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Dedicated to Academician Štefan Schwarz on the occasion of his 80th birthday

THE TRANSLATIONAL HULL OF A NORMAL CRYPTOGROUP

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ABSTRACT. Given a normal cryptogroup S, we write it as $[Y; S_\alpha, \chi_{\alpha,\beta}]$, that is as a strong semilattice of completely simple semigroups. We construct the translational hull $\Omega(S)$ of S when S is given in this form and specialize this construction when all $\chi_{\alpha,\beta}$ are injective, that is when S is a subdirect product of a semilattice and a completely simple semigroup. We consider threads in S which, under the multiplication of complexes, provide an isomorphic copy of a remarkable ideal $\Omega_i(S)$ of $\Omega(S)$. We also consider some other ideals of $\Omega(S)$. These results are then used to establish several properties of the semigroup $\Omega_i(S)$ including its position within $\Omega(S)$.

1. Introduction and summary

For any semigroup S, the translational hull $\Omega(S)$ of S consists of all bitranslations $\omega = (\lambda, \rho)$, where λ is a left translation of S, ρ is a right translation of S, and they are linked in the sense that $(a\rho)b = a(\lambda b)$ for all $a, b \in S$. We write λ on the left and ρ on the right. The product of two bitranslations is by components, where left (respectively right) translations are composed as left (respectively right) operators. The semigroup $\Omega(S)$ plays an essential role in the study of ideal extensions. For an extensive discussion, see [1].

If we restrict the above semigroup S to belong to a class of semigroups for which there is a sufficiently explicit structure theorem, we may be able to use the ingredients in that theorem for constructing $\Omega(S)$ in a relatively transparent way. Let S be a completely regular semigroup in which $\mathcal{H}$ is a congruence and $S/\mathcal{H}$ is a normal band; in short we refer to S as to a normal cryptogroup

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(or normal band of groups). These are precisely semigroups which can be expressed as strong semilattices of completely simple semigroups, and we may set $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$. This is a special case of a semilattice of weakly reductive semigroups whose translational hull was studied in [2]. When each S_α is a group, a construction of $\Omega(S)$ can be found in [1; Section V.6].

We concentrate here on the translational hull of a normal cryptogroup written in the form $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$, that is as a strong semilattice of completely simple semigroups. The main novelty here is the introduction of certain ideals of $\Omega(S)$ which exhibit remarkable properties. For we may ask for that part of the translational hull of S which plays the same role for ideal extensions of S which are normal cryptogroups as $\Omega(S)$ for arbitrary ideal extensions of S. We succeed in finding such an object, $\Omega_i(S)$, and for it provide an alternative construction.

Section 2 contains a brief compendium of concepts and notation used throughout the paper. In Section 3, we extract a construction of the translational hull of a strong semilattice of weakly reductive semigroups from two results in [2]. In addition, we specialize that construction to regular semigroups which form a subdirect product of a semilattice and a completely simple semigroup. Threads in a strong semilattice of (arbitrary) semigroups are introduced in Section 4 and are applied to our situation giving some new insights into the nature of the ideal $\Omega_i(S)$ of $\Omega(S)$. Some of these results are used in Section 5 to establish several interesting properties of $\Omega_i(S)$ including statements concerning its position within $\Omega(S)$.

2. Terminology and notation

For any set X, ι_X denotes the identity map on X.

Let Y be a semilattice. For every $\alpha \in Y$ let S_α be a semigroup and assume that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For any $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, let $\chi_{\alpha,\beta} \colon S_\alpha \to S_\beta$ be a homomorphism satisfying

$$\chi_{\alpha,\alpha} = \iota_{S_\alpha}, \qquad \chi_{\alpha,\beta}\chi_{\beta,\gamma} = \chi_{\alpha,\gamma} \quad \text{if} \quad \alpha \geq \beta \geq \gamma .$$

On the set $S = \bigcup_{\alpha \in Y} S_\alpha$, define a product by

$$a * b = (a\chi_{\alpha,\alpha\beta})(b\chi_{\beta,\alpha\beta}) \qquad \text{if} \quad a \in S_\alpha , \quad b \in S_\beta .$$

Then S is a semigroup called a *strong semilattice* Y of semigroups S_α with structure homomorphisms $\chi_{\alpha,\beta}$. We denote S by $[Y; S_\alpha, \chi_{\alpha,\beta}]$ and its product by juxtaposition.

If $\alpha \in Y$, (α) denotes the principal ideal of Y generated by α. An ideal I of Y is a *retract ideal* if $(\alpha) \cap I$ is a principal ideal of Y for every $\alpha \in Y$. We shall use the notation:

$\mathcal{I}_Y$ – ideals of Y, $\mathcal{R}_Y$ – retract ideals of Y, $\mathcal{P}_Y$ – principal ideals of Y

under the operation of set theoretical intersection.

Now let S be an arbitrary semigroup. A mapping λ (respectively ρ) written on the left (respectively right) is a *left* (respectively *right*) *translation* of S if $\lambda(xy) = (\lambda x)y$ (respectively $(xy)\rho = x(y\rho)$) for all $x, y \in S$. If also λ and ρ are *linked* in the sense that $(x\rho)y = x(\lambda y)$ for all $x, y \in S$, then $\omega = (\lambda, \rho)$ is a *bitranslation* of S. We shall consider ω as a bioperator on S with $\omega x = \lambda x$ and $x\omega = x\rho$ for all $x \in S$. The set $\Omega(S)$ of all bitranslations of S with componentwise composition is the *translational hull* of S (evidently $\Omega(S)$ is a semigroup).

For every $a \in S$, define λ_a, ρ_a and π_a by

$$\lambda_a x = ax, \quad x\rho_a = xa \qquad (x \in S),$$
$$\pi_a = (\lambda_a, \rho_a).$$

Then π_a is an *inner bitranslation* of S and the set $\Pi(S)$ of all π_a with $a \in S$ is the *inner part* of $\Omega(S)$.

We shall be concerned with the translational hull $\Omega(S)$ of a semigroup S which is a strong semilattice Y of semigroups S_α. These semigroups S_α will sometimes satisfy certain familiar conditions: weak reductivity, weak cancellation or complete simplicity. In order to simplify the notation, we shall write Ω and Π instead of $\Omega(S)$ and $\Pi(S)$, but for all other semigroups T, S_α, etc. we shall write the full notation $\Omega(T)$, $\Omega(S_\alpha)$, $\Pi(T)$, $\Pi(S_\alpha)$, etc.

For any subsemigroup T of a semigroup S,

$$i_S(T) = \{s \in S \mid st, ts \in T \text{ for all } t \in T\}$$

is the *idealizer* of T in S (the greatest subsemigroup of S having T as an ideal). We denote by S^0 the semigroup S with a zero adjoined.

Let S be a completely regular semigroup, that is a semigroup which is the union of its (maximal) subgroups. For $a \in S$ denote by a^{-1} the inverse of a in the $\mathcal{H}$-class of a, and let $a^0 = aa^{-1}$. Completely regular semigroups under multiplication and this inversion form a variety denoted by $\mathcal{CR}$. The lattice of all subvarieties of $\mathcal{CR}$ is denoted by $\mathcal{L}(\mathcal{CR})$; $\langle S \rangle$ is the variety generated by S. The member of $\mathcal{L}(\mathcal{CR})$ consisting of all semilattices is denoted by $\mathcal{S}$. A completely regular semigroup S in which $\mathcal{H}$ is a congruence and $S/\mathcal{H}$ is a normal band (that is satisfies the identity $axya = ayxa$) is a *normal cryptogroup*; the variety of all such is denoted by $\mathcal{NBG}$.

