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Robert Šulka

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NILPOTENCY IN SEMIGROUPS AND SUBLATTICES OF THEIR BOOLEANS

ROBERT ŠULKA

1. Introduction.

Let S be a semigroup, S' a subsemigroup of S, $M \subseteq S'$, N the set of all positive integers and $\langle \mathscr{P}(S), \subseteq \rangle$ the Boolean of S. We introduce the following notations

$N_1(S', M) = \{x \in S' | x^n \in M$ for almost all $n \in N\}$,
$N_2(S', M) = \{x \in S' | x^n \in M$ for infinitely many $n \in N\}$,
$N_3(S', M) = \{x \in S' | x^n \in M$ fore some $n \in N\}$.

With respect to the notations in the paper [5] if $M \subseteq S$, then $N_i(M) = N_i(S, M)$ for $i = 1, 2, 3$, $N_1(S', M)$ is the set of all strongly M-potent elements of S', $N_2(S', M)$ is the set of all weakly M-potent elements of S' and $N_3(S', M)$ is the set of all almost M-potent elements of S'.

Further let

$\mathscr{N}_{1\,2}(S') = \{M \subseteq S' | N_1(S', M) = N_2(S', M)\}$,
$\mathscr{N}_{1\,3}(S') = \{M \subseteq S' | N_1(S', M) = N_3(S', M)\}$ and
$\mathscr{N}_{2\,3}(S') = \{M \subseteq S' | N_2(S', M) = N_3(S', M)\}$.

With respect to the notation in the paper [5] if $M \subseteq S$, then $\mathscr{N}_{i\,j}(S) = \mathscr{N}_{i\,j}$ for $i < j$, $i, j = 1, 2, 3$.

From the paper [5] it follows that $\langle \mathscr{N}_{1\,2}(S'), \subseteq \rangle$ is a lattice and $\langle \mathscr{N}_{1\,3}(S'), \subseteq \rangle$ and $\langle \mathscr{N}_{2\,3}(S'), \subseteq \rangle$ are complete lattices. In the mentioned paper the structure of $\mathscr{N}_{1\,2}(S)$, $\mathscr{N}_{1\,3}(S)$ and $\mathscr{N}_{2\,3}(S)$ was studied in the case of a cyclic semigroup S.

The purpose of this paper is to elucidate the connections between the lattices $\mathscr{N}_{i\,j}(S)$ and the lattices $\mathscr{N}_{i\,j}(S_k)$ $(k \in K)$ where S_k are subsemigroups of the semigroup S, to elucidate the connections between the lattices $\mathscr{N}_{i\,j}(S)$ and the lattices $\mathscr{N}_{j\,j}(S')$, if S' is a homomorphic image of S and to give characterizations of some classes of periodic semigroups by means of the notions mentioned above.

It will be shown that if $S = \cup \{S_k | k \in K\}$, S_k are subsemigroups of S and $M \subseteq S$, then $M \in \mathcal{N}_{i\ j}(S)$ iff for all $k \in K$ $M \cap S_k \in \mathcal{N}_{i\ j}(S_k)$ holds. Hence the knowledge of the lattices $\mathcal{N}_{i\ j}(S_k\ k \in K)$ allows to test, whether the set M belongs to $\mathcal{N}_{i\ j}(S)$ or not. Therefore the knowledge of the lattices $\mathcal{N}_{i\ j}(S_k)$ $(k \in K)$ allows to construct the lattices $\mathcal{N}_{i\ j}(S)$.

Since every semigroup S is a union of some system of its cyclic subsemigroups $\langle a_k \rangle$ $(k \in K)$ and the structure of lattices $\mathcal{N}_{i\ j}(\langle a_k \rangle)$ is known, we get a tool for the construction of the lattices $\mathcal{N}_{i\ j}(S)$ of an arbitrary semigroup S.

As we shall see the above mentioned construction of the lattices $\mathcal{N}_{i\ j}(S)$ can be essentialy simplified if $S = \cup \{S_k | k \in K\}$, where every two subsemigroups S_k, S_l, $k, l \in K$, $k \neq l$ are disjoint. In this case $M \in \mathcal{N}_{i\ j}(S)$ iff $M = \cup \{M_k | k \in K\}$ and $M_k \in \mathcal{N}_{i\ j}(S_k)$ for every $k \in K$. This will be particularly true in the case of a free semigroup $\mathcal{F}_X$ on a set X, because this semigroup is a union of a system of its cyclic subsemigroups that are mutually disjoint.

If φ: $S \to S'$ is a homomorphism of a semigroup S onto a semigroup S', then $\mathcal{N}_{i\ j}(S') = \{M' \subseteq S' | \varphi^{-1}(M') \in \mathcal{N}_{i\ j}(S)\}$ holds for $i, j = 1, 2, 3$, $i < j$.

This result may be also applied to the free semigroup $\mathcal{F}_X$ on a set X and its arbitrary homomorphic image.

2. $\mathcal{N}_{i\ j}(S)$ for a cyclic semigroup S.

For completeness we have to mention that it follows from the paper [5]

Proposition 1. *Let $S = \langle a \rangle$ be the cyclic semigroup generated by the element a. Then $\square \neq M \in \mathcal{N}_{2\ 3}(S)$ iff M is a union of countably many sets* $\{x, x^{k_1}, x^{k_1 k_2}, \ldots, x^{k_1 k_2 \ldots k_n}, \ldots\}$, $x \in S$, where $(k_n)_{n=1}^{\infty}$ is a sequence of positive integers k_n, $k_n > 1$.

Proposition 2 and Proposition 3 are also consequences of the paper [5].

