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LOWER BOUND ON THE DISTANCE k-DOMINATION NUMBER OF A TREE

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ABSTRACT. A subset D of vertices of a graph $G = (V, E)$ is said to be a distance k-dominating set of G if every vertex of $V - D$ is at distance at most k from some vertex of D. The minimum size of a distance k-dominating set of G is called the distance k-domination number of G. We prove that for each tree T of order n with n_1 end-vertices, the distance k-domination number is bounded below by $(n+2k-k\cdot n_1)/(2k+1)$ and we characterize the corresponding extremal trees.

1. Introduction

Let G be a finite simple graph and let $k \geq 1$ be an integer. A set D of vertices of G is said to be *distance k-dominating* if any vertex not in D is within distance k from some vertex of D. The *distance k-domination number* $\gamma_k(G)$ of G is the smallest number of vertices of a k-dominating set in G. Note that the distance 1-domination number is the *domination number* $\gamma(G)$.

This kind of domination was defined by Henning, Oellermann and Swart [1]. As an illustration, let G be a graph associated with the road grid of a city, the vertices of G corresponding to the street intersections. Two vertices of G are adjacent if and only if the corresponding street intersections are adjacent (i.e. block apart). Then $\gamma(G)$ is the smallest number of policemen who may be placed at intersections so that every intersection is at most one block away from a policeman. If the prescribed distance of each intersection (or intersection and policeman) from a policeman is changed from at most one block to at most $k-1$ blocks, $k \geq 2$, then the minimum number of policemen required is $\gamma_k(G)$.

In this paper we consider the distance domination number of non-trivial trees. Let $n = n(T)$ be the order of T and let $n_1 = n_1(T)$ denote the number of end-vertices of T.

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Lemańska [2] proved that each non-trivial tree T satisfies the inequality $\gamma(T) \geq (n + 2 - n_1)/3$ and characterized the extremal trees. The purpose of this paper is to generalize this result to the distance version of the domination number. In particular, we prove that for each tree T of order n the distance k-domination number is bounded below by $(n + 2k - k \cdot n_1)/(2k + 1)$ and we characterize the corresponding extremal trees.

2. Proof of the bound

Our aim in this section is to present and prove a lower bound on the distance k-domination number of a tree.

Lemma 1. *If T is a tree with $\gamma_k(T) = 1$, then $k \cdot n_1(T) \geq n(T) - 1$.*

Proof. We proceed by induction on the number of end-vertices of a tree T. If $n_1(T) = 2$, then T is a path P. As $\gamma_k(T) = 1$, it follows that the path has at most $2k + 1$ vertices. Thus

$$k \cdot n_1(P) = k \cdot 2 = (2k + 1) - 1 \geq n(P) - 1.$$

Assume now that the result is true for all trees T' with $n_1(T') = 2, \ldots, j$ and $\gamma_k(T') = 1$. Let T be a tree with $\gamma_k(T) = 1$ and $n_1(T) = j + 1$. Let p be the smallest integer such that $(x_0, \ldots, x_p)$ is a path in T with $d_T(x_0) = 1$ and $d_T(x_p) > 2$. Observe that $1 \leq p \leq k$, as $\operatorname{diam}(T) \leq 2k$, and T is not a path. Let $V_1 = \{x_0, x_1, \ldots, x_{p-1}\}$. Obviously $|V_1| \leq k$. Let us remove from T all vertices of V_1. By induction we have

$$k \cdot n_1(T - V_1) \geq n(T - V_1) - 1.$$

Since $n_1(T) - 1 = n_1(T - V_1)$ and $n(T - V_1) \geq n(T) - k$, it follows that

$$k \cdot \big(n_1(T) - 1\big) \geq n(T) - k - 1.$$

Thus

$$k \cdot n_1(T) \geq n(T) - 1,$$

which completes the induction step. □

The *open k-neighbourhood* of a vertex $x \in V(G)$, denoted $N_G^k(x)$, is the set $\{v \in V(G) : 0 < d_G(v, x) \leq k\}$. The set $N_G^k[x] = N_G^k(x) \cup \{x\}$ is called the *closed k-neighbourhood* of v in G. Let us define $PN_G^k[x, D] = N_G^k[x] - N_G^k[D - \{x\}]$ to be the *private distance k-neighbourhood of a vertex x, with respect to a set D*. If $y \in PN_G^k[x, D]$, we say that y is a *private distance k-neighbour of x*. The set of end-vertices of G is denoted by $\Omega(G)$.