All the undefined concepts and notation can be found in [1].

3. The translational hull of two special semigroups

We are generally interested in a construction of the translational hull of a normal cryptogroup S in terms of the translational hull of the underlying semilattice Y and the translational hulls of the completely simple components S_α of S. To this end, we represent S as $[Y; S_\alpha, \chi_{\alpha,\beta}]$ – the strong semilattice of completely simple semigroups, and note that the translational hull of Y can be represented by retract ideals of Y (see [1; Lemma V.6.1]). Moreover, we have considered in [2] the more general situation of the translational hull of a semilattice of weakly reductive semigroups. From the results of that paper we extract an explicit construction of the translational hull of a strong semilattice of weakly reductive semigroups in the first result of this section. We then specialize sharply to an even more explicit construction of the translational hull of a regular semigroup which is a subdirect product of a semilattice and a completely simple semigroup.

CONSTRUCTION 3.1. Let $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$, where S_α is an arbitrary semigroup for every $\alpha \in Y$. Let Γ be the set of all $(I; \omega_\alpha)$ [1], where $I \in \mathcal{I}_Y$ and $\omega_\alpha \in \Omega(S_\alpha)$ for each $\alpha \in I$, satisfying the following condition: for every $\alpha \in Y$, write $\omega_\alpha = (\lambda_\alpha, \rho_\alpha)$, and for any $\alpha > \beta$ in Y and $\theta \in \{\lambda, \rho\}$, the diagram

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\theta_\alpha} & S_\alpha \\ {\scriptstyle \chi_{\alpha,\beta}}\downarrow & & \downarrow{\scriptstyle \chi_{\alpha,\beta}} \\ S_\beta & \xrightarrow{\theta_\beta} & S_\beta \end{array}$$

commutes. Define a product in Γ by

$$(I; \omega_\alpha)(I'; \omega'_\alpha) = (I \cap I'; \omega_\alpha \omega'_\alpha).$$

Simple verification shows that Γ is a semigroup.

We now single out the following subsets of Γ:

$$\begin{aligned} \Gamma_\Omega &= \{(I; \omega_\alpha) \in \Gamma \mid I \in \mathcal{R}_Y\}, \\ \Gamma_p &= \{(I; \omega_\alpha) \in \Gamma \mid I \in \mathcal{P}_Y\}, \\ \Gamma_i &= \{(I; \omega_\alpha) \in \Gamma \mid I \in \mathcal{R}_Y,\ \omega_\alpha \in \Pi(S_\alpha) \text{ for all } \alpha \in I\}, \\ \Gamma_\Pi &= \Gamma_p \cap \Gamma_i. \end{aligned}$$

[1] We shall use the simplify notation $(I; \omega_\alpha)$ instead of $(I; (\omega_\alpha)_{\alpha \in I})$, which is more precise.

Since $\mathcal{R}_Y$ is a subsemigroup of $\mathcal{I}_Y$, we have that Γ_Ω is a subsemigroup of Γ. Also, since $\mathcal{P}_Y$ is an ideal of $\mathcal{R}_Y$, we get that Γ_p is an ideal of Γ_Ω. Finally, since $\Pi(S_\alpha)$ is an ideal of $\Omega(S_\alpha)$ for every $\alpha \in I$, Γ_i is an ideal of Γ_Ω. The relationship between Γ_Π and Γ_Ω is the content of the next lemma.

LEMMA 3.2. $\Gamma_\Omega = i_\Gamma(\Gamma_\Pi)$.

P r o o f. We remarked above that both Γ_p and Γ_i are ideals of Γ_Ω, and hence Γ_Π is an ideal of Γ_Ω. Let $(I;\omega_\alpha) \in i_\Gamma(\Gamma_\Pi)$. Let $\beta \in Y$ and $a \in S_\beta$. Then $\big((\beta);\pi_{a\chi_{\beta,\alpha}}\big) \in \Gamma_\Pi$ and by hypothesis $\big(I \cap (\beta), \omega_\alpha \pi_{a\chi_{\beta,\alpha}}\big) \in \Gamma_\Pi$. Hence $I \cap (\beta) \in \mathcal{P}_Y$, and since $\beta \in Y$ is arbitrary, we obtain that $I \in \mathcal{R}_Y$. But then $(I;\omega_\alpha) \in \Gamma_\Omega$, as required.

With the notation in Construction 3.1, we now extract from [2; Theorems 1 and 2, and their proofs] the following result.

Recall that a semigroup S is *weakly cancellative* if $ax = bx$, $ya = yb$ in S implies $a = b$. We have used in the proof of Theorem 3.5 that completely simple semigroups are weakly cancellative.

THEOREM 3.3. *Let $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$, where S_α is a weakly reductive semigroup for every $\alpha \in Y$. For any $\omega \in \Omega$, let $I_\omega = \{\alpha \in Y \mid \omega S \cap S_\alpha \neq \emptyset\}$, and define a mapping φ by*

$$\varphi\colon \omega \to \big(I_\omega; \omega|_{S_\alpha}\big) \qquad (\omega \in \Omega(S)).$$

Let $(I;\omega_\alpha) \in \Gamma_\Omega$, and for any $\alpha \in Y$ write $(\alpha) \cap I = (\overline{\alpha})$. Define a bioperator ω by

$$\omega a = \omega_{\overline{\alpha}}(a\chi_{\alpha,\overline{\alpha}}), \quad a\omega = (a\chi_{\alpha,\overline{\alpha}})\omega_{\overline{\alpha}} \qquad (a \in S_\alpha,\ \alpha \in Y).$$

With this notation define a mapping ψ by

$$\psi\colon (I;\omega_\alpha) \to \omega \qquad ((I;\omega_\alpha) \in \Gamma_\Omega).$$

Then the mappings φ and ψ are mutually inverse isomorphisms between Ω and Γ_Ω. In this association, Π corresponds to Γ_Π.

P r o o f. The derivation from the reference cited above is left to the interested reader.

Theorem 3.3 makes it possible to single out two remarkable ideals of Ω:

$$\Omega_p = \Gamma_p\psi, \qquad \Omega_i = \Gamma_i\psi.$$

The last assertion of Theorem 3.3 yields $\Omega_p \cap \Omega_i = \Pi$. Note that Ω_p consists of those bitranslations which induce a principal ideal on Y, whereas Ω_i consists of the bitranslations which restricted to S_α for each $\alpha \in I_\omega$ is an inner bitranslation of S_α.

We shall now consider a special case which will help illustrate the general situation as treated above. It is the case of a sturdy composition of completely simple semigroups, that is $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$ as above with S_α a completely simple semigroup for every $\alpha \in Y$, and $\chi_{\alpha,\beta}$ is injective for all choices $\alpha \geq \beta$. According to [1; Theorem IV.51], sturdy compositions of completely simple semigroups coincide with regular semigroups which are subdirect products of a semilattice and a completely simple semigroup. The latter, by [1; Corollary IV.5.3], are precisely those given by the following device.