Proposition 2. *Let $S = \langle a \rangle$ be a cyclic semigroup of infinite order Then $\square \neq M \in \mathcal{N}_{1\ 3}(S)$ iff M is the complement of a finite subset of S.*

Let S = ⟨a⟩ *be a cyclic semigroup of finite order. Then* $\square \neq M \in \mathcal{N}_{1\ 3}(S)$ iff M *contains the maximal subgroup G of S.*

Proposition 3. *Let $S = \langle a \rangle$ be a cyclic semigroup of infinite order. Then $M \in \mathcal{N}_{1\ 2}(S)$ iff either M is a finite subset of S or M is the complement of a finite subset* of S.

Let $S = \langle a \rangle$ be a cyclic semigroup of finite order. Then $M \in \mathcal{N}_{1\ 2}(S)$ iff either $M \cap G = \square$ or $M \supseteq G$.

3. $\mathcal{N}_{2\,3}(\langle a\rangle)$ in the case if $\langle a\rangle$ is a cyclic semigroup of finite order

Proposition 4. *Let G be a group. Then every finite cyclic subsemigroup of G is a group.*

Proof. Let $\langle a\rangle = \{a, a^2, \ldots, a^{r-1}, a^r, \ldots, a^{r+m-1}\} \subseteq G$ and r be the index and m the period of the semigroup $\langle a\rangle$. Then $a^{r+1} = a^{r+m+1}$. In G there exists $(a^r)^{-1}$, hence $a = a^{m+1}$. This means that $\langle a\rangle$ is a group.

Let $\langle a\rangle = \{a, a^2, \ldots, a^{r-1}, a^r, \ldots, a^{r+m-1}\}$ be the cyclic semigroup of finite order with index r and with period m. We denote $P(a) = \{a, a^2, \ldots, a^{r-1}\}$ and $G(a) = \{a^r, \ldots, a^{r+m-1}\}$. It is known that $G(a)$ is the maximal subgroup of the semigroup $\langle a\rangle$ and $G(a)$ is a cyclic group.

Proposition 5. *Let $\langle a\rangle$ be a cyclic semigroup of finite order. Then for every cyclic semigroup $\langle b\rangle$, $b \in \langle a\rangle$ there holds: $P(b) \subseteq P(a)$, $G(b) \subseteq G(a)$.*

Proof. Since $G(b)$ is a cyclic group of finite order, $\langle x\rangle$ is a cyclic group for all $x \in G(b)$. Hence for every $x \in G(b)$ there exists a $t \in N$ such that $x^t = x$, therefore $G(b) \cap P(a) = \square$. This implies that $G(b) \subseteq G(a)$.

If $x \in G(a) \cap P(b)$, then $\langle x\rangle \subseteq G(a) \cap \langle b\rangle$. Therefore $\langle x\rangle$ is a cyclic group of finite order of $\langle b\rangle$, hence $x \in G(b)$. However, this is a contradiction with the assumption $x \in P(b)$. This means that $G(a) \cap P(b) = \square$, hence $P(b) \subseteq P(a)$.

Theorem 1. *Let $S = \langle a\rangle$ be a cyclic semigroup of finite order. Then the following statements hold:*

i) *The lattice $\mathcal{N}_{2\,3}(S)$ is atomic.*
ii) *The atoms of $\mathcal{N}_{2\,3}(S)$ are exactly all one-element sets $\{b\}$, $b \in G(a)$.*
iii) *The lattice $\mathcal{N}_{2\,3}(S)$ contains all sets of the form $\{b, b^k\}$, $b \in P(a)$, $b^k \in G(a)$.*
iv) *The lattice $\mathcal{N}_{2\,3}(S)$ contains exactly all unions of all subsystems of the system of all sets mentioned in* ii) *and* iii).

Proof. i) is evident, since $\mathcal{N}_{2\,3}(S)$ is finite.

a) We shall prove that all sets mentioned in ii) belong to $\mathcal{N}_{2\,3}(S)$. Let $b \in G(a)$ and $x \in N_3(S, \{b\})$ hold. Then there exists a $p \in N$ such that $x^p = b$. Since $\langle b\rangle$ is a cyclic group of finite order, there exists a $q \in N$, $q > 1$ such that for all $s \in N$ we have $(b)^{q^s} = b$. Hence $x^{pq^s} = (x^p)^{q^s} = b$, for all $s \in N$. This means that infinitely many powers of x are equal to b, therefore $x \in N_2(S, \{b\})$. We have $N_3(S, \{b\}) = N_2(S, \{b\})$, hence $\{b\} \in \mathcal{N}_{2\,3}(S)$.

b) Now we shall prove that all sets mentioned in iii) belong to $\mathcal{N}_{2\,3}(S)$.
Let $x \in N_3(S, \{b, b^k\})$, $b \in P(a)$ and $b^k \in G(a)$. Then either there exsits a $p \in N$ such that $x^p = b \in P(a)$ or there exists a $p \in N$ such that $x^p = b^k \in G(a)$.
α) Let $x^p = b^k \in G(a)$. Then like in a) infinitely many powers of x are equal to b^k, hence $x \in N_2(S, \{b, b^k\})$.

β) Let $x^p = b \in P(a)$. Then $x^{kp} = b^k \in G(a)$ and again like in a) infinitely many powers of x are equal to b^k. Hence $x \in N_2(S, \{b, b^k\})$.
We have $N_3(S, \{b, b^k\}) = N_2(S, \{b, b^k\})$, i.e. $\{b, b^k\} \in \mathcal{N}_{2\,3}(S)$.

c) Since $\langle \mathcal{N}_{2\,3}(S), \subseteq \rangle$ is a complete upper subsemilattice of the complete semilattice $\langle \mathcal{P}(S), \subseteq \rangle$, the unions of arbitrary subsystems of the system of sets mentioned in ii) and iii) are elements of $\mathcal{N}_{2\,3}(S)$.

d) Finally we shall prove that $\mathcal{N}_{2\,3}(S)$ does not contain sets that are not unions of a subsystem of the system of sets mentioned in ii) and iii).