We continue with a basic property of minimal k-dominating sets, due to Henning, Oellerman and Swart [1].

PROPOSITION 2. *Let D be a distance k-dominating set of a graph G for some $k \geq 1$. Then D is a minimal distance k-dominating set if and only if each vertex $u \in D$ satisfies at least one of the following conditions:*

1. *there exists a vertex $v \in V(G) - D$ for which $N_G^k(v) \cap D = \{u\}$;*
2. *$d_G(u,w) > k$ for every vertex $w \in D - \{u\}$.*

For a given tree T, let $S = (s_0, s_1, \dots, s_l)$ be a longest path in T. Assume $l > 2k$. Let $P_0, P_1, \dots, P_l$ be a partition of $V(T)$ such that

$$P_i = \Big\{ v \in V(T) : \ d_T(v, s_i) = d_T\big(v, V(S)\big) \Big\}.$$

Observe that $d_T(s_i, x) \leq \min\{i, l-i\}$ only if $x \in P_i$, as otherwise there would be a path longer than S in T.

Let D be a minimum distance k-dominating set in T. We say that D has a *property $\mathcal{F}$* if $s_k \in D$ and $\sum_{v \in D} d_T\big(v, V(S)\big)$ is minimum.

LEMMA 3. *If D has the property $\mathcal{F}$, then s_k distance k-dominates all vertices in $P_0 \cup \dots \cup P_k$.*

Proof. As S is a longest path in T and $s_k \in V(S)$, the result is straightforward. □

LEMMA 4. *If D has the property $\mathcal{F}$, then $(P_0 \cup \dots \cup P_k) \cap D = \{s_k\}$.*

Proof. Suppose that the result is not true, i.e. let $x \in (P_0 \cup \dots \cup P_k) \cap D$ and $x \neq s_k$. By Lemma 3, s_k distance k-dominates x. Thus, by Proposition 2, x has a private distance k-neighbour, say y. As y is not distance k-dominated by s_k, we have $y \notin P_0 \cup \dots \cup P_k$ and $d_T(s_k, y) > k$. It follows that $d_T(x,y) = d_T(x, s_k) + d_T(s_k, y) > k$, which is a contradiction with the fact, that y is a private distance k-neighbour of x. Thus such a vertex x does not exist, as claimed. □

LEMMA 5. *If D has the property $\mathcal{F}$ and $x \in D \cap P_i$, then $d_T(s_i, x) \leq i - k$.*

Proof. Suppose to the contrary that D is a minimum distance k-dominating set having the property $\mathcal{F}$ in T and let $x \in D \cap P_i$ such that $d_T(s_i, x) > i - k$. Thus $i - k < d_T(s_i, x) \leq i$. It follows that there exists a vertex y such that $d_T(s_i, x) = d_T(s_i, y) + d_T(y, x)$ and $d_T(s_i, y) = i - k$. Hence $d_T(y, x) = d_T(s_i, x) - d_T(s_i, y) \leq k$.

We claim that $N_T^k[x] \subseteq N_T^k[y]$. Suppose, contrary to our claim, that there exists a vertex z belonging to $N_T^k[x] - N_T^k[y]$. Then from $d_T(z, y) > k$ and $d_T(s_i, y) = i - k$ we find that either $d_T(s_i, z) > i$ or $d_T(s_i, z) < i - 2k$. The inequality $d_T(s_i, z) > i$ implies that there is a longer path than S in T, which is a contradiction. If $d_T(s_i, z) < i - 2k$, then $d_T(x, z) \geq d_T(x, s_i) - d_T(s_i, z) > i - k - (i - 2k) = k$, which contradicts $z \in N_T^k[x]$.

