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Persistent URL: http://dml.cz/dmlcz/130712

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PERIODIC ORBITS OF CERTAIN HÉNON–LIKE MAPS

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(Communicated by Milan Medved*)

ABSTRACT. The existence of periodic orbits for certain two-dimensional Hénon-like maps is shown. For this purpose, critical point theorems are used.

1. Introduction

The purpose of this brief report is to show the existence of periodic orbits of Hénon-like maps of the forms

\[ r_p(x, y) = (b \cdot x + d \cdot y - f(p, x), c \cdot x) \quad (1.1) \]

and

\[ r(x, y) = (b \cdot x + d \cdot y - q(x), c \cdot x), \quad (1.2) \]

where \( b, d, c \) are constant satisfying \( c \cdot d = -1 \).

We assume \( f \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R}), \quad f(\cdot, 0) = 0 \). We shall study the existence of periodic orbits of (1.1) near \( x = 0, y = 0 \) considering \( p \) as a bifurcation parameter. Under additional conditions for \( f \) we show the existence of a closed interval \( I \) such that for \( p \notin I \) the point \( x = 0, y = 0 \) is a hyperbolic fixed point of (1.1). Hence there is no periodic orbit of (1.1) near \((0, 0)\). On the other hand, the set of bifurcation values \( p \) of periodic orbits of (1.1) near \((0, 0)\) is dense in \( I \). Thus for each open neighbourhood \( U \) of \((0, 0)\) it holds: each \( s \in I \) can be approximated by a sequence \( \{p_n\}_{n=3}^\infty \subset I \), \( p_n \to s \), such that the map (1.1) with \( p = p_n \) has an \( n \)-periodic nontrivial orbit in \( U \). (The trivial orbit is the fixed point \((0, 0)\).


Key words: Hénon-like maps, Asymptotically linear maps, Periodic orbits.
We study the map (1.2) globally when \( q \) is asymptotically linear at the infinity. We show the existence of an infinite number of periodic orbits whose minimal periods tend to the infinity.

We see that the orbit \( \{(x_n, y_n)\}_{n=0}^{\infty} \) of (1.1) satisfies

\[
x_{n+2} - bx_{n+1} + x_n + f(p, x_{n+1}) = 0 \quad (1.3)
\]

and similarly for (1.2). Hence we study the difference equation (1.3). Note that there is a relation between (1.3) and the area preserving twist maps (see Angenent [1]). Indeed, let us put

\[
h(x, z) = \frac{-1}{2}(bx - 1/bz)^2 + \int_0^z f(p, s) \, ds - (b - b^2 - 1/b^2)z^2/2,
\]

and following [1, p. 355] we define a map \( F \) in the following way:

\[
F(x, y) = (x_1, y_1) \iff y = \frac{\partial}{\partial x} h(x, x_1), \quad y_1 = -\frac{\partial}{\partial z} h(x, x_1).
\]

Hence

\[
y = -b^2 x + x_1,
\]

\[
y_1 = -x + x_1/b^2 - f(p, x_1) + (b - b^2 - 1/b^2)x_1,
\]

and

\[
x_1 = y + b^2 x,
\]

\[
y_1 = y/b^2 - f(p, y + b^2 x) + (b - b^2 - 1/b^2) \cdot (y + b^2 x).
\]

But the orbit \( \{(x_n, y_n)\}_{n=\infty}^{-\infty} \) of (1.4) satisfies precisely the equation (1.3).

Essentially, our approach to the problem is similar to [1]. We shall define a functional as in [1, p. 354], whose critical points are periodic orbits of (1.3) or of a similar equation corresponding to (1.2). Then we apply theorems of [2] and [5] to prove our results. The author of this paper has recently used the same approach for studying discretizations of higher dimensional variational problems [4]. We note that for \( b = 2 \) the equation (1.3) is the Euler discretization of \( z'' + f(p, z) = 0 \).

2. Local results

We study the existence of periodic orbits of (1.1) near \((0,0)\). We assume \( f \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R}), \ f(\cdot, 0) = 0 \) and \( g(\cdot) = \frac{\partial f}{\partial x}(\cdot, 0) \) satisfies \( g'(\cdot) > 0 \),

\[
\inf_{\mathbb{R}} g < -2 + b < 2 + b < \sup_{\mathbb{R}} g.
\]
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**THEOREM 2.1.** For each \( s \in \langle g^{-1}(b-2), g^{-1}(b+2) \rangle \), \( \delta > 0 \) there exists a sequence \( \{p_n\}_{n=2}^{\infty} \subset \langle g^{-1}(b-2) - \delta, g^{-1}(b+2) + \delta \rangle \) with the properties:

i) \( p_n \to s \) as \( n \to \infty \),

ii) for \( p = p_n \) the map (1.1) has a nontrivial \( n \)-periodic orbit \( \{y_1, \ldots, y_n\} \) such that \( \max_i |y_i| < \delta \).