Let $M \subseteq S$ not be a union of a subsystem of the system of sets mentioned in ii) and iii). Then M contains an element $x \in P(a)$, but M contains no power of x that is in $G(a)$. Therefore $x \in N_3(S, M)$ and M can contain only powers of x that belong to $P(a)$. This means that M contains only a finite number of powers of x, hence $x \notin N_2(S, M)$. This implies that $M \notin \mathcal{N}_{2\,3}(S)$.

From these results it follows immediately that all sets $\{b\}$, $b \in G(a)$ are exactly all atoms of the lattice $\mathcal{N}_{2\,3}(S)$.

Corollary. *Let $S = \langle a \rangle$ be a cyclic group of finite order. Then $\mathcal{N}_{2\,3}(S) = \mathcal{P}(S)$.*

Proof. Evidently all atoms of $\mathcal{N}_{2\,3}(S)$ are exactly all sets $\{b\}$, $b \in \langle a \rangle$, hence $\mathcal{N}_{2\,3}(S) = \mathcal{P}(S)$.

Example 1. Let $S = \langle a \rangle = \{a, a^2, a^3, a^4, a^5\}$ be the cyclic semigroup of finite order with index 3 and period 3.

Then $P(a) = \{a, a^2\}$ and $G(a) = \{a^3, a^4, a^5\}$. Further $\langle a^2 \rangle = \{a^2, a^3, a^4, a^5\}$, $P(a^2) = \{a^2\}$ and $G(a^2) = \{a^3, a^4, a^5\}$.

The atoms of $\mathcal{N}_{2\,3}(S)$ are: $\{a^3\}$, $\{a^4\}$, $\{a^5\}$.

Other elements of $\mathcal{N}_{2\,3}(S)$ are:

$\{a, a^3\}$, $\{a, a^4\}$, $\{a, a^5\}$,
$\{a^2, a^3\}$, $\{a^2, a^4\}$, $\{a^2, a^5\}$.

Any element of $\mathcal{N}_{2\,3}(S)$ is a union of a subsystem of the system of the above mentioned sets.

In this case all apirs $\{b, c\}$, $b \in P(a)$, $c \in G(a)$ belong to $\mathcal{N}_{2\,3}(S)$.

Example 2. Let $S = \langle a \rangle = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}\}$ be the cyclic semigroup of finite order with index 5 and period 6.

Then $P(a) = \{a, a^2, a^3, a^4\}$ and $G(a) = \{a^5, a^6, a^7, a^8, a^9, a^{10}\}$. Further $\langle a^2 \rangle = \{a^2, a^4, a^6, a^8, a^{10}\}$, $P(a^2) = \{a^2, a^4\}$ and $G(a^2) = \{a^6, a^8, a^{10}\}$,
$\langle a^3 \rangle = \{a^3, a^6, a^9\}$, $P(a^3) = \{a^3\}$ and $G(a^3) = \{a^6, a^9\}$,
$\{a^4\} = \{a^4, a^6, a^8, a^{10}\}$, $P(a^4) = \{a^4\}$ and $G(a^4) = \{u^6, a^8, a^{10}\}$. The atoms of $\mathcal{N}_{2\,3}(S)$ are:

$\{a^5\}$, $\{a^6\}$, $\{a^7\}$, $\{a^8\}$, $\{a^9\}$ and $\{a^{10}\}$.

Other elements of $\mathcal{N}_{2\,3}(S)$ are:

$\{a, a^5\}, \{a, a^6\}, \{a, a^7\}, \{a, a^8\}, \{a, a^9\}, \{a, a^{10}\},$
$\{a^2, a^6\}, \{a^2, a^8\}, \{a^2, a^{10}\},$
$\{a^3, a^6\}, \{a^3, a^9\},$
$\{a^4, a^6\}, \{a^4, a^8\}, \{a^4, a^{10}\}.$

All elements of $\mathcal{N}_{2\,3}(S)$ are unions of a subsystem of the system of all sets mentioned above.

The set $\{a^2, a^9\} \notin \mathcal{N}_{2\,3}(S)$ because $a^2 \in N_3(S, \{a^2, a^9\})$ but $a^2 \notin N_2(S, \{a^2, a^9\})$, since $a^9 \notin \langle a^2 \rangle$.

We can see that not all pairs $\{b, c\}$, $b \in P(a)$, $c \in G(a)$ belong to $\mathcal{N}_{2\,3}(S)$.

4. Semigroup and its subsemigroups

Theorem 2. *Let S be a semigroup, M a subset of S, $S_k (k \in K)$ subsemigroups of S and let $S = \cup \{S_k | k \in K\}$. Then $N_i(S, M) = \cup \{N_i(S_k, M \cap S_k) | k \in K\}$ for $i = 1, 2, 3$.*

Proof. We give the proof only for $i = 3$. For $i = 1, 2$ the proofs are similar.

a) Let $x \in N_3(S, M)$ hold. Then there exists an $n \in N$ such that $x^n \in M$. Since $S = \cup \{S_k | k \in K\}$, there exists a $k \in K$ such that $x \in S_k$, hence for all $n \in N$ we have $x^n \in S_k$. This means that there exists an $n \in N$ such that $x^n \in M \cap S_k$. However, since $x \in S_k$, this implies that $x \in N_3(S_k, M \cap S_k) \subseteq \subseteq \cup \{N_3(S_k, M \cap S_k) | k \in K\}$ and we have $N_3(S, M) \subseteq \subseteq \cup \{N_3(S_k, M \cap S_k) | k \in K\}$.

b) Let $x \in \cup \{N_3(S_k, M \cap S_k) | k \in K\}$ hold. Then there exists a $k \in N$ such that $x \in N_3(S_k, M \cap S_k)$. Hence there exists an $n \in N$ such that $x^n \in M \cap S_k \subseteq M$. This means that $x \in N_3(S, M)$ holds and we have $\cup \{N_3(S_k, M \cap S_k) | k \in K\} \subseteq \subseteq N_3(S, M)$.