Consequently, the set $D' = D - \{x\} \cup \{y\}$ is a minimum distance k-dominating set in T, where $\sum_{v \in D'} d_T(v, V(S)) < \sum_{v \in D} d_T(v, V(S))$, again a contradiction. □

LEMMA 6. *If $PN_T^k[s_k, D] \cap P_i \neq \emptyset$ for $i \in \{k+1, \dots, 2k\}$, then $D \cap (P_{k+1} \cup \dots \cup P_i) = \emptyset$.*

P r o o f. For a contradiction let us suppose that $x \in PN_T^k[s_k, D] \cap P_i$ for $i \in \{k+1, \dots, 2k\}$ and let there exist a vertex $z \in D \cap P_r$, where $r \in \{k+1, \dots, i\}$. By Lemma 5, $d_T(z, s_r) \leq r - k$. As s_k distance k-dominates x, we have

$$d_T(s_k, x) = d_T(s_k, s_r) + d_T(s_r, x) = r - k + d_T(s_r, x) \leq k\,.$$

Hence, $d_T(s_r, x) \leq 2k - r$. On the other hand

$$d_T(z, x) = d_T(z, s_r) + d_T(s_r, x) \leq (r - k) + (2k - r) = k\,,$$

which means that z distance k-dominates x, a contradiction, as x is a private distance k-neighbour of s_k. □

By Lemmas 4 and 6 we find the following corollary.

COROLLARY 7. *If $PN_k[s_k, D] \cap P_i \neq \emptyset$ for $i \in \{k+1, \dots, 2k\}$, then $D \cap (P_0 \cup \dots \cup P_i) = \{s_k\}$.*

LEMMA 8. *If $PN_T^k[s_k, D] \cap P_i \neq \emptyset$ for $i \in \{k+1, \dots, 2k\}$, then s_k distance k-dominates all vertices in $P_{k+1} \cup \dots \cup P_i$.*

P r o o f. Suppose that the result is not true, i.e. suppose that $x \in PN_T^k[s_k, D] \cap P_i$ for $i \in \{k+1, \dots, 2k\}$ and s_k does not distance k-dominate a vertex $y \in P_j$, where $j \in \{k+1, \dots, i\}$. Then there exists a vertex $z \in D \cap P_r$ such that z distance k-dominates y and z does not distance k-dominate x. Corollary 7 implies that $i < r$. We need to consider the inequality chain: $k < j \leq i < r$. As z distance k-dominates y, we have

$$d_T(z, y) = d_T(z, s_r) + d_T(s_r, s_j) + d_T(s_j, y) = d_T(z, s_r) + r - j + d_T(s_j, y) \leq k\,.$$

As z does not distance k-dominate x we can write that

$$d_T(z, x) = d_T(z, s_r) + d_T(s_r, s_i) + d_T(s_i, x) = d_T(z, s_r) + r - i + d_T(s_i, x) > k\,.$$

Combining these inequalities we obtain

$$d_T(s_j, y) - j < d_T(s_i, x) - i.$$

As s_k distance k-dominates x, we have $d_T(s_k, x) \leq k$ and thus $d_T(s_i, x) \leq 2k - i$. Similarly, as s_k does not distance k-dominate y, we have $d_T(s_j, y) > 2k - j$. Thus

$$-2j + 2k < d_T(s_j, y) - j < d_T(s_i, x) - i \leq 2k - 2i\,,$$

which gives $i < j$, which is impossible. □

From Lemma 3 and Lemma 8 we have:

COROLLARY 9. *If $PN_T^k[s_k, D] \cap P_i \neq \emptyset$ for $i \in \{k+1, \dots, 2k\}$, then s_k distance k-dominates all vertices in $P_0 \cup \dots \cup P_i$.*

Here and subsequently, for a tree T and an edge uv, let T_1 and T_2 be the components of $T - uv$ to which vertices u and v belong, respectively.

LEMMA 10. *If $\gamma_k(T) > 1$, then there exists an edge uv in T such that $\gamma_k(T_1) + \gamma_k(T_2) = \gamma_k(T)$.*