CONSTRUCTION 3.4. Let Y be a semilattice, T be a completely simple semigroup, $\mathcal{R}(T)$ be the partially ordered set of all regular subsemigroups of T ordered by inclusion, $\eta\colon Y \to \mathcal{R}(T)$ be an order inverting function for which $\bigcup\limits_{\alpha\in Y} \alpha\eta = T$, and set

$$S = \{(\alpha, a) \in Y \times T \mid a \in \alpha\eta\}.$$

Denote S by (Y, η, T).

Let Δ be the set of all (I, θ), where $I \in \mathcal{R}_Y$, and $\theta \in \Omega(T)$ satisfying the following condition: for every $\alpha \in Y$, let $(\alpha) \cap I = (\overline{\alpha})$, then $t \in \alpha\eta$ implies $\theta t, t\theta \in \overline{\alpha}\eta$. Define a product in Δ by the formula

$$(I, \omega)(I', \omega') = (I \cap I', \omega\omega').$$

It follows easily that Δ is a semigroup. Also let

$$\Delta_\Omega = \{(I, \theta) \in \Delta \mid I \in \mathcal{R}_Y\},$$
$$\Delta_p = \{(I, \theta) \in \Delta \mid I \in \mathcal{P}_Y\},$$
$$\Delta_i = \{(I, \theta) \in \Delta \mid I \in \mathcal{R}_Y,\ \theta|_{\alpha\eta} \in \Pi(\alpha\eta) \text{ for all } \alpha \in Y\},$$
$$\Delta_\Pi = \Delta_p \cap \Delta_i.$$

Analogous statements to those after Construction 3.1 are valid for the sets Δ_q for $q \in \{\Omega, p, i, \Pi\}$. Below blank spaces indicate that the omitted entry is of no importance for our purposes.

THEOREM 3.5. *Let $S = (Y, \eta, T)$ be as above. For $\omega \in \Omega$ define*

$$I = \{\alpha \in Y \mid \omega(\beta, t) = (\alpha, \ ) \text{ for some } (\beta, t) \in S\}$$

and a bioperator θ on T by the formulae

$$\omega(\alpha, t) = (\ , \theta t), \qquad (\alpha, t)\omega = (\ , t\theta)$$

for some $(\alpha, t) \in S$. With this notation define a mapping φ by

$$\varphi \colon \omega \to (I, \theta) \qquad (\omega \in \Omega).$$

For $(I, \theta) \in \Delta_\Omega$ and $\alpha \in Y$, let $(\alpha) \cap I = (\overline{\alpha})$, and define a bioperator ω by

$$\omega(\alpha, t) = (\overline{\alpha}, \theta t), \quad (\alpha, t)\omega = (\overline{\alpha}, t\theta) \qquad ((\alpha, t) \in S).$$

With this notation define a mapping ψ by

$$\psi \colon (I, \theta) \to \omega \qquad ((I, \theta) \in \Delta_\Omega).$$

Then φ and ψ are mutually inverse isomorphisms between Ω and Δ_Ω. In this association, we have the correspondence

$$\Omega_p \longleftrightarrow \Delta_p, \qquad \Omega_i \longleftrightarrow \Delta_i, \qquad \Pi \longleftrightarrow \Delta_\Pi.$$

Proof.

1. *φ is single valued.* In order to make use of Theorem 3.3, we now convert the given notation to that of the cited theorem by introducing the following symbolism. For every $\alpha \in Y$ let

$$S_\alpha = \{(\alpha, t) \mid t \in \alpha\eta\},$$

and for $\alpha \geq \beta$ in Y

$$\chi_{\alpha,\beta} \colon (\alpha, t) \to (\beta, t) \qquad (t \in \alpha\eta).$$

Simple verification shows that these define a strong semilattice and $[Y; S_\alpha, \chi_{\alpha,\beta}] = S$.

Let $\omega \in \Omega$. It is easy to see that I defined in the statement of the theorem coincides with I_ω. Let $t \in T$. By the condition on η, there exists $\alpha \in Y$ such that $(\alpha, t) \in S$. In order to prove that φ is single valued, we suppose that also

$(\beta, t) \in S$. Letting $(\gamma) \cap I = (\overline{\gamma})$ for all $\gamma \in Y$, by Theorem 3.3, we have that $\omega(\alpha, t) = (\overline{\alpha}, a)$ and $\omega(\beta, t) = (\overline{\beta}, b)$ for some $a, b \in T$. Then

$$\begin{aligned}
[\omega(\alpha, t)](\beta, t) &= (\overline{\alpha}, a)(\beta, t) = (\overline{\alpha}\beta, at), \\
[\omega(\beta, t)](\alpha, t) &= (\overline{\beta}, b)(\alpha, t) = (\overline{\beta}\alpha, bt), \\
(\alpha, t)(\beta, t) &= (\alpha\beta, t^2) = (\beta, t)(\alpha, t),
\end{aligned}$$

so that $at = bt$. We also have $(\alpha, t)\omega = (\overline{\alpha}, a')$ and $(\beta, t)\omega = (\overline{\beta}, b')$ for some $a', b' \in T$. By an argument similar to the one above, we obtain $ta' = tb'$. Further,

$$\begin{aligned}
[(\alpha, t)\omega](\beta, t) &= (\overline{\alpha}, a')(\beta, t) = (\overline{\alpha}\beta, a't), \\
(\alpha, t)[\omega(\beta, t)] &= (\alpha, t)(\overline{\beta}, b) = (\alpha\overline{\beta}, tb),
\end{aligned}$$

and thus $a't = tb$. We analogously get $b't = ta$. Hence

$$t^2 a = t(ta) = t(b't) = (tb')t = (ta')t = t(a't) = t(tb) = t^2 b,$$

which together with $at = bt$ in the completely simple semigroup T implies that $a = b$. Therefore φ is single valued.

2. *φ maps Ω into Δ_Ω.* We have noted above that $I = I_\omega$, and hence, by Theorem 3.3, we have $I \in \mathcal{R}_Y$. Simple verification shows that $\theta \in \Omega(T)$. Let $\alpha \in Y$ and $t \in \alpha\eta$. Then $(\alpha, t) \in S$ and $\omega(\alpha, t) = (\overline{\alpha}, \theta t) \in S$, so that $\theta t \in \overline{\alpha}\eta$; analogously $t\theta \in \overline{\alpha}\eta$. Therefore $(I, \theta) \in \Delta_\Omega$.

3. *ψ maps Δ_Ω into Ω.* For $(\alpha, t) \in S$, we have $\omega(\alpha, t) = (\overline{\alpha}, \theta t) \in S$ and $(\alpha, t)\omega = (\overline{\alpha}, t\theta) \in S$ by the hypothesis that $(I, \theta) \in \Delta_\Omega$. Now for $(\alpha, s), (\beta, t) \in S$, we obtain

$$\begin{aligned}
[\omega(\alpha, s)](\beta, t) &= (\overline{\alpha}, \theta s)(\beta, t) = (\overline{\alpha}\beta, (\theta s)t) = (\overline{\alpha\beta}, \theta(st)) \\
&= \omega(\alpha\beta, st) = \omega[(\alpha, s)(\beta, t)],
\end{aligned}$$

similarly $(\alpha, s)[(\beta, t)\omega] = [(\alpha, s)(\beta, t)]\omega$, and

$$\begin{aligned}
[(\alpha, s)\omega](\beta, t) &= (\overline{\alpha}, s\theta)(\beta, t) = (\overline{\alpha}\beta, (s\theta)t) = (\alpha\overline{\beta}, s(\theta t)) \\
&= (\alpha, s)(\overline{\beta}, \theta t) = (\alpha, s)[\omega(\beta, t)].
\end{aligned}$$

Therefore $\omega \in \Omega$.