We see that \( \{x_1, \ldots, x_{n+1}\}, \ x_{n+1} = x_1 \) is the \( n \)-periodic orbit of (1.3) if and only if for \( \tilde{f}(p, z) = (2 - b)z + f(p, z) \) there holds:

\[
\begin{align*}
x_2 - 2x_1 + x_n + \tilde{f}(p, x_1) &= 0, \\
& \quad \vdots \\
1 - 2x_n + x_{n-1} + \tilde{f}(p, x_n) &= 0,
\end{align*}
\]

We put

\[
\begin{align*}
D : \mathbb{R}^n &\to \mathbb{R}^n, \quad D(x_1, \ldots, x_n) = (x_2 + x_n - 2x_1, \ldots, x_1 + x_{n-1} - 2x_n), \\
F(p, \cdot) : \mathbb{R}^n &\to \mathbb{R}^n, \quad F(p, x_1, \ldots, x_n) = (\tilde{f}(p, x_1), \ldots, \tilde{f}(p, x_n)).
\end{align*}
\]

Then the above equation has the form

\[
Dx + F(p, x) = 0, \quad x = (x_1, \ldots, x_n). \tag{2.1}
\]

Note that \( \text{grad}((Dx, x)/2 + \bar{q}(p, x_1) + \cdots + \bar{q}(p, x_n)) = Dx + F(p, x) \), where \( \bar{q}(p, z) = \int_0^z \tilde{f}(p, s) \, ds \).

**LEMMA 2.2.** The spectrum of \( D \) is \( \{-4\sin^2 \frac{\pi}{n} j, \ j = 0, \ldots, n - 1\} \).

**P r o o f.** See [3]. \( \Box \)

**P r o o f of T h e o r e m 2.1.** The linearization of (2.1) at \( x = 0 \) has the form

\[
A(p) = D + (2 - b + g(p)) \cdot \text{Id}.
\]

Hence the matrix \( A(p) \) has eigenvalues

\[
\{-4\sin^2 \frac{\pi}{n} j + 2 - b + g(p), \ j = 0, \ldots, n - 1\}.
\]

If \( 2 - b + g(p) \neq 4\sin^2 \frac{\pi}{n} j \) for each \( j = 0, \ldots, n - 1 \), then \( A(p) \) is invertible and we can define the positive Morse index (see [5, pp. 53]) \( M(p) \) of \( A(p) \). Moreover,
if \( p \) passes through the numbers \( g^{-1}(4\sin^2\frac{\pi}{n}j + b - 2) \), then there is a change of the numbers \( M(p) \). Hence by a result of Chow and Lauterbach \([2]\) the numbers \( g^{-1}(4\sin^2\frac{\pi}{n}j + b - 2) \) are bifurcation values of \( p \) for (2.1).

Finally, we see that the set \( \{ g^{-1}(4\sin^2\frac{\pi}{n}j + b - 2), \ j \in \{0,\ldots, n - 1\}, \ n \in \{2,3\ldots\}\} \), is dense in \( \langle g^{-1}(b-2), g^{-1}(b+2) \rangle \). Note that \( b-2 > \inf g \) and \( \sup g > b+2 \).

It is clear that for \( p \notin \langle g^{-1}(b-2), g^{-1}(b+2) \rangle \) the fixed point \((0,0)\) of (1.1) is hyperbolic, i.e., the eigenvalues of \( Dr_p(0,0) \) lie off the unit circle. For \( p \in \langle g^{-1}(b-2), g^{-1}(b+2) \rangle \) the eigenvalues of \( Dr_p(0,0) \) lie on the unit circle. The following theorem is the consequence of this fact.

**Theorem 2.3.** For \( p \notin \langle g^{-1}(b-2), g^{-1}(b+2) \rangle \) there is a \( \delta > 0 \) such that for each \( s \in (p - \delta, p + \delta) \) the map (1.1) with \( p = s \) has no nontrivial periodic orbits \( \{y_1, \ldots, y_n\} \) satisfying \( \max |y_i| < \delta \).