Proof. Let $S = (s_0, s_1, \dots, s_l)$ be a longest path in T. As $\gamma_k(T) > 1$, we have $l > 2k$. Let D be a minimum distance k-dominating set having property $\mathcal{F}$ in T. According to Lemma 4, we conclude that $s_0 \in PN_T^k[s_k, D]$. Let $i = \max\{j : P_j \cap PN_T^k[s_k, D] \neq \emptyset\}$. We need to consider two cases:
Case 1: $i \leq k$.
If $i \leq k$, then we remove the edge $s_k s_{k+1}$ and we obtain two trees: $T_1 = \langle P_0 \cup \dots \cup P_k \rangle$ and $T_2 = \langle P_{k+1} \cup \dots \cup P_l \rangle$. By Lemma 4, $(P_0 \cup \dots \cup P_k) \cap D = \{s_k\}$ and by Lemma 3, s_k distance k-dominates all vertices in T_1. Moreover, s_k has no private distance k-neighbour among vertices of $P_{k+1} \cup \dots \cup P_l$ in T. Therefore $\gamma_k(T_1) = 1$ and $\gamma_k(T_2) = \gamma_k(T) - 1$.
Case 2: $k < i \leq 2k$.
If $k < i \leq 2k$, then we remove the edge $s_i s_{i+1}$ and we obtain two trees: $T_1 = \langle P_0 \cup \dots \cup P_i \rangle$ and $T_2 = \langle P_{i+1} \cup \dots \cup P_l \rangle$. According to Corollary 7, $(P_0 \cup \dots \cup P_i) \cap D = \{s_k\}$ and by Corollary 9, s_k distance k-dominates all vertices in T_1 and s_k has no private distance k-neighbour among vertices of $P_{i+1} \cup \dots \cup P_l$ in T. Therefore $\gamma_k(T_1) = 1$ and $\gamma_k(T_2) = \gamma_k(T) - 1$. □

We are now in position to prove the main result of this paper.

THEOREM 11. *If T is a tree, then*

$$k \cdot n_1(T) \geq n(T) + 2k - (2k+1)\gamma_k(T).$$

Proof. We proceed by induction on $\gamma_k(T)$. If $\gamma_k(T) = 1$, the result follows by Lemma 1.

We assume now that the result is true for all trees T' with $\gamma_k(T') = 1, \dots, j$. Let T be a tree with $\gamma_k(T) = j + 1$. Let $S = (s_0, s_1, \dots, s_l)$ be a longest path in T. Note that $l > 2k$, as $\gamma_k(T) > 1$. Let D be a minimum distance k-dominating set in T such that $s_k \in D$ and $\sum_{v \in D} d_T(v, V(S))$ is minimum. From Lemma 10 there exists an edge uv of T such that for the two components T_1, T_2 of $T - uv$, we have $\gamma_k(T) = \gamma_k(T_1) + \gamma_k(T_2)$. By induction we have the following inequalities for T_1 and T_2:

$$k \cdot n_1(T_1) \geq n(T_1) + 2k - (2k+1)\gamma_k(T_1)$$

and

$$k \cdot n_1(T_2) \geq n(T_2) + 2k - (2k+1)\gamma_k(T_2).$$

Summing those inequalities we obtain

$$k \cdot \big(n_1(T_1) + n_1(T_2)\big) - 2k \geq n(T) + 2k - (2k+1)\gamma_k(T).$$

Observe that $n_1(T) \geq n_1(T_1) + n_1(T_2) - 2$, so

$$k \cdot n_1(T) \geq n(T) + 2k - (2k+1)\gamma_k(T),$$

which completes the proof of the inequality. □

3. Characterization of the extremal trees

We are now able to provide a characterization of all the trees for which $k \cdot n_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$. For this purpose, we define a family $\mathcal{R}$ to be a family of all trees for which $d_T(u,v) = 2k \mod (2k+1)$ for each two end-vertices u, v, where $u \neq v$.

For a given integer $j \geq 2$, a *k-spider* is a graph obtained by attaching j disjoint paths of length k to a single vertex of K_1.

Let T be a tree belonging to the family $\mathcal{R}$ and let $S = (s_0, s_1, \ldots, s_l)$ be a longest path in T. Since s_0, s_l are end-vertices of T, we conclude that $l = t(2k+1) + 2k$ for some non-negative integer t. Let $I = \{0, l\} \cup \big\{j(2k+1) + k : j = 0, 1, \ldots, t\big\}$ be a set of indexes.

Lemma 12. *If T is a tree belonging to the family $\mathcal{R}$ and $S = (s_0, s_1, \ldots, s_l)$ is a longest path in T, then $P_i \cap \Omega(T) = \emptyset$ for $i \notin I$.*