4. ψ *is a homomorphism.* Let $(I,\theta),(I',\theta')\in\Delta_\Omega$ and $(I,\theta)\psi=\omega$, $(I',\theta')\psi=\omega'$. For any $\alpha\in Y$ let $(\alpha)\cap I=(\overline{\alpha})$ and $(\alpha)\cap I'=(\widehat{\alpha})$. Hence $(\alpha)\cap(I\cap I')=(\widehat{\overline{\alpha}})$ with $\widehat{\overline{\alpha}}=\overline{\widehat{\alpha}}$. Now, for any $(\alpha,t)\in S$ we obtain

$$\omega\big(\omega'(\alpha,t)\big)=\omega(\widehat{\alpha},\theta't)=\big(\overline{\widehat{\alpha}},\theta\theta't\big)=(\omega\omega')(\alpha,t),$$

$$\big((\alpha,t)\omega\big)\omega'=(\overline{\alpha},t\theta)\omega'=\big(\widehat{\overline{\alpha}},t\theta\theta'\big)=(\alpha,t)(\omega\omega'),$$

which implies that ψ is a homomorphism.

5. $\varphi\psi=\iota_\Omega$. Indeed, for $\omega\in\Omega$ we have $\omega\varphi\psi=(I,\theta)\psi=\omega'$, where, with $(\alpha)\cap I=(\overline{\alpha})$, for $(\alpha,t)\in S$ we get $\omega'(\alpha,t)=(\overline{\alpha},\theta t)=\omega(\alpha,t)$, and similarly $(\alpha,t)\omega'=(\alpha,t)\omega$. Therefore $\omega=\omega'$, and thus $\varphi\psi=\iota_\Omega$.

6. $\psi\varphi=\iota_{\Delta_\Omega}$. Indeed, for $(I,\theta)\in\Delta_\Omega$ we have $(I,\theta)\psi\varphi=\omega\varphi=(I',\theta')$, whence we easily obtain that $I=I'$ and $\theta=\theta'$. Therefore $\psi\varphi=\iota_{\Delta_\Omega}$.

7. We now deduce that φ and ψ are mutually inverse isomorphisms between $\mathcal{I}$ and Δ_Ω.

8. The claim concerning the correspondence of the three sets follows without difficulty.

COROLLARY 3.6. *Let S be a regular semigroup which is a subdirect product of a semilattice Y and a completely simple semigroup T. Then $\Omega(S)$ is isomorphic to a subsemigroup of $\mathcal{R}_Y\times\Omega(T)$ whose projection into $\mathcal{R}_Y$ is surjective.*

Proof. In view of Theorem 3.5, it remains to prove only the last assertion of the corollary. Indeed, for any $I\in\mathcal{R}_Y$ the pair (I,ι_T) satisfies the condition for membership in Δ_Ω.

4. Threads

For a semigroup $S=[Y;S_\alpha,\chi_{\alpha,\beta}]$ given as a strong semilattice we shall construct a semigroup, based on this decomposition of S, which has some remarkable properties. Then we shall establish its relationship with the translational hull of S.

DEFINITION 4.1. *Let $S=[Y;S_\alpha,\chi_{\alpha,\beta}]$, where S_α is an arbitrary semigroup for every $\alpha\in Y$. A thread T in S is a nonempty subset of S satisfying*

(i) $a\in S_\alpha\cap T$, $\alpha\geq\beta\implies a\chi_{\alpha,\beta}\in T$,

(ii) $|T\cap S_\alpha|\leq 1$ *for every* $\alpha\in Y$.

Denote by $\mathcal{T}$ the set of all threads in S with the multiplication of complexes. For each $T\in\mathcal{T}$, let

$$\overline{T}=\{\alpha\in Y\mid T\cap S_\alpha\neq\emptyset\},$$

$$\mathcal{T}_\mathcal{R}=\{T\in\mathcal{T}\mid \overline{T}\in\mathcal{R}_Y\},\qquad \mathcal{T}_\mathcal{P}=\{T\in\mathcal{T}\mid \overline{T}\in\mathcal{P}_Y\}.$$

Note that for $T \in \mathcal{T}$, $\overline{T}$ is an ideal of Y in view of condition (i) in its definition. That $\mathcal{T}$ is closed under its multiplication will follow from the next result.

LEMMA 4.2. *For $T = (t_\alpha)_{\alpha\in I}$ and $K = (k_\beta)_{\beta\in J}$ in $\mathcal{T}$ we have $TK = (t_\gamma k_\gamma)_{\gamma\in I\cap J}$. For every $I \in \mathcal{I}_Y$ let*

$$\widehat{I} = \{T \in \mathcal{T} \mid \overline{T} = I\}, \qquad \widehat{\mathcal{I}}_Y = \{I \in \mathcal{I}_Y \mid \widehat{I} \neq \emptyset\}.$$

If $\widehat{I} \in \widehat{\mathcal{I}}_Y$, then $\widehat{I}$ is a subsemigroup of $\prod\limits_{\alpha\in I} S_\alpha$. For $I, J \in \widehat{\mathcal{I}}_Y$ such that $I \supseteq J$, define a function $\Psi_{I,J}$ by

$$\Psi_{I,J}\colon (t_\alpha)_{\alpha\in I} \to (t_\alpha)_{\alpha\in J} \qquad ((t_\alpha)_{\alpha\in I} \in \widehat{I}).$$

We can construct a strong semilattice of semigroups $Q = [\widehat{\mathcal{I}}_Y; \widehat{I}, \Psi_{I,J}]$. Then $\mathcal{T}$ is a semigroup which coincides with Q.

P r o o f. Let T and K be as in the statement of the lemma. Trivially, $(t_\gamma k_\gamma)_{\gamma\in I\cap J} \subseteq TK$. Conversely, let $t_\alpha \in T \cap S_\alpha$ and $k_\beta \in K \cap S_\beta$. Then $t_\alpha k_\beta = (t\chi_{\alpha,\alpha\beta})(k_\beta\chi_{\beta,\alpha\beta})$, where $t\chi_{\alpha,\alpha\beta} \in T \cap S_{\alpha\beta}$ and $k_\beta\chi_{\beta,\alpha\beta} \in K \cap S_{\alpha\beta}$ with $\alpha\beta \in I \cap J$. Therefore $t_\alpha k_\beta \in (t_\gamma k_\gamma)_{\gamma\in I\cap J}$. This establishes the first assertion of the lemma.

