.1 and let \( \alpha > \text{card } H. \) Then \( H_a \in B. \)
Proof. By way of contradiction, suppose that $H_a$ does not belong to $B$. Since $H \in A$ and because $H_a$ is a transfinite extension of $H$, we have $H_a \in A$. From this and from the fact that $B$ is covered by $A$ we infer that $T(H_a) \lor B = A$ is valid. Hence $H \not\in T(H_a) \lor B$. In view of 1.4, $H \in \operatorname{Ext}(\operatorname{Hom}\{H_a\} \lor B)$. If $G_1 \in \operatorname{Hom}\{H_a\}$, $G_1 \neq \{0\}$, then $\operatorname{card}G_1 > \operatorname{card}H$. Hence $H \not\in \operatorname{Ext}B = B$, which is a contradiction.

3.3. Lemma. There exists $G_1 \in \operatorname{Hom}(G)$, $G_1 \neq \{0\}$ such that $G_1$ can be expressed as a lexicographic product $G_1 = \Gamma_{j \in J}H_j$ of factors $H_j$ isomorphic to $H$, where $J$ is a well-ordered set.

Proof. According to 3.2 we have $H_a \in B = T(G)$. Hence in view of 1.3 $H_a \in \operatorname{Ext} \operatorname{Hom}\{G\}$. Thus there exists a convex subgroup $G_1$ of $H_a$ with $G_1 \neq \{0\}$ such that $G_1 \in \operatorname{Hom}(G)$.

From the definition of $H_a$ it follows that $G_1$ can be expressed uniquely as

$$G_1 = P \circ Q$$

such that (i) either $P = \{0\}$ or $P = \Gamma_{j \in J}H_j$, where $J$ is a well-ordered set and each $H_j$ is isomorphic to $H$, and (ii) either $Q = \{0\}$, or $Q$ is isomorphic to a convex subgroup of $H$ and $Q$ is not isomorphic to $H$.

Suppose that $Q \neq \{0\}$. Since $G_1 \in \operatorname{Hom}(G)$, we have $G_1 \in B$ and hence $Q \in B$ (because of $Q \in \operatorname{Hom}(G_1)$). Thus $B(Q) = Q$ and hence $B(H) \neq \{0\}$; in view of 3.1, this is a contradiction. Hence $Q = \{0\}$ and therefore $P \neq \{0\}$, completing the proof.

Now let $K_1$ and $K_2$ be non-zero archimedean linearly ordered groups such that $K_1$ is not isomorphic to $K_2$. Let $\beta$ and $\gamma$ be infinite cardinals. Denote

$$G(\beta, \gamma) = (K_1)_{\beta} \circ (K_2)_{\gamma} K_2.$$  

3.4. Lemma. $T(G(\beta, \gamma))$ has no cover in the lattice $\mathcal{P}_a$.

Proof. Put $G(\beta, \gamma) = G$, $B = T(G)$ and assume that $A$ is a cover of $B$. Let $H$ be as in 3.1 and let $G_1$ be as in 3.3. Since both $K_1$ and $K_2$ are archimedean, $G_1$ can be expressed in the form

$$G_1 = P_1 \circ Q_1$$

such that (i) either $P_1 = \{0\}$ or $P_1 = \Gamma_{i \in I}K_i$, where $I$ is a well-ordered set and each $K_i$ is isomorphic to $K_1$; (ii) $Q_1 = \Gamma_{s \in S}K'_s$, where $S$ is a well-ordered set and each $K'_s$ is isomorphic to $K_2$.

First assume that $P_1 \neq \{0\}$. Then from 3.3 and 2.2 we obtain that $G_1 = H$ implying $H \in B$, which is a contradiction.

Now suppose that $P_1 = \{0\}$. Thus $G_1 = Q_1$. From this and from the Malcev Theorem on the existence of isomorphic refinements (cf. [7] or [2]) it follows that $H$ can be written as $H = \Gamma_{m \in M}K_m$, where $M$ is a well-ordered set and each $K_m$ is
isomorphic to $K_2$. Hence $H \in T(K_2)$. From the definition of $G$ we infer that $K_2 \in T(G)$, hence $H \in T(G)$, which is impossible.

3.5. Lemma. Let $\beta_1, \beta_2$ be distinct infinite cardinals, $\beta > \text{card} K_1$ ($i = 1, 2$). Then $T(G(\beta_1, \gamma)) \neq T(G(\beta_2, \gamma))$.

Proof. Let $\beta_1 < \beta_2$. It suffices to verify that $G(\beta_1, \gamma)$ does not belong to $T(G(\beta_2, \gamma))$. By way of contradiction assume that $G(\beta_1, \gamma) \in T(G(\beta_2, \gamma))$. Thus $G(\beta_1, \gamma) \in \text{Ext Hom}(G(\beta_2, \gamma))$. Hence there exists a convex subgroup $G_1 \neq \{0\}$ of $G(\beta_1, \gamma)$ and a homomorphic image $G_2$ of $G(\beta_2, \gamma)$ such that $G_1$ is isomorphic to $G_2$.

From the structure of $G(\beta_1, \gamma)$ and $G(\beta_2, \gamma)$ it follows that

$$G_1 = P_1 \circ Q_1,$$

such that either (i) $P_1 = (K_1)_{\beta_1}$ and $Q_1$ is a convex subgroup of $(K_2)_{\gamma \circ K_2}$, or (ii) $P_1$ is a convex subgroup of $(K_1)_{\beta_1}$ and $Q_1 = \{0\}$; further $G_2$ is isomorphic to

$$P_2 \circ Q_2$$

such that either (i') $P_2$ is a homomorphic image of $(K_1)_{\beta_2}$ and $Q_2$ is isomorphic to $(K_2)_{\gamma \circ K_2}$, or (ii') $P_2 = \{0\}$ and $Q_2$ is a homomorphic image of $(K_2)_{\gamma \circ K_2}$.

In view of (i) and (ii), the condition (ii') cannot hold. From (i') it follows that the condition (i) must be valid and that $G_1$ is isomorphic to $G(\beta_1, \gamma)$; this implies that $P_1$ is isomorphic to $P_2$. However, this is impossible, because $\text{card} P_1 = \beta_1 < \beta_2 = \text{card} P_2$.

From 3.4, 3.5 and from the fact that $K_2$ belongs to $T(G(\beta, \gamma))$ we obtain:

3.6. Theorem. Let $K \neq \{0\}$ be an archimedean linearly ordered group. There exists $C \subset R_a$ such that:

(i) if $X \in C$, then $X$ is principal and has no cover in $R_a$;

(ii) $T(K) < X$ for each $X \in C$;

(iii) there exists an injective mapping of the class of all cardinals into $C$.

3.7. Corollary. There exists an injective mapping of the class of all cardinals into $R_1$.

We conclude by remarking that for each $X \in C$ there exists $X' \in R_a$ such that $X$ covers $X'$; moreover, $X'$ is the join of all radical classes less than $X$ (the proof is analogous to that of Propos. 4.7, [5] and therefore it will be omitted).
REFERENCES


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ПОКРЫТИЯ В РЕШЕТКЕ РАДИКАЛЬНЫХ КЛАССОВ ЛИНЕЙНО УПОРЯДОЧЕННЫХ ГРУПП

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Резюме

Пусть $R_a$ — решетка всех радикальных классов абелевых линейно упорядоченных групп. Пусть, далее, $R_1$ — класс всех $X \in R_a$ таких, что а) $X$ является главным радикальным классом, и б) $X$ не имеет покрытий в $R_a$. В статье доказано, что $R_1$ — собственный класс.