### 3. A global result

We shall study the map (1.2). For this purpose we need the following result:

**Theorem A.** (see Li and Liu \([5]\)) Let \( \vec{a}: \mathbb{R}^m \rightarrow \mathbb{R} \) be a \( C^2 \)-function satisfying \( |\text{grad} \vec{a}(x) - A_\infty x|/|x| \rightarrow 0 \) as \( |x| \rightarrow \infty \) for a symmetric nonsingular matrix \( A_\infty \in L(\mathbb{R}^m) \). Suppose that \( \vec{a} \) has critical points \( \vec{x}_1, \ldots, \vec{x}_k \) and all of them are nondegenerate. If \( M(\vec{a}''(\vec{x}_i)) \neq M(B) \) for each \( i \), then \( \vec{a} \) has another critical point. Here \( \vec{a}'' \) is the Hessian of \( \vec{a} \), \( M(B) \) is the positive Morse index of the symmetric matrix \( B \).

**Theorem 3.1.** Let us assume:

i) \( \lim_{|x| \rightarrow \infty} q(x)/x = s \),

ii) \( q \) has only a finite number of roots \( x_1, \ldots, x_m \),

i.e., \( q(x_i) = 0 \), \( m \geq 1 \),

iii) \( s \in (b-2, b+2) \), \( q'(x_i) \neq b-2 \), \( s \neq q'(x_i) \),

for \( i = 1, \ldots, m \).

Then the map \( r \) has an infinite number of nontrivial periodic orbits whose minimal periods tend to \( \infty \), i.e., there is a sequence of natural numbers \( \{n_i\}_{i=1}^\infty \), \( n_{i+1} > n_i \), such that \( r \) has a periodic orbit with the minimal period \( n_i \) for any \( i \). (Here the trivial periodic orbits are fixed points of \( r \).)

**Proof.** We take a sequence of prime numbers \( \{p_t\}_{t=1}^\infty \) such that

\[
2 - b + s, \quad 2 - b + q(x_j) \neq 4\sin^2\frac{\pi}{p_t}k, \quad p_t > 2,
\]
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for each natural number $t$ and $j = 1, \ldots, m$, $0 < k < p_t$. Then we solve (2.1) for $n = p_k$, $f(a, x) = q(x)$. We shall apply Theorem A with

$$\tilde{a}(x) = (Dx, x)/2 + \tilde{q}(x_1) + \cdots + \tilde{q}(x_n),$$

$$\tilde{q}(z) = (2 - b)z^2/2 + \int_0^z q(s) \, ds,$$

$$\tilde{x}_i = (x_i, \ldots, x_i),$$

$$A_\infty = D + (2 - b + s) \cdot \text{Id}.$$

In this case we have

$$\tilde{a}''(\tilde{x}_i) = D + (q'(x_i) + 2 - b) \cdot \text{Id},$$

and eigenvalues of $\tilde{a}''(\tilde{x}_i)$ and $A_\infty$ are the following:

$$\left\{-4 \sin^2 \frac{\pi}{p_k} j + 2 - b + q'(x_i), \quad 0 \leq j \leq p_k - 1 \right\}$$

and

$$\left\{-4 \sin^2 \frac{\pi}{p_k} j + 2 - b + s, \quad 0 \leq j \leq p_k - 1 \right\},$$

respectively.

By the choice of $\{p_k\}$ we see that $\tilde{a}''(\tilde{x}_i)$, $A_\infty$ are nonsingular. We can define the positive Morse indexes $M(\tilde{a}''(\tilde{x}_i))$ and $M(A_\infty)$. By [3] we know that

$0, -4 \sin^2 \frac{\pi}{p_k} j, \quad 0 < j \leq (p_k - 1)/2$

have the geometric multiplicities 1, 2 in $D$, respectively. Hence

$$M(\tilde{a}''(\tilde{x}_i)) = 2\# \{0 < j \leq (p_k - 1)/2, \quad -4 \sin^2 \frac{\pi}{p_k} j + 2 - b + q'(x_i) > 0 \} + 1,$$

$$M(A_\infty) = 2\# \{0 < j \leq (p_k - 1)/2, \quad -4 \sin^2 \frac{\pi}{p_k} j + 2 - b + s > 0 \} + 1,$$

$(\# \text{ means the cardinality})$.

Using the assumption iii) we see that

$$M(\tilde{a}''(\tilde{x}_i)) \neq M(A_\infty), \quad i = 1, \ldots, m,$$

for $p_k$ large. Hence Theorem A implies the existence of a critical point, i.e. a solution of (2.1) for our case $f(\cdot, x) = q(x)$, $n = p_k$, different from $x_i$, $i = 1, \ldots, m$. This gives a $p_k$-periodic nontrivial orbit of $r$, for $k$ large. Since $\{p_t\}_{t=1}^\infty$ is a sequence of prime numbers, we can conclude the proof.

□
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Received November 7, 1991