It now follows that if $\widehat{I} \in \widehat{\mathcal{I}}_Y$, then $\widehat{I} \subseteq \prod\limits_{\alpha\in I} S_\alpha$, and that $\widehat{\mathcal{I}}_Y$ is a subsemilattice of $\mathcal{I}_Y$. Obviously, $\mathcal{T}$ and Q coincide as sets, and with the above notation,

$$TK = (t_\gamma k_\gamma)_{\gamma\in I\cap J} = (t_\gamma)_{\gamma\in I\cap J}(k_\gamma)_{\gamma\in I\cap J} = (T\Psi_{I,I\cap J})(K\Psi_{J,I\cap J}),$$

so that their multiplications agree.

LEMMA 4.3. *The mapping*

$$\tau\colon a \to (a\chi_{\alpha,\beta})_{\beta\le\alpha} \qquad (a \in S_\alpha, \ \ \alpha \in Y)$$

is an isomorphism of S onto $\mathcal{T}_\mathcal{P}$.

P r o o f. If $a \in S_\alpha$, then clearly $a\tau$ is a thread in S and $\overline{a\tau} = (\alpha)$. Hence τ maps S into $\mathcal{T}_\mathcal{P}$. Now let $a \in S_\alpha$ and $b \in S_\beta$. Then

$$\begin{aligned}(a\tau)(b\tau) &= (a\chi_{\alpha,\gamma})_{\gamma\le\alpha}(b\chi_{\beta,\delta})_{\delta\le\beta}\\ &= \{(a\chi_{\alpha,\gamma})(b\chi_{\beta,\gamma}) \mid \gamma \le \alpha\beta\}\\ &= ((ab)\chi_{\alpha\beta,\gamma})_{\gamma\le\alpha\beta} = (ab)\tau,\end{aligned}$$

and τ is a homomorphism. If $a\tau = b\tau$, then $\alpha = \beta$, and hence $a = b$, so τ is injective. If $(a_\beta)_{\beta\le\alpha} \in \mathcal{T}_\mathcal{P}$, then $a_\alpha\tau = (a_\beta)_{\beta\le\alpha}$, so that τ is also surjective.

LEMMA 4.4. $\mathcal{T}_{\mathcal{R}} = i_{\mathcal{T}}(\mathcal{T}_{\mathcal{P}})$.

P r o o f. The argument here is almost identical to that in the proof of Lemma 3.2 and is omitted.

LEMMA 4.5. *Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$, where S_α is an arbitrary semigroup for every $\alpha \in Y$. Then S is a subdirect product of semigroups S_α with a zero possibly adjoined.*

P r o o f. Define a mapping χ by

$$\chi\colon a \to (a_\alpha) \qquad (a \in S),$$

where for $a \in S_\alpha$

$$a_\gamma = \begin{cases} a\varphi_{\alpha,\gamma} & \text{if } \alpha \geq \gamma, \\ 0 & \text{otherwise}. \end{cases}$$

Then χ maps S into $\prod_{\alpha \in Y} T_\alpha$, where $T_\alpha = S_\alpha$ if α is the zero of Y, and $T_\alpha = S_\alpha^0$ otherwise. For $a \in S_\alpha$ and $b \in S_\beta$ we have $(a\chi)(b\chi) = (a_\gamma)(b_\gamma)$, where

$$\begin{aligned} a_\gamma b_\gamma &= \begin{Bmatrix} a\varphi_{\alpha,\gamma} & \text{if } \alpha \geq \gamma \\ 0 & \text{otherwise} \end{Bmatrix} \begin{Bmatrix} b\varphi_{\beta,\gamma} & \text{if } \beta \geq \gamma \\ 0 & \text{otherwise} \end{Bmatrix} \\ &= \begin{cases} (a\varphi_{\alpha,\gamma})(b\varphi_{\beta,\gamma}) & \text{if } \alpha\beta \geq \gamma, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} (ab)\varphi_{\alpha\beta,\gamma} & \text{if } \alpha\beta \geq \gamma, \\ 0 & \text{otherwise}, \end{cases} \end{aligned}$$

so that $(a\chi)(b\chi) = (ab)\chi$. Therefore χ is a homomorphism, and it is easy to see that it is injective and that the image of S under χ is a subdirect product of semigroups T_α.

LEMMA 4.6. *Let $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ and $S \in \mathcal{V}$. Let T be the semigroup S with a zero adjoined. Then $T \in \mathcal{V} \vee \mathcal{S}$.*

P r o o f. Let $Y_2 = \{0, 1\}$ be a two-element semilattice. Then

$$T \cong (S \times Y_2)/\{(s, 0) \mid s \in S\},$$

and hence $T \in \mathcal{V} \vee \mathcal{S}$.

PROPOSITION 4.7. *Let S be a normal cryptogroup. Then $\mathcal{T}, \mathcal{T}_{\mathcal{R}} \in \langle S \rangle$.*

Proof. We may let $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$, where S_α is a completely simple semigroup for every $\alpha \in Y$. If Y is trivial, then obviously $\mathcal{T} \cong \mathcal{S}$. Assume that Y has at least two elements. Let $T \in \mathcal{T}$ and $I = \overline{T}$. In the notation of Lemma 4.2, we have $\widehat{I} \subseteq \prod_{\alpha \in I} S_\alpha$. But for any $\alpha \in Y$, S_α is a subsemigroup of S, so that $S_\alpha \in \langle S \rangle$. Hence $\prod_{\alpha \in I} S_\alpha \in \langle S \rangle$, and since $\widehat{I}$ is a regular subsemigroup of the completely simple semigroup $\prod_{\alpha \in I} S_\alpha$, it is itself completely simple and thus completely regular. Therefore $\widehat{I} \in \langle S \rangle$. By Lemma 4.2, $\mathcal{T}$ is a strong semilattice $\widehat{\mathcal{I}}_Y$ of completely simple semigroups $\widehat{I}$. Thus, according to Lemma 4.6, $\mathcal{T}$ is a subdirect product of semigroups $\widehat{I}$ with a zero possibly adjoined. Now Lemma 4.6 gives that $\widehat{I}^0 \in \langle \widehat{I} \rangle \vee \mathcal{S}$. It follows that $\widehat{I}^0 \in \langle S \rangle$ since $\langle \widehat{I} \rangle$, $\mathcal{S} \subseteq \langle S \rangle$ for Y is assumed to be nontrivial. Therefore $I, \widehat{I}^0 \in \langle S \rangle$, and hence $\mathcal{T} \in \langle S \rangle$. Since $\mathcal{T}_{\mathcal{R}}$ is closed under taking of inverses, it follows that $\mathcal{T}_{\mathcal{R}} \in \langle S \rangle$.

COROLLARY 4.8. *If S is a normal cryptogroup, so are $\mathcal{T}$ and $\mathcal{T}_{\mathcal{R}}$.*

Proof. Let S be a normal cryptogroup. Then $\langle S \rangle$ is a variety of normal cryptogroups. By Proposition 4.7, $\mathcal{T}$ is a normal cryptogroup. Clearly, $\mathcal{T}_{\mathcal{R}}$ is closed under taking of inverses which makes it a completely regular semigroup. Since $\mathcal{T}$ is a normal cryptogroup, so must be $\mathcal{T}_{\mathcal{R}}$.