From a) and b) we get $N_3(S, M) = \cup \{N_3(S_k, M \cap S_k) | k \in K\}$. Next we shall need the following statement of paper [5].

Proposition 6. *Let S be a semigroup, S' a subsemigroup of S and M a subset of S. Then $N_i(S, M) \cap S' = N_i(S', S' \cap M)$ holds for $i = 1, 2, 3$.*

Now we can prove

Theorem 3. *Let S be a semigroup, $S_k (k \in K)$ subsemigroups of S, $S = = \cup \{S_k | k \in K\}$, $i, j = 1, 2, 3$, $i < j$. Then $M \in \mathcal{N}_{i\,j}(S)$ iff $M \cap S_k \in \mathcal{N}_{i\,j}(S_k)$ holds for all $k \in K$.*

Proof. a) Let $M \in \mathcal{N}_{i\,j}(S)$, i.e. $N_i(S, M) = N_j(S, M)$. Then Proposition 6 implies that $N_i(S_k, M \cap S_k)$ for all $k \in K$. This means that $M \cap S_k \in \mathcal{N}_{i\,j}(S)$ for all $k \in K$.

b) Let $M \cap S_k \in \mathcal{N}_{i\,j}(S_k)$ for all $k \in K$, i.e. $N_i(S_k, M \cap S_k) = N_j(S_k, M \cap S_k)$ for all $k \in K$. Then Theorem 2 implies that $N_i(S, M) = \cup \{N_i(S_k, M \cap S_k | k \in K\} =$

$= \cup \{N_j(S_k, M \cap S_k | k \in K\} = N_j(S, M)$. This means that $M \in \mathcal{N}_{i\,j}(S)$ holds.

From the paper [5] we have

Proposition 7. *Let S be a semigroup, S' a subsemigroup of S and $M \subseteq S'$. Then $M \in \mathcal{N}_{2\,3}(S')$ implies $M \in \mathcal{N}_{2\,3}(S)$.*

Now we can prove

Theorem 4. *Let S be a semigroup, $S_k (k \in K)$ subsemigroups of S, $S = \cup \{S_k | k \in K\}$ and $M_k \in \mathcal{N}_{2\,3}(S_k)$ for all $k \in K$. Then $M = \cup \{M_k | k \in K\} \in \mathcal{N}_{2\,3}(S)$.*

Proof. By the assumption $M_k \in \mathcal{N}_{2\,3}(S_k)$ holds for all $k \in K$. Hence Proposition 7 implies that $M_k \in \mathcal{N}_{2\,3}(S)$ for all $k \in K$. Since $\langle \mathcal{N}_{2\,3}(S), \subseteq \rangle$, is a complete upper sublattice of $\langle \mathcal{P}(S), \subseteq >$, $M = n \cup \{M_k | k \in K\} \in \mathcal{N}_{2\,3}(S)$ holds.

Corollary 1. *Let S be a periodic semigroup and every cyclic subsemigroup of S a group. Then $\mathcal{N}_{2\,3}(S) = \mathcal{P}(S)$.*

Proof. $S = \cup \{\langle a \rangle | a \in S\}$, where $\langle a \rangle$ is a cyclic group of finite order. By Corollary of Theorem 1 and by Theorem 4 $\mathcal{N}_{2\,3}(S)$ contains all sets $\{a\}$, $a \in S$. Since $\langle \mathcal{N}_{2\,3}(S), \subseteq \rangle$ is a complete upper sublattice of $\langle \mathcal{P}(S) \rangle, \subseteq$, $\mathcal{N}_{2\,3}(S)$ contains all elements of $\mathcal{P}(S)$.

Corollary 2. *Let S be a band. Then $\mathcal{N}_{2\,3}(S) = \mathcal{P}(S)$.*

Theorem 5. *Let S be a semigroup, $S_k (k \in K)$ subsemigroups of S, $S = \cup \{S_k | k \in K\}$ and $M \subseteq S$. Then $M \in \mathcal{N}_{2\,3}(S)$ iff $M = \cup \{M_k | k \in K\}$ and $M_k \in \mathcal{N}_{2\,3}(S_k)$ for every $k \in K$.*

The proof follows from Theorem 3 and Theorem 4. In the following example it is shown that a similar Theorem does not hold for the other two kinds of lattices.

Example 3. Let $S = S_1 = \langle a \rangle$ be the cyclic semigroup of infinite order and $S_2 = \langle a^2 \rangle = \{a^{2k} | k = 1, 2, 3, \ldots\}$. Then $S = S_1 \cup S_2$. Further let $M_1 = \square$ and $M_2 = \{a^{2n} | n = 2, 3, 4, \ldots\}$. Then $M_1 \subseteq S_1$, $M_2 \subseteq S_2$ and $M = M_1 \cup M_2 = M_2 = \{a^{2n} | n = 2, 3, 4, \ldots\}$.

Since $M_1 = \square$, we have $M_1 \in \mathcal{N}_{1\,3}(S_1)$ and $M_1 \in \mathcal{N}_{1\,2}(S_1)$. The fact that M_2 is a complement of a finite set in S_2 implies that $M_2 \in \mathcal{N}_{1\,3}(S_2)$ and $M_2 \in \mathcal{N}_{1\,2}(S_2)$. But since M is neither a finite set nor a complement of a finite set in S we have $M \notin \mathcal{N}_{1\,3}(S)$ and $M \notin \mathcal{N}_{1\,2}(S)$. Nevertheless the following Thorem holds.

