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ON EQUIVALENCE OF SUMMABILITY METHODS

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1. Definitions and Notation.

Let Σa_n be an infinite series with a sequence of its partial sums (s_n) and let $\mathbf{A} = (a_{nk})$ be an infinite matrix. Suppose that

$$T_n = \sum_{v=0}^{\infty} a_{nv} s_v, \qquad (n = 0, 1, \ldots), \tag{1}$$

exists (i.e. the series on the right-hand side converges for each n). If $(T_n) \in bv$, i.e.

$$\sum_{n=0}^{\infty} |T_n - T_{n-1}| < \infty, \ (T_{-1} = 0), \tag{2}$$

then the series Σa_n is said to be absolutely summable by the matrix $\mathbf{A}$ or simply summable $|\mathbf{A}|$. As known, the series Σa_n is said to be $|\bar{N}, p_n|$ summable if (2) holds when $\mathbf{A}$ is a Riesz matrix. By a Riesz matrix we mean one such that

$$a_{nv} = p_v/P_n, \quad \text{for} \quad 0 \leqslant v \leqslant n, \quad \text{and} \quad a_{nv} = 0 \quad \text{for} \quad v > n,$$

where (p_n) is a sequence of positive real numbers and

$$P_n = p_0 + p_1 + \ldots + p_n, \ P_{-1} = 0.$$

Let (T_n) be given by (1). If

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty, \tag{3}$$

then Σa_n is said to be $|\mathbf{A}|_k$ summable, where $k \geqslant 1$. Some results on $|\mathbf{A}|_k$, $(k \geqslant 1)$ summability may be found in [2].

Throughout the paper, the matrix $\mathbf{A} = (a_{nv})$ will be a Riesz matrix with $P_n \to \infty$ as $n \to \infty$. Hence, if no confusion is likely to arise, we say that Σa_n is summable $|R, p_n|_k$, $k \geqslant 1$ if (3) holds.

Concerning the $|\bar{N}, p_n|$ summability the following result is due to Sunouchi [3].

Theorem 1. *Let (p_n) and (g_n) be positive sequences such that*

$$\frac{q_n}{Q_n} \leqslant K \frac{p_n}{P_n}. \tag{4}$$

Then $|\bar{N}, p_n| \Rightarrow |\bar{N}, q_n|$.

In 1950, while reviewing this paper, Bosanquet [1], observed that condition (4) is not only sufficient but also necessary for $|\bar{N}, p_n| \Rightarrow |\bar{N}, q_n|$.

In this paper we give sufficient conditions on the sequences (p_n) and (q_n) for the summability methods $|R, p_n|_k$ and $|R, q_n|_k$, $(k \geqslant 1)$ to be equivalent and therefore we extend the known results of [1], [3] to the cases $k > 1$.

2. Equivalence of the Summability Methods $|R, p_n|_k$ and $|R, q_n|_k$.

Let (p_n) and (q_n) be positive sequences such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty, \ (n \to \infty)$$

$$Q_n = \sum_{v=0}^{n} q_v \to \infty, \ (n \to \infty).$$

Now we have the following.

Theorem 2.1. *The $|R, p_n|_k$, $(k \geqslant 1)$ summability implies the $|R, q_n|_k$, $(k \geqslant 1)$ summability provided that*

(i) $nq_n = O(Q_n)$

(ii) $P_n = O(np_n)$

(iii) $Q_n = Q(nq_n)$.

Proof. Suppose that Σa_n is summable $|R, p_n|_k$, $(k \geqslant 1)$. Then

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty, \ (k \geqslant 1), \tag{5}$$

where

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad \text{and} \quad s_v = a_0 + a_1 + \dots + a_v.$$

On the other hand we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v.$$

Some calculation reveals that

$$\Delta T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v,$$

where $n \geqslant 1$ and $\Delta T_{-1} = T_0 = a_0$.
and

$$a_n = \frac{P_n}{p_n} \Delta T_{n-1} - \frac{P_{n-2}}{p_{n-1}} \Delta T_{n-2}, \tag{6}$$

where $n \geqslant 0$, $P_{-2} = P_{-1} = 0$, $p_{-1} = 1$ and $\Delta T_{-2} = 0$. Similarly we get that

$$t_n = \frac{1}{Q_n} \sum_{v=0}^{n} q_v s_v = \frac{1}{Q_n} \sum_{v=1}^{n} (Q_n - Q_{v-1}) a_v$$

and

$$\Delta t_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v, \quad (n \geqslant 1), \tag{7}$$

If follows from (6) and (7) that

$$\Delta t_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} \left(\frac{P_v}{p_v} \Delta T_{v-1} - \frac{P_{v-2}}{p_{v-1}} \Delta T_{v-2} \right) =$$

$$= \frac{q_n P_n}{p_n Q_n} \Delta T_{n-1} + \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{\Delta T_{v-1}}{p_v} (Q_{v-1} P_v - Q_v P_{v-1}).$$

But, since

$$Q_{v-1} P_v - Q_v P_{v-1} = Q_{v-1} P_v - Q_v (P_v - p_v) = (Q_{v-1} - Q_v) P_v + p_v Q_v =$$
$$= -q_v P_v + p_v Q_v$$

we get

$$\Delta t_{n-1} = \frac{q_n P_n}{p_n Q_n} \Delta T_{n-1} - \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{p_v} q_v \Delta T_{v-1} + \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} Q_v \Delta T_{v-1} =$$
$$= \omega_{n1} + \omega_{n2} + \omega_{n3}, \text{ say.}$$

To prove the theorem, by Minkowski's inequality it is enough to show that

$$\sum_{n=1}^{\infty} n^{k-1} |\omega_{ni}|^k < \infty, \quad \text{for} \quad i = 1, 2, 3.$$