THEOREM 4.9. *Let $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$, where S_α is a weakly cancellative semigroup for every $\alpha \in Y$. For $\omega \in \Omega_i$ let I_ω be as in Theorem 3.3, and for $\alpha \in I_\omega$ let $\omega\big|_{S_\alpha} = \pi_{a_\alpha}$. With this notation define a mapping φ by*

$$\varphi \colon \omega \to (a_\alpha)_{\alpha \in I_\omega} \qquad (\omega \in \Omega_i(S)).$$

For $(a_\alpha)_{\alpha \in I} \in \mathcal{T}_{\mathcal{R}}$, define a bioperator ω by

$$\omega x = a_{\overline{\alpha}} x, \quad x\omega = x a_{\overline{\alpha}} \qquad (x \in S_\alpha, \ \alpha \in Y, \ (\alpha) \cap I = (\overline{\alpha})).$$

With this notation define a mapping ψ by

$$\psi \colon (a_\alpha)_{\alpha \in I} \to \omega \qquad ((a_\alpha)_{\alpha \in I} \in \mathcal{T}_{\mathcal{R}}).$$

Then the mappings φ and ψ are mutually inverse isomorphisms between Ω_i and $\mathcal{T}_{\mathcal{R}}$. In this association, Π corresponds to $\mathcal{T}_{\mathcal{P}}$. Moreover, with τ in Lemma 4.3 and $\pi \colon S \to \Omega$ the canonical homomorphism, we have $\tau\psi = \pi$.

Proof.

1. φ *maps* Ω_i *into* $\mathcal{T}_{\mathcal{R}}$. First note that, by Theorem 3.3, we have $I_\omega \in \mathcal{R}_Y$. For $\alpha \in I_\omega$, since $\omega \in \Omega_i$, we have $\omega\big|_{S_\alpha} \in \Pi(S_\alpha)$. By hypothesis, S_α is weakly cancellative and thus weakly reductive, so that $\omega\big|_{S_\alpha} = \pi_{a_\alpha}$ within S_α for a unique a_α. Therefore φ is single valued.

Now let $\omega\big|_{S_\alpha} = \pi_{a_\alpha}$, $\omega\big|_{S_\beta} = \pi_{a_\beta}$ and $\alpha \geq \beta$. In view of Theorem 3.3, for any $x \in S_\alpha$ we have $(\omega x)\chi_{\alpha,\beta} = \omega(x\chi_{\alpha,\beta})$, which implies $(\pi_{a_\alpha}x)\chi_{\alpha,\beta} = \pi_{a_\beta}(x\chi_{\alpha,\beta})$, whence $(a_\alpha x)\chi_{\alpha,\beta} = a_\beta(x\chi_{\alpha,\beta})$, and finally $(a_\alpha\chi_{\alpha,\beta})(x\chi_{\alpha,\beta}) = a_\beta(x\chi_{\alpha,\beta})$. One proves similarly that $(x\chi_{\alpha,\beta})(a_\alpha\chi_{\alpha,\beta}) = (x\chi_{\alpha,\beta})a_\beta$, which, by weak cancellation in S_β, yields $a_\alpha\chi_{\alpha,\beta} = a_\beta$. Therefore $(a_\alpha)_{\alpha\in I_\omega} \in \mathcal{T}_{\mathcal{R}}$.

2. ψ *maps* $\mathcal{T}_{\mathcal{R}}$ *into* Ω_i. Let $(a_\alpha)_{\alpha\in I} \in \mathcal{T}_{\mathcal{R}}$, $x \in S_\alpha$, $y \in S_\beta$. Then $\overline{\alpha}\beta = \overline{\alpha\beta}$ since $I \in \mathcal{R}_Y$, and thus

$$(\omega x)y = (a_{\overline{\alpha}}x)y = a_{\overline{\alpha}}(xy) = (a_{\overline{\alpha}}\chi_{\overline{\alpha},\overline{\alpha}\beta})xy = a_{\overline{\alpha\beta}}xy = \omega(xy)$$

since $(a_\alpha)_{\alpha\in I} \in \mathcal{T}_{\mathcal{R}}$. Similarly, $x(y\omega) = (xy)\omega$, and

$$\begin{aligned}(x\omega)y = (xa_{\overline{\alpha}})y = x(a_{\overline{\alpha}}y) = x(a_{\overline{\alpha}}\chi_{\overline{\alpha},\overline{\alpha}\beta}y) = x(a_{\overline{\alpha}\beta}y)\\ = x(a_{\alpha\overline{\beta}}y) = x(a_{\overline{\beta}}\chi_{\overline{\beta},\alpha\overline{\beta}}y) = x(a_{\overline{\beta}}y) = x(\omega y),\end{aligned}$$

which proves that $\omega \in \Omega$. It follows easily that $I_\omega = I$, and thus $I \in \mathcal{R}_Y$ implies that $I_\omega \in \mathcal{R}_Y$, and therefore $\omega \in \Omega_i$.

3. ψ *is a homomorphism*. Let $(a_\alpha)_{\alpha\in I}, (b_\beta)_{\beta\in J} \in \mathcal{T}_{\mathcal{R}}$, and let $(\gamma)\cap I = (\overline{\gamma})$, $(\gamma)\cap J = (\widehat{\gamma})$ for all $\gamma \in Y$. Hence

$$(\gamma)\cap(I\cap J) = (\gamma\cap I)\cap J = (\overline{\gamma})\cap J = \big(\widehat{\overline{\gamma}}\big),$$

and similarly $(\gamma)\cap(I\cap J) = \big(\overline{\widehat{\gamma}}\big)$, so that $\widehat{\overline{\gamma}} = \overline{\widehat{\gamma}}$. For every $x \in S_\alpha$ we obtain

$$\omega(\omega' x) = \omega(b_{\widehat{\alpha}}x) = a_{\overline{\widehat{\alpha}}}b_{\widehat{\alpha}}x = a_{\overline{\widehat{\alpha}}}(b_{\widehat{\alpha}}\chi_{\widehat{\alpha},\overline{\widehat{\alpha}}})x = a_{\overline{\widehat{\alpha}}}b_{\overline{\widehat{\alpha}}}x = (\omega\omega')x,$$

and similarly $(x\omega)\omega' = x(\omega\omega')$. It follows that

$$\big((a_\alpha)_{\alpha\in I}\psi\big)\big((b_\beta)_{\beta\in J}\psi\big) = \omega\omega' = \big((a_\gamma b_\gamma)_{\gamma\in I\cap J}\big)\psi,$$

and therefore ψ is a homomorphism.

4. $\varphi\psi = \iota_{\Omega_i}(S)$. Indeed, let $\omega \in \Omega_i$, $\omega\varphi = (a_\alpha)_{\alpha\in I_\omega}$ and $\omega\varphi\psi = \omega'$. Further let $x \in S_\alpha$ and $(\alpha)\cap I_\omega = (\overline{\alpha})$. Then

$$\omega' x = a_{\overline{\alpha}}x = \pi_{a_\alpha}x = \omega x,$$

and similarly $x\omega' = x\omega$, so that $\omega = \omega'$.