Theorem 6. *Let S be a semigroup, $S_k (k \in K)$ subsemigroups of S. Let $S = \cup \{S_k | k \in K\}$ and every two subsemigroups S_k, $S_l (k, l \in K)$, $k \neq l$ be disjoint. Then $M \in \mathcal{N}_{i\,j}(S)$ iff $M = \cup \{M_k | k \in K\}$ and $M_k \in \mathcal{N}_{i\,j}(S_k)$ for every $k \in K$.*

Proof. With respect to the fact that the subsemigroups $S_k (k \in K)$ are mutually disjoint it is clear that if $M = \cup \{M_k | k \in K\}$, then $M_k = S_k \cap M$ for all $k \in K$. Now it is sufficient to use Theorem 3.

Corollary 1. *Let $\{S_k | k \in K\}$ be a semilattice decomposition of a semigroup S, $M \subseteq S$ and $i, j = 1, 2, 3$, $i < j$. Then $M \in \mathcal{N}_{i\,j}(S)$ iff $M = \cup \{M_k | k \in K\}$ and $M_k \in \mathcal{N}_{i\,j}(S_k)$ for every $k \in K$.*

Proposition 8. *Let $\mathcal{F}_X$ be the free semigroup on a set X. Let $A = \{a \in \mathcal{F}_X | a$ not be a power of any other element of $\mathcal{F}_X\}$. Then $\mathcal{F}_X = \cup \{\langle a \rangle | a \in A\}$ and if $a_1, a_2 \in A$, $a_1 \neq a_2$, then $\langle a_1 \rangle \cap \langle a_2 \rangle = \square$.*

Proof. Let $a_1 = u_1 u_2 \ldots u_m$, $u_1, u_2, \ldots, u_m \in X$, $a_2 = v_1 v_2 \ldots v_n$, $v_1, v_2, \ldots, v_n \in X$, $a_1 \neq a_2$. Let a_1 be a power of no other element of $\mathcal{F}_X$ and a_2 be a power of no other element of $\mathcal{F}_X$.

Let us suppose that $a_1^k = a_2^l$ for some $k, l \in N$. We shall prove that this is impossible. This will imply that $\langle a_1 \rangle \cap \langle a_2 \rangle = \square$.

Let Z be the set of all integers.

We can define two functions:

$f\colon Z \to X$, $f(1) = u_1$, $f(2) = u_2$, $\ldots$, $f(m) = u_m$ and $f(s) = f(s+m)$, for all $s \in Z$. This function is periodic with a positive period m.

$g\colon Z \to X$, $g(1) = v_1$, $g(2) = v_2$, $\ldots$, $g(n) = v_n$ and $g(s) = g(s+n)$, for all $s \in Z$. This function is periodic with a positive period n.

With respect to the condition

$a_1^k = (u_1 u_2 \ \ldots \ u_m)^k = (v_1 v_2 \ \ldots \ v_n)^l = a_2^l$ and since $\mathcal{F}_X$ is a free semigroup, $f(i) = g(i)$, for all $i \in Z$.

The function $f\colon Z \to X$ is therefore periodic and has positive periods m and n.

Since a_1 is not a power of another element of $\mathcal{F}_X$ and a_2 is not a power of another element of $\mathcal{F}_X$, both m and n are the smallest positive periods of the function $f\colon Z \to X$, hence $m = n$.

This means that $a_1 = u_1 u_2 \ldots u_m = v_1 v_2 \ldots v_m = a_2$. But this is a contradiction because we have supposed that $a_1 \neq a_2$.

Corollary 2. *Let $\mathcal{F}_X$ be the free semigroup on a set X, $M \subseteq \mathcal{F}_X$ and $i, j = 1, 2, 3$, $i < j$.*
Then $M \in \mathcal{N}_{i\,j}(\mathcal{F}_X)$ iff $M = \cup \{M_a | a \in A\}$ and $M_a \in \mathcal{N}_{i\,j}(\langle a \rangle)$ for every $a \in A$.

Corollary 3. *Let S be a union of mutually disjoint, cyclic groups of finite order, $G_k = \langle a_k \rangle$ $(k \in K)$ and $j = 2, 3$. Then the following statements hold:*

i) *$M \in \mathcal{N}_{1\,j}(S)$ iff $M = \cup \{\langle a_l \rangle | l \in L\}$ and L is an arbitraly subset of K.*

ii) *$\mathcal{N}_{2\,3}(S) = \mathcal{P}(S)$.*

Proof. The proof of i) follows from the fact that if $\langle a \rangle$ is a cyclic group of finite order, then $\mathcal{N}_{1j}(\langle a \rangle) = \{\square, \langle a \rangle\}$. ii) is a direct consequence of the Corollary of Theorem 1. (See also Corollary 1 of Theorem 4.)

Corollary 4. Let S be a band. Then $\mathcal{N}_{i\,j}(S) = \mathcal{P}(S)$ for $i, j = 1, 2, 3$, $i < j$. (See also Corollary 2 of Theorem 4.)

5. Examples.

We shall give some examples showing how Theorem 6 can be uused.

Example 4. Let G be the cyclic group of infinite order generated by the element a with the identity e. Then G is the union of mutually disjoint, cyclic semigroups $\langle a\rangle$, $\langle e\rangle$, $\langle a^{-1}\rangle$, i. e $G = \langle a\rangle \cup \langle e\rangle \cup \langle a^{-1}\rangle$. We can use Theorem 6 and we get the following results. $M \in \mathcal{N}'_{1,3}(G)$ iff $M = M_1 \cup M_2 \cup M_3$, where $M_1 \in \mathcal{N}'_{1,3}(\langle a\rangle)$, $M_2 \in \mathcal{N}'_{1,3}(\langle e\rangle)$ and $M_3 \in \mathcal{N}'_{1,3}(\langle a^{-1}\rangle)$ holds. This means that $M \in \mathcal{N}'_{1,3}(G)$ iff $M = M_1 \cup M_2 \cup M_3$, where M_1 is either the empty set or M_1 is a complement of a finite subset of $\langle a\rangle$, M_2 is either the empty set or $M_2 = \{e\}$ and M_3 is either the empty set or M_3 is a complement of a finite subset of $\langle a^{-1}\rangle$. $M \in \mathcal{N}'_{1,2}(G)$ iff $M = M_1 \cup M_2 \cup M_3$, where $M_1 \in \mathcal{N}'_{1,2}(\langle a\rangle)$, $M_2 \in \mathcal{N}'_{1,2}(\langle e\rangle)$ and $M_3 \in \mathcal{N}'_{1,3}(\langle a^{-1}\rangle)$. This means that $M \in \mathcal{N}'_{1,2}(G)$ iff $M = M_1 \cup M_2 \cup M_3$, where M_1 is either a finite subset of $\langle a\rangle$ or M_1 is a complement of a finite subset of $\langle a\rangle$, M_2 is either the empty set or $M_2 = \{e\}$ and M_3 is either a finite subset of $\langle a^{-1}\rangle$ or M_3 is a complement of a finite subset of $\langle a^{-1}\rangle$.