Proof. Suppose that x is an end-vertex belonging to P_i and $i \notin I$. As $T \in \mathcal{R}$, we conclude that $d_T(s_0, x) = t_1(2k+1) + 2k$ for some non-negative integer t_1 and thus $d_T(s_i, x) = t_1(2k+1) + 2k - i$. On the other hand we have

$$\begin{aligned} d_T(s_l, x) &= d_T(s_l, s_i) + d_T(s_i, x) = t(2k+1) + 2k - i + t_1(2k+1) + 2k - i \\ &= t_2(2k+1) + 4k - 2i, \end{aligned}$$

where $t_2 = t + t_1$. As $T \in \mathcal{R}$, it follows that $d_T(s_l, x) = 2k \mod (2k+1)$. Hence, it is required to be $4k - 2i = t_3(2k+1) + 2k$, where t_3 is an integer and thus $i = k - \frac{t_3}{2}(2k+1)$. Since i is a positive integer, t_3 must be 0 or a negative even number. For $t_3 = 0, -2, -4, \ldots$, we obtain $i = k, (2k+1)+k, 2(2k+1)+k, \ldots$, respectively, which are elements of I, and it is a contradiction with our assumptions. □

COROLLARY 13. *If $T \in \mathcal{R}$, then for all vertices of a longest path $S = (s_0, \dots, s_l)$ there holds $d_T(s_i) = 2$ only if $i \notin I$.*

LEMMA 14. *If $T \in \mathcal{R}$ and $\gamma_k(T) = 1$, then*

$$k \cdot n_1(T) = n(T) - 1 .$$

Proof. Let $S = (s_0, s_1, \dots, s_l)$ be a longest path in a tree T belonging to the family $\mathcal{R}$. Since $\gamma_k(T) = 1$, we have $d_T(s_0, s_l) \leq 2k$. As $T \in \mathcal{R}$, we conclude that $d_T(s_0, s_l) = 2k$ and thus $l = 2k$. Moreover, $d_T(s_0) = d_T(s_{2k}) = 1$ and, by Corollary 13, $d_T(s_i) = 2$ for $i \notin \{0, k, 2k\}$. Hence, if $d_T(s_k) = 2$, then T is a path P on $2k+1$ vertices. In this case

$$k \cdot n_1(T) = k \cdot 2 = (2k+1) - 1 = n(T) - 1 ,$$

so the equality holds. Otherwise, $d_T(s_k) = j > 2$ and $T \in \mathcal{R}$ imply that T is a k-spider. In this case we have

$$k \cdot n_1(T) = k \cdot j = (k \cdot j + 1) - 1 = n(T) - 1 ,$$

and the equality holds as well. □

LEMMA 15. *If $T \in \mathcal{R}$, then*

$$k \cdot n_1(T) = n(T) + 2k - (2k+1)\gamma_k(T) .$$

Proof. We proceed by induction on $\gamma_k(T)$. If $\gamma_k(T) = 1$, then by Lemma 14 the equality holds.

Assume now that the result is true for all tress T' belonging to the family $\mathcal{R}$ with $\gamma_k(T') = 1, \dots, j$. Let $T \in \mathcal{R}$ be a tree with $\gamma_k(T) = j+1$ and let $S = (s_0, s_1, \dots, s_l)$ be a longest path in T. Let D be a minimum distance k-dominating set with property $\mathcal{F}$ in T. Lemma 12 implies that each vertex s_i where $i \notin I$ has degree 2. Thus, without loss of generality, we may assume that $\{s_{k+1}, \dots, s_{3k}\} \cap D = \emptyset$ and $s_{3k+1} \in D$. Now we remove the edge $s_{2k}s_{2k+1}$ from T to obtain trees $T_1 = \langle P_0 \cup \dots \cup P_{2k} \rangle$ and $T_2 = \langle P_{2k+1} \cup \dots \cup P_l \rangle$. It is clear that $\gamma_k(T_1) = 1$ as s_k distance k-dominates all vertices in T_1. Moreover, $\gamma_k(T_2) = \gamma_k(T) - 1$ as s_k has no private distance k-neighbour in $P_{2k+1} \cup \dots \cup P_l$. Furthermore, as $d_T(s_{2k}) = d_T(s_{2k+1}) = 2$ and $d_{T_1}(s_{2k}) = d_{T_2}(s_{2k+1}) = 1$, we conclude that $n_1(T_1) + n_1(T_2) = n_1(T) + 2$.

We claim that $T_1 \in \mathcal{R}$. Indeed, $d_T(s_0, s_{2k}) = 2k$. Moreover, as $T \in \mathcal{R}$, we have $d_T(s_0, x) = 2k$ only if $x \in \Omega(T) \cap P_k$. Thus $d_T(x, s_{2k}) = 2k$. In this case T_1 is a k-spider. If $\Omega(T) \cap P_k = \emptyset$, then T is a path on $2k+1$ vertices. Hence $T_1 \in \mathcal{R}$.