5. $\psi\varphi = \iota_{\mathcal{T}_\mathcal{R}}$. Indeed, let $(a_\alpha)_{\alpha\in I} \in \mathcal{T}_\mathcal{R}$, $(a_\alpha)_{\alpha\in I}\psi = \omega$ and $(a_\alpha)_{\alpha\in I}\psi\varphi = (b_\beta)_{\beta\in J}$. From Theorem 3.3, we know that $I = I_\omega$, and here also $I_\omega = J$, so that $I = J$. For any $\alpha \in Y$ let $(\alpha) \cap I = (\overline{\alpha})$. Then for any $x \in S_\alpha$ we have $\omega x = a_{\overline{\alpha}} x$ and $x\omega = xa_{\overline{\alpha}}$. If now $\alpha \in I$, it follows that $\omega\big|_{S_\alpha} = \pi_{a_\alpha}$. On the other hand, $\pi_{b_\alpha} = \omega\big|_{S_\alpha}$ by the definition of φ, so that $a_\alpha = b_\alpha$. Consequently, $\psi\varphi = \iota_{\mathcal{T}_\mathcal{R}}$.

6. We deduce that φ and ψ are mutually inverse isomorphisms between Ω_ι and $\mathcal{T}_\mathcal{R}$. We have seen above that for $(a_\alpha)_{\alpha\in I}\psi = \omega$ we have $I = I_\omega$, which evidently implies that ψ maps $\mathcal{T}_\mathcal{P}$ onto Π. Therefore, under both φ and ψ, Π corresponds to $\mathcal{T}_\mathcal{P}$.

7. For any $a \in S_\alpha$ we have

$$a\tau\psi = (a\chi_{\alpha,\beta})_{\beta\leq\alpha}\psi = \omega\,,$$

where for any $x \in S_\gamma$, $(\gamma) \cap (\alpha) = (\gamma\alpha)$, and

$$\omega x = a_{\gamma\alpha}x = (a_\alpha\chi_{\alpha,\alpha\gamma})x\,, = ax = \pi_a x\,,$$

and similarly $x\omega = x\pi_a$. Therefore $\omega = \pi_a$, so that $a\tau\psi = \pi_a$, and thus $\tau\psi = \pi$.

For the sake of completeness, we introduce

$$\Gamma_\mathcal{T} = \big\{(I;\omega_\alpha) \in \Gamma \mid \omega_\alpha \in \Pi(S_\alpha) \text{ for all } \alpha \in I\big\}.$$

For the case that S_α is weakly cancellative for all $\alpha \in Y$, the mappings

$$(I;\pi_{a_\alpha}) \to (a_\alpha)_{\alpha\in I}\,, \qquad (a_\alpha)_{\alpha\in I} \to (I;\pi_{a_\alpha})$$

are mutually inverse isomorphisms between $\Gamma_\mathcal{T}$ and $\mathcal{T}$. The proof of this assertion follows along the same lines as the arguments above. Moreover, $\Gamma_\mathcal{T} \cap \Gamma_\Omega = \Gamma_i$. We present the mappings and inclusions of the various semigroups in Diagram 1.[2)]

Let $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$, where S_α is an arbitrary semigroup for all $\alpha \in Y$. If $\mathcal{R}_Y = \mathcal{P}_Y$, that is every retract ideal of Y is principal, then $\Omega_i = \Pi$ for S. The converse does not hold in general.

E x a m p l e 4.10. Let $Y = \{1, 2, \dots\}$ with the operation of min. For each $n \in Y$ let $S_n = \{n, n+1, \dots\}$ with left zero multiplication, and for $m \geq n$

[2)] In Diagram 1, S is a strong semilattice of weakly cancentlative semigroups.

let $\chi_{m,n}\colon S_m \to S_n$ be the embedding map. Then $S = \bigcup_{n\geq 1} S_n$ is a subdirect product of the semilattice Y and the left zero semigroup S, so it is a normal cryptogroup. There exists no thread T for which $\overline{T} = Y$. Since $\{Y\} = \mathcal{R}_Y \setminus \mathcal{P}_Y$, we conclude that $\mathcal{R}_Y \neq \mathcal{P}_Y$, but $\Omega_i = \Pi$ for S.

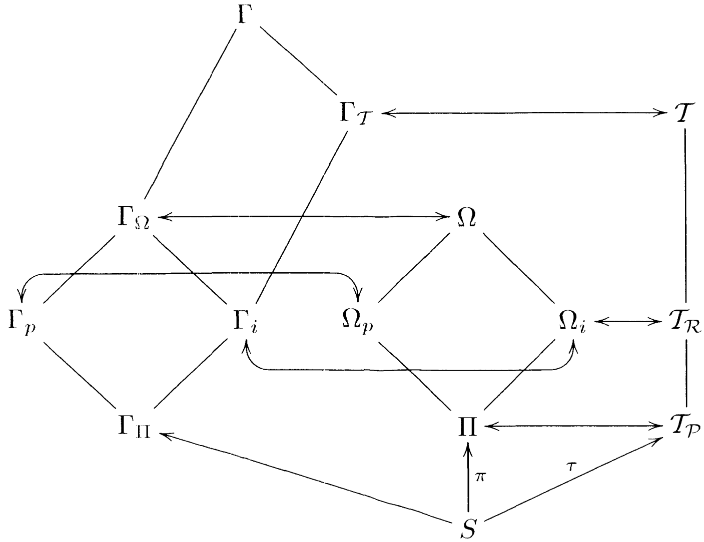

Diagram 1.

5. Characterizations of Ω_i for normal cryptogroups

For $S = [Y; S_\alpha, \chi_{\alpha\beta}]$, where S_α is an arbitrary semigroup for every $\alpha \in Y$, we have defined $\Omega_i = \Gamma_i\psi$ after Theorem 3.3 with Γ_i defined in Construction 3.1. Combining all these definitions, we see that Ω_i consists of those bitranslations ω of S for which $\omega|_{S_\alpha} \in \Pi(S_\alpha)$ for all α such that $\omega S \cap S_\alpha \neq \emptyset$. It should be noted that Ω_i depends on the way in which S is decomposed into its subsemigroups S_α. In the case such a decomposition may be chosen in a natural way, we may omit the reference to the semigroups S_α. This is the case when S is a normal cryptogroup for, in this instance, we take the Green relation $\mathcal{D}$ which coincides with the least semilattice congruence on S, and in fact, S is a strong semilattice of completely simple semigroups.

For a normal cryptogroup S, we shall characterize Ω_i in two interesting ways. In addition, we shall see that Ω_i plays the same role for ideal extensions of S which are themselves normal cryptogroups as Ω does for arbitrary ideal extensions. The results here also suggest a generalization of the concept of a densely embedded ideal and of a dense embedding.

THEOREM 5.1. *Let $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$, where S_α is a completely simple semigroup for every $\alpha \in Y$. Then Ω_i is the greatest normal subcryptogroup of Ω containing Π.*

Proof. We have observed in Corollary 4.8 that $\mathcal{T}_\mathcal{R}$ is a normal cryptogroup. According to Theorem 4.9, $\mathcal{T}_\mathcal{R}$ and Ω_i are isomorphic. Therefore Ω_i is a normal cryptogroup containing Π.

In order to establish maximality, let T be a normal subcryptogroup of Ω containing Π, and let $\omega \in T$. In view of Theorem 3.3, we have $\omega\varphi = (I; \omega_\alpha)$, where $I \in \mathcal{R}_Y$. It remains to show that $\omega_\alpha \in \Pi(S_\alpha)$ for all $\alpha \in I$.