Remark 1. Let S_k, $k \in K$, $0 \notin K$ be mutually disjoint semigroups and $S_0 = \{0\}$ be a semigroup disjoint with every semigroup S_k, $k \in K$. Let $S = \cup\{S_k | k \in K\} \cup S_0$. Then S is a semigroup and $\{S_k | k \in K\} \cup S_0$ is a semilattice decomposition of S if $xy = yx = 0$ for $x \in S_i$, $y \in S_j$, $i \neq j$, $i, j \in K \cup \{0\}$ and the multiplication in every semigroup S_l, $l \in K \cup \{0\}$ remains as before.

Example 5. Let the semigroup $S = \cup\{\langle a_i\rangle | i \in I\} \cup \{0\}$ be the union of mutually disjoint cyclic semigroups $\langle a_i\rangle$, $i \in I$ and of the semigroup $\langle 0\rangle$ that is disjoint with every semigroup $\langle a_i\rangle$, $i \in I$. Let $xy = 0$ in the case if x and y belong to distinct subsemigroups of the partition $\{\langle a_i\rangle | i \in I\} \cup \langle 0\rangle$ of the semigroup S. Then we can use Theorem 6 and we have:

i) $M \in \mathcal{N}'_{1,3}(S)$ iff $M = \cup\{M_i | i \in I\} \cup M_0$, where $M_i \in \mathcal{N}'_{1,3}(\langle a_i\rangle)$ and $M_0 \in \mathcal{N}'_{1,3}(\langle 0\rangle)$,

ii) $M \in \mathcal{N}'_{1,2}(S)$ iff $M = \cup\{M_i | i \in I\} \cup M_0$, where $M_i \in \mathcal{N}'_{1,3}(\langle a_i\rangle)$ and $M_0 \in \mathcal{N}'_{1,2}(\langle 0\rangle)$.

Remark 2. Let S_1 and S_2 be disjoint semigroups and $S = S_1 \cup S_2$. Then S is a semigroup and $\{S_1, S_2\}$ is a semilattice decomposition of S if $xy = yx = x$ for all $x \in S_1$ and $y \in S_2$ and the multiplication of two elements of S_1, resp. of two elements of S_2 remains as before.

Example 6. Let $S = \langle a\rangle \cup \langle b\rangle$, where $\langle a\rangle$ and $\langle b\rangle$ are disjoint cyclic semigroups, generated by a and b, respepcively, and $xy = yx = x$ in the case if $x \in \langle a\rangle$ and $y \in \langle b\rangle$.

In this case Theorem 6 can be also used.

Remark 3. Combining the constructions of Remark 1 and Remark 2 we get other semigroups, where Theorem 6 may be used.

6. Semigroup and its homorphic image

Proposition 9. *Let S and S' be semigroups and let $\varphi\colon S \to S'$ be a surjective homomorphism. Let $M \subseteq S$.* If $x \in S$ and $x^n \in M$ *for some* $n \in N$, *then* $(\varphi(x))^n \in \varphi(M)$.

Proof. $(\varphi(x))^n = \varphi(x^n) \in \varphi(M)$.

Proposition 10. *Let S and S' be semigroups and let $\varphi\colon S \to S'$ be a surjective homomorphism.* Let $M' \subseteq S'$. *If $x' \in S'$, $(x')^n \in M'$ for some $n \in N$ and $\varphi(x) = x'$, then $x^n \in \varphi^{-1}(M')$.*

Proof. $\varphi(x^n) = (\varphi(x))^n = (x')^n \in M'$, hence $x^n \in \varphi^{-1}(M')$.

Corollary 1. *Let S and S' be semigroups and $\varphi\colon S \to S'$ be a surjective homomorphism. Let $M' \subseteq S'$. Then for $i = 1, 2, 3$ there holds:*

i) $N_i(S, \varphi^{-1}(M')) = \varphi^{-1}(N_i(S', M'))$,

ii) $N_i(S', M') = \varphi(N_i(S, \varphi^{-1}(M')))$.

We give the proof only for $i = 2$. Let $\varphi^{-1}(M') = M$, then $\varphi(M) = M'$.

a) If $x \in N_2(S, M)$, then $x^n \in M$ holds for infinitely many $n \in N$, hence by Proposition 9 we have $(\varphi(x))^n \in \varphi(M) = M'$ for infinitely many $n \in N$ i.e. $\varphi(x) \in N_2(S', M')$, hence $x \in \varphi^{-1}(N_2(S', M'))$. Therefore $N_2(S, \varphi^{-1}(M')) \subseteq \subseteq \varphi^{-1}(N_2(S', M'))$.

b) Let $x \in \varphi^{-1}(N_2(S', M'))$. This means that $x' = \varphi(x) \in N_2(S', M')$, i.e. $(x')^n = = (\varphi(x))^n \in M'$ for infinitely many $n \in N$. By Proposition 10 we have $x^n \in \varphi^{-1}(M')$ for infinitely many $n \in N$ i.e. $x \in N_2(S, \varphi^{-1}(M'))$. Hence $\varphi^{-1}(N_2(S', M')) \subseteq N_2(S, \varphi^{-1}(M'))$.