Using conditions (i) and (ii), we get that

$$\sum_{n=1}^{\infty} n^{k-1}|\omega_{n1}|^k = O\left\{\sum_{n=1}^{\infty} n^{k-1}|\Delta T_{n-1}|^k\right\}$$

$$< \infty, \quad \text{by (5).}$$

Let $i = 2$. By Hölder's inequality when $k > 1$ (and trivially when $k = 1$), we have

$$|\omega_{n2}|^k \leqslant \left(\frac{q_n}{Q_n Q_{n-1}}\right)^k \left\{\sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k q_v|\Delta T_{v-1}|^k\right\} \left(\sum_{v=1}^{n-1} q_v\right)^{k-1} =$$

$$= O\left\{\frac{1}{n^{k-1}} \cdot \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^k q_v|\Delta T_{v-1}|^k\right\}, \text{ by (i) and (ii).}$$

Hence

$$\sum_{n=1}^{\infty} n^{k-1}|\omega_{n2}|^k = O\left\{\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^k q_v|\Delta T_{v-1}|^k\right\} =$$

$$= O\left\{\sum_{v=1}^{\infty} \left(\frac{P_v}{p_v}\right)^k q_v|\Delta T_{v-1}|^k \sum_{n=v}^{\infty} \frac{q_n}{Q_n Q_{n-1}}\right\} = O\left\{\sum_{v=1}^{\infty} \left(\frac{P_v}{p_v}\right)^k \frac{q_v}{Q_v} |\Delta T_{v-1}|^k\right\} =$$

$$= O\left\{\sum_{v=1}^{\infty} v^{k-1}|\Delta T_{v-1}|^k\right\}, \text{ by (i) and (ii)}$$

$$< \infty, \text{ by (5).}$$

Now let $i = 3$
Writing

$$\sum_{v=1}^{n-1} Q_v|\Delta T_{v-1}| = \sum_{v=1}^{n-1} \frac{Q_v}{q_v} \frac{Q_v}{q_v} |\Delta T_{v-1}| q_v$$

and using Hölder's inequality we obtain that

$$|\omega_{n3}|^k \leqslant \left(\frac{q_n}{Q_n Q_{n-1}}\right)^k \left(\sum_{v=1}^{n-1} \left(\frac{Q_v}{q_v}\right)^k q_v|\Delta T_{v-1}|^k\right) \left(\sum_{v=1}^{n-1} q_v\right)^{k-1} =$$

$$O\left\{\frac{1}{n^{k-1}} \cdot \frac{q_n}{Q_n} \cdot \frac{1}{Q_{n-1}} \sum_{v=1}^{n} \left(\frac{Q_v}{q_v}\right)^k q_v|\Delta T_{v-1}|^k\right\}.$$

This gives us that

$$\sum_{n=1}^{\infty} n^{k-1}|\omega_{n3}|^k = O\left\{\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} \left(\frac{Q_v}{q_v}\right)^k q_v|\Delta T_{v-1}|^k\right\} =$$

$$= O\left\{\sum_{v=1}^{\infty}\left(\frac{Q_v}{q_v}\right)^k q_v|\Delta T_{v-1}|^k \sum_{n=v}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\right\} = O\left\{\sum_{v=1}^{\infty}\left(\frac{Q_v}{q_v}\right)^k \frac{q_v}{Q_v}|\Delta T_{v-1}|^k\right\} =$$

$$= O\left\{\sum_{v=1}^{\infty}\left(\frac{Q_v}{q_v}\right)^{k-1}|\Delta T_{v-1}|^k\right\} = O\left\{\sum_{v=1}^{\infty} v^{k-1}|\Delta T_{v-1}|^k\right\}, \quad \text{by (iii)}$$

$$< \infty, \quad \text{by (5)}$$

which proves the theorem.

We note that, interchanging the roles of (p_n) and (q_n) in Theorem 2.1, we get the next theorem immediately.

Theorem 2.2. *Suppose that conditions* (ii) *abd* (iii) *of Theorem* 2.1 *hold and that* (iv) $np_n = O(P_n)$. *Then the* $|R, q_n|_k$, $(k \geqslant 1)$ *summability implies the* $|R, p_n|_k$, $(k \geqslant 1)$ *summability.*

Our final result follows from Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *Suppose that* (p_n) *and* (q_n) *are positive sequences such that*

(i) $nq_n = O(Q_n)$, (ii) $Q_n = O(nq_n)$, (iii) $P_n = O(np_n)$, (iv) $np_n = O(P_n)$.

Then the $|R, p_n|_k$ *summability is equivalent to the* $|R, q_n|_k$ *summability, where* $k \geqslant 1$.

We remark that if we take $p_n = 1$ (for all n), then $P_n = n$. In this case the $|R, p_n|_k$ summability is the same as the $|C, 1|_k$ summability. Therefore the following corollary can be derived from Theorem 2.3.

Corollary. *Suppose that* (q_n) *is a positive sesquence for which*

(i) $nq_n = O(Q_n)$, (ii) $Q_n = O(nq_n)$.

Then the $|R, q_n|_k$ *summability is equivalent to the* $|C, 1|_k$, $(k \geqslant 1)$ *summability.*

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О ЭКВИВАЛЕНТНОСТИ МЕТОДОВ СУММИРОВАНИЯ

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Резюме

В этой статье даются достаточные условия, накладываемые на последовательности (p_n) и (q_n), для того, чтобы методы суммирования $|R, p_n|_k$ и $|R, q_n|_k$, $k \geqslant 1$, были эквивалентны. Тем самым мы расширяем известные результаты из работ [1], [3] на случай $k > 1$.