Let $\alpha \in I$. Then $\omega|_{S_\alpha} = \omega_\alpha$. We may represent S_α by a Rees matrix semigroup and ω_α by a quadruple $(\alpha, \varphi, \psi, \beta)$ as in [1; Section V.3]. Let $e = (i, p_{\lambda i}^{-1}, \lambda) \in S_\alpha$. Then

$$(\omega_\alpha e)^0\omega_\alpha = \big(\alpha i, (\varphi i)p_{\lambda i}^{-1}, \lambda\big)^0\omega = \big(\alpha i, p_{\lambda(\alpha i)}^{-1}, \lambda\big)\omega = \big(\alpha i, p_{\lambda(\alpha i)}^{-1}(\lambda\psi), \lambda\beta\big), \tag{1}$$
$$\omega_\alpha(e\omega_\alpha)^0 = \omega\big(i, p_{\lambda i}^{-1}(\lambda\psi), \lambda\beta\big)^0 = \omega\big(i, p_{(\lambda\beta)i}^{-1}, \lambda\beta\big) = \big(\alpha i, (\varphi i)p_{(\lambda\beta)i}^{-1}, \lambda\beta\big),$$

and in view of [1; Proposition V.3.7], we get $(\omega_\alpha e)^0\omega_\alpha = \omega_\alpha(e\omega_\alpha)^0$. Now letting $a = (\omega_\alpha e)^0\omega_\alpha$, by [1; Lemma III.1.6 ii)], we obtain $\pi_a = \pi_{(\omega_\alpha e)^0}\omega_\alpha = \omega_\alpha\pi_{(e\omega_\alpha)^0}$ within $\Omega(S_\alpha)$. In order to obtain the same type of formula in all of S, we let $x \in S_\beta$. Then

$$\big(\pi_{(\omega e)^0}\omega\big)x = \pi_{(\omega e)^0}(\omega x) = (\omega e)^0(\omega x) = \big((\omega e)^0\omega\big)x = ax = \pi_a x,$$
$$x\big(\pi_{(\omega e)^0}\omega\big) = \big(x\pi_{(\omega e)^0}\big)\omega = \big(x(\omega e)^0\big)\omega = x\big((\omega e)^0\omega\big) = xa = x\pi_a,$$

and since S is weakly reductive, we deduce that $\pi_{(\omega e)^0}\omega = \pi_a$. A similar argument will yield $\omega\pi_{(e\omega)^0} = \pi_a$. It follows that $\pi_a \leq \omega$ in the natural partial order on T.

By hypothesis $\omega \in T$, T is a normal cryptogroup and $\Pi \subseteq T$. Hence $\pi_a \in T$, and in view of [1; Theorem IV.4.3], π_a is the only element of the $\mathcal{D}$-class of π_a majorized by ω. If we now start with any idempotent f in S_α and form $b = (\omega f)^0\omega$, we get that $\pi_a \mathrel{\mathcal{D}} \pi_b$ and, as above, $\pi_b \leq \omega$, so that $\pi_a = \pi_b$. Since S_α is weakly reductive, this yields that $a = b$. In particular, $(\omega e)^0\omega = (\omega f)^0\omega$. Now writing $f = (j, p_{\mu j}^{-1}, \mu)$, by (1), we obtain that $\alpha i = \alpha j$ and $\lambda\beta = \mu\beta$. Since f is an arbitrary idempotent of S_α, we conclude that both α and β are constant. Now [1; Theorem V.3.8] gives that $\omega_\alpha \in \Pi(S_\alpha)$, as required. Consequently $\omega \in \Omega_i$, which finally gives $T \subseteq \Omega_i$ establishing the maximality of Ω_i.

COROLLARY 5.2. *With the hypothesis of Theorem* 5.1, Ω_i *is the greatest completely regular subsemigroup* T *of* Ω *containing* Π *for which* $T \in \langle S \rangle$.

P r o o f . By Theorem 4.9, Ω_i is isomorphic to $\mathcal{T}_{\mathcal{R}}$, and, by Proposition 4.7, we have $\mathcal{T}_{\mathcal{R}} \in \langle S \rangle$. Therefore $\Omega_i \in \langle S \rangle$. Let T be a completely regular subsemigroup of Ω containing Π for which $T \in \langle S \rangle$. Since S is a normal cryptogroup, $T \in \langle S \rangle$ implies that T is also a normal cryptogroup. Now T fits the specifications in Theorem 5.1 and is thus contained in Ω_i . This establishes the desired maximality of Ω_i .

Recall the notation $\tau(V : S)$ in [1; Theorem III.1.12].

THEOREM 5.3. *Let* V *be a normal cryptogroup and an ideal extension of* S . *Then* $\tau = \tau(V : S)$ *maps* V *into* $\Omega_i(S)$. *Moreover,* V *is a maximal normal cryptogroup dense extension of* S *if and only if* τ *is an isomorphism of* V *onto* $\Omega_i(S)$.

P r o o f . First $V\tau$ is a normal cryptogroup and a subsemigroup of $\Omega(S)$ containing $\Pi(S)$. Now Theorem 5.1 implies that $V\tau \subseteq \Omega_i(S)$.

Next assume that V is a maximal cryptogroup dense extension of S . By [1; Corollary III.5.5], τ is injective. If $V\tau$ is a proper subsemigroup of $\Omega_i(S)$, we can define a multiplication on $V \cup \big(\Omega_i(S) \setminus V\tau\big)$ in an obvious way making it a normal cryptogroup dense extension of S which contradicts the assumed maximality of V . Therefore τ is an isomorphism of V onto $\Omega_i(S)$.

Conversely, suppose that τ is an isomorphism of V onto $\Omega_i(S)$. According to [1; Corollary III.5.5], V is a dense extension of S . Let U be any normal cryptogroup dense extension of S containing V . By the above, τ is an isomorphism of U into $\Omega_i(S)$. Since $\tau|_V = \tau$, we must have that $U = V$. This establishes the maximality of V .

Recall the concept of a densely embedded ideal in [1; Definition III.5.8].

A monomorphism φ of a semigroup S into a semigroup T is a *dense embedding* if $S\varphi$ is a densely embedded ideal of its idealizer in T .

The above results suggest a natural extension of these notions involving varieties of (completely regular) semigroups as follows.

DEFINITION 5.4. Let $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ and $S \in \mathcal{V}$. If T is a dense extension of S with the properties: $T \in \mathcal{V}$, and if $V \in \mathcal{V}$ is a dense extension of S which contains T , then $T = V$, and then S is a $\mathcal{V}$*-densely embedded ideal of* T . If $P, Q \in \mathcal{V}$, and $\varphi \colon P \to Q$ is an embedding such that $P\varphi$ is a $\mathcal{V}$-densely embedded ideal of its idealizer in Q , then φ is a $\mathcal{V}$*-dense embedding of* P *into* Q .

THEOREM 5.5. *Let $\mathcal{V} \in \mathcal{L}(\mathcal{NBG})$ and $S \in \mathcal{V}$. Then* Π *is a densely embedded ideal of* Ω_i, *and the mapping* τ *in Lemma* 4.3 *is a* $\mathcal{V}$*-dense embedding of* S *in* $\mathcal{T}$.

P r o o f . This follows directly from Theorem 5.3, Lemmas 4.3 and 4.4, and Proposition 4.7.

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