This means that $N_2(S, \varphi^{-1}(M')) = \varphi^{-1}(N_2(S', M'))$ and evidently also $N_2(S', M') = \varphi(N_2(S, \varphi^{-1}(M')))$.

Corollary 2. *Let S* and *S' be semigroups and $\varphi\colon S \to S'$ be a surjective homomorphism. Then $\mathcal{N}_{i\,j}(S') = \{M' \subseteq S' \mid \varphi^{-1}(M') \in \mathcal{N}_{i\,j}(S)\}$ holds for $i, j = 1, 2, 3$, $i < j$.*

Proof. a) Let $M' \in \mathcal{N}_{i\,j}(S')$, i.e. $N_i(S', M') = N_j(S', M')$. Then Corollary 1 i) implies that $N_i(S, \varphi^{-1}(M')) = \varphi^{-1}(N_i(S', M')) = \varphi^{-1}(N_j(S', M')) = = N_j(S, \varphi^{-1}(M'))$, hence $\varphi^{-1}(M') \in \mathcal{N}_{i\,j}(S)$.

b) Let $\varphi^{-1}(M') \in \mathcal{N}_{i\,j}(S)$ i.e. $N_i(S, \varphi^{-1}(M')) = N_j(S, \varphi^{-1}(M'))$. Then Corollary 1 ii) implies that $N_i(S', M') = \varphi(N_i(S, \varphi^{-1}(M'))) = \varphi(N_j(S, \varphi^{-1}(M'))) = = N_j(S', M')$ hence $M' \in \mathcal{N}_{i\,j}(S')$.

Remark 4. It is known that every semigroup S is a homomorphic image of some free semigroup $\mathcal{F}_X$ on a set X. This implies the following

Theorem 7. *Let S be a homomorphic image of a free semigroup $\mathcal{F}_X$ on a set X by the homomorphism $\varphi\colon \mathcal{F}_X \to S$. Let $M \subseteq S$. Then the following statements hold:*

i) $N_i(\mathcal{F}_X, \varphi^{-1}(M)) = \varphi^{-1}(N_i(S, M))$,

ii) $N_i(S, M) = \varphi(N_i(\mathcal{F}_X, \varphi^{-1}(M)))$,

iii) $\mathcal{N}_{i,j}(S) = \{M \subseteq S | \varphi^{-1}(M) \in \mathcal{N}_{i,j}(\mathcal{F}_X)\}$, for $i, j = 1, 2, 3$, $i < j$.

7. Application to characterizations of some classes of semigroups

Theorem 8. *Let S be a semigroup. Then the following statements are equivalent:*

i) $\{a\} \in \mathcal{N}_{1\,3}(S)$ *for all* $a \in S$.

ii) *S is a band.*

iii) $\mathcal{N}_{1\,3}(S) = \mathcal{P}(S)$.

Proof. i) $\Rightarrow$ ii). Since $a \in \{a\} \in \mathcal{N}_{1\,3}(S)$, we have $a \in N_3(S, \{a\}) = N_1(S, \{a\})$, hence there exists an $n_0 \in N$ such that $a^n \in \{a\}$ holds for all $n \geq n_0$, i.e. $a^n = a$ for all $n \geq n_0$. Therefore $a = a^{n_0} = a^{n_0+1} = a^{n_0}a = a^2$; i.e. a is an idempotent.
ii) $\Rightarrow$ iii). Let S be a band and let $M \subseteq S$. If $x \in N_3(S, M)$, then $x^{n_0} \in M$ for some $n_0 \in N$. Since x is an idempotent, we have $x^n = x^{n_0} \in M$ for all $n \in N$, hence $x \in N_1(S, M)$. This means that $N_1(S, M) = N_3(S, M)$, therefore $M \in \mathcal{N}_{1\,3}(S)$. (See also Corollary 4 of Theorem 6.)
iii) $\Rightarrow$ i) is evident.

Theorem 9. *Let S be a semigroup. Then the following statements are equivalent:*

i) $\{a\} \in \mathcal{N}_{2\,3}(S)$ *for all* $a \in S$.

ii) *S is a periodic semigroup and each cyclic subsemigroup of it is a group.*

iii) $\mathcal{N}_{2\,3}(S) = \mathcal{P}(S)$.

Proof. i) $\Rightarrow$ ii). From $\{a\} \in \mathcal{N}_{2\,3}S)$ we have $N_2(S, \{a\}) = N_3(S, \{a\})$. Since $a \in N_3(S, \{a\}) = N_2(S, \{a\})$, $a^n = a$ holds for infinitely many $n \in N$, hence $a^n = a$ holds for at least one $n > 1$. This means that $\langle a \rangle$ is a finite, cyclic group.
ii) $\Rightarrow$ iii). Let $M \subseteq S$ and let $x \in N_3(S, M)$. Then $x^{n_0} \in M$ for some $n_0 \in N$. By the assumption $\langle x^{n_0} \rangle$ is a finite cyclic group. Hence there exists an $m \in N$, $m > 1$, such that $x^{n_0 m^k} = (x^{n_0})^{m^k} = x^{n_0} \in M$ for all $k \in N$. This means that $x^n \in M$ for infinitely many $n \in N$, therefore $x \in N_2(S, M)$.

We have $N_3(S, M) \subseteq N_2(S, M)$. This together with $N_2(S, M) \subseteq N_3(S, M)$ gives $N_3(S, M) = N_2(S, M)$ for all subsets $M \subseteq S$. Hence $\mathcal{N}_{2\,3}(S) = \mathcal{P}(S)$.
iii) $\Rightarrow$ i) is evident.

Theorem 10. *Let S be a semigroup. Then the ffollowing statements are equivalent:*

i) $\langle a \rangle \in \mathcal{N}_{1\,2}(S)$ *all* $a \in S$.

ii) *S is a periodic semigroup and each its cyclic subsemigroup has period* 1.

iii) $\mathcal{N}_{1\,2}(S) = \mathcal{P}(S)$.

Proof. i) $\Rightarrow$ ii). Let $a \in S$. By the assumption we have $\langle a^2 \rangle \in \mathcal{N}_{1\,2}(S)$, i.e. $N_1(S, \langle a^2 \rangle) = N_2(S, \langle a^2 \rangle)$. Hence $a \in N_2(S, \langle a^2 \rangle) = N_1(S, D\langle a^2 \rangle)$, therefore there exists an $n_0 \in N$ such that $a^n \in \langle a^2 \rangle$ holds for all $n \geq n_0$. This means that an even power of a is equal to an odd power of a, hence the cyclic semigroup $\langle a \rangle$ is of finite order for every $a \in S$. This means that the semigroup S is periodic.

Since every subsemigroup $\langle a \rangle$ of the semigroup S is of finite order, $\langle a \rangle$ contains an idempotent $e = a^s$. Evidently $\langle a^s \rangle = \{a^s\}$ and by the assumption $\langle a^s \rangle \in \mathcal{N}_{1\,2}(S)$. Hence $a \in N_2(S, \langle a^s \rangle) = N_1(S, \langle a^s \rangle) = N_1(S, \{a^s\})$. Therefore there exists an $n_0 \in N$ such that $a^n = a^s$ holds for all $n \geq n_0$, i.e. $a^{n_0} = a^s = a^{n_0+1}$. This implies that the period of $\langle a \rangle$ is equal to 1, for all $a \in S$.

ii) $\Rightarrow$ iii). Let $M \subseteq S$, S be a periodic semigroup and let every cyclic subsemigroup of S have a period $m = 1$. Let $x \in N_2(S, M)$, i.e. $x^n \in M$ for infinitely many $n \in N$. Let r the index of the semigroup $\langle x \rangle$. Then there exists a $k_0 \in N$ such that $x^r = x^{r+k_0} \in M$. Since the semigroup $\langle x \rangle$ has period $m = 1$, $x^r = x^{r+k} \in M$ for all $k \in N$, hence $x \in N_1(S, M)$. Therefore we have $N_2(S, M) \subseteq N_1(S, M)$. This together with $N_1(S, M) \subseteq N_2(S, M)$ gives $N_1(S, M) = N_2(S, M)$. This means that $M \in \mathcal{N}_{1\,2}(S)$ for all $M \subseteq S$, i.e. $\mathcal{N}_{1\,2}(S) = \mathcal{P}(S)$.

iii) $\Rightarrow$ i) is evident.

Remark 5. If $S = \langle a \rangle$ is the cyclic semigroup of infinite order, then $\{a\} \in \mathcal{N}_{1\,2}(S)$ for all $a \in S$, but $\mathcal{N}_{1\,2}(S) \neq \mathcal{P}(S)$.

Theorem 11. *Let S be a semigroup. Then the following statements are equivalent*:

i) $N_3(S, \{a^2\}) = \{a^2\}$ *for all* $a \in S$.

ii) *S is a band.*

iii) $N_3(S, M) = M$ *for all* $M \subseteq S$.

iv) $N_3(S, \{a\}) = \{a\}$ *for all* $a \in S$.

Proof. i) $\Rightarrow$ ii). If $a \in S$, then $a \in N_3(S, \{a^2\}) = \{a^2\}$. Hence $a = a^2$.

ii) $\Rightarrow$ iii) $\Rightarrow$ iv) $\Rightarrow$ i) is evident.

Theorem 12: *Let S be a semigroup. Then the following statements are equivalent*:

i) *For every $a \in S$ there exists a $k \in N$ such that* $N_2(S, \{a^k\}) = \{a^k\}$.

ii) *S is a band.*

iii) $N_2(S, M) = M$ *for all* $M \subseteq S$.

Proof. Let $a \in S$. Then $a^k \in N_2(S, \{a^k\}) = \{a^k\}$ for some $k \in N$. This means that $a^{kn} = (a^k)^n = a^k$ holds for infinitely many $n \in N$. Hence the semigroup $\langle a^k \rangle$ is a cyclic group with a unit $e = a^{kr} = a^{knr}$ and this equality holds for infinitely many $n \in N$. Therefore $a \in N_2(S, \{e\}) = \{e\}$. Hence $a = e$ and this implies $a^2 = a$.

We have proved i) $\Rightarrow$ ii).

ii) $\Rightarrow$ iii) $\Rightarrow$ i) is evident.

Theorem 13. *Let S be a semigroup. Then the following statements are equivalent*:

i) $N_1(S, \{a\}) = \{a\}$ *for all* $a \in S$.

ii) S *is a band.*

iii) $N_1(S, M) = M$ *for all* $M \subseteq S$.

Proof. i) $\Rightarrow$ ii). If $a \in S$, then $a \in N_1(S, \{a\}) = \{a\}$. Therefore there exists an $n_0 \in N$ such that $a^n = a$ for all $n \geq n_0$. Hence
$a = a^{n_0} = a^{n_0+1} = a^{n_0} a = a a = a^2$. We have $a = a^2$ for all $a \in S$, therefore S is a band.

ii) $\Rightarrow$ iii) $\Rightarrow$ i) is evident.

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Katedra matematiky
Elektrotechnickej fakulty SVŠT
Mlynská dolina, blok A
812 19 Bratislava

НИЛЬПОТЕНТНОСТЬ В ПОЛУГРУППАХ И ТРИ РЕШЕТКИ ИХ БУЛЕАНОВ

Robert Šulka

Резюме

С помощью понятия нильпотентности определены три решетки. Дана конструкция этих решеток для полугрупп, являющихся объединением непересекающихся циклических полугрупп, и характеризация некоторых классов периодических полугрупп.