We also claim that $T_2 \in \mathcal{R}$. From $T \in \mathcal{R}$ it follows that $d_T(s_0, s_l) = 2k \mod (2k+1)$. Thus $d_T(s_{2k+1}, x) = d_T(s_0, x) - (2k+1) = 2k \mod (2k+1)$ for each $x \in \Omega(T_2)$. Hence $T_2 \in \mathcal{R}$.

By induction, for T_1 and T_2 we have equalities

$$k \cdot n_1(T_1) = n(T_1) + 2k - (2k+1)\gamma_k(T_1)$$

and

$$k \cdot n_1(T_2) = n(T_2) + 2k - (2k+1)\gamma_k(T_2).$$

Summing these equalities, we obtain

$$k \cdot \big(n_1(T_1) + n_1(T_2)\big) - 2k = n(T) + 2k - (2k+1)\gamma_k(T)$$

and from $n_1(T_1) + n_1(T_2) - 2 = n_1(T)$,

$$k \cdot n_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$$

and the induction is completed. □

Lemma 16. *If $\gamma_k(T) = 1$ and $k \cdot n_1(T) = n(T) - 1$, then $T \in \mathcal{R}$.*

Proof. We use induction on the number of end-vertices of a tree T. If $n_1(T) = 2$ and $k \cdot 2 = n(T) - 1$, T is a path P on $2k+1$ vertices. In this case $d_T(u,v) = 2k$ for the two end-vertices in P, so $T \in \mathcal{R}$.

Assume now that the result is true for all trees T' with $n_1(T') = 2, \dots, j$. Let T be a tree with $n_1(T) = j+1$, $\gamma_k(T) = 1$ and $k \cdot n_1(T) = n(T) - 1$. Let p be the smallest integer such that $(x_0, \dots, x_p)$ is a path in T, where $d_T(x_0) = 1$ and $d_T(x_p) > 2$. Observe that $1 \le p \le k$, as $\operatorname{diam}(T) \le 2k$ and T is not a path. Let $V_1 = \{x_0, x_1, \dots, x_{p-1}\}$. Obviously $|V_1| \le k$. Let T' be a tree obtained from T by removing of all vertices belonging to V_1. According to Theorem 11, we have inequality

$$k \cdot n_1(T') \ge n(T') + 2k - (2k+1)\gamma_k(T').$$

As $n_1(T') = n_1(T) - 1$, $\gamma_k(T') = 1$ and $|V_1| \le k$, we obtain

$$k \cdot n_1(T) - k = k \cdot n_1(T') \ge n(T') - 1 \ge n(T) - k - 1.$$

Since $k \cdot n_1(T) = n(T) - 1$, it follows that

$$k \cdot n_1(T) - k = k \cdot n_1(T') = n(T') - 1 = n(T) - k - 1.$$

Thus $|V_1| = k$ and $k \cdot n_1(T') = n(T') - 1$. By induction we find that $T' \in \mathcal{R}$. As $\gamma_k(T) = \gamma_k(T') = 1$ and $|V_1| = k$, we conclude that T is a k-spider and thus $T \in \mathcal{R}$. □

Lemma 17. *If*

$$k \cdot n_1(T) = n(T) + 2k - (2k+1)\gamma_k(T),$$

then $T \in \mathcal{R}$.

Proof. We proceed by induction on $\gamma_k(T)$. If $\gamma_k(T) = 1$, then by Lemma 16 the equality holds.

Assume now that the result is true for all trees T' with $\gamma_k(T') = 1, \dots, j$. Let T be a tree with $\gamma_k(T) = j+1$ and let $k \cdot n_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$.

By Lemma 10, there exists an edge $uv \in E(T)$ such that $T - uv$ has two components T_1 and T_2 and $\gamma_k(T_1) + \gamma_k(T_2) = \gamma_k(T)$. Theorem 11 implies that

$$k \cdot n_1(T_1) \geq n(T_1) + 2k - (2k+1)\gamma_k(T_1) \tag{1}$$

and

$$k \cdot n_1(T_2) \geq n(T_2) + 2k - (2k+1)\gamma_k(T_2). \tag{2}$$

By summing (1) and (2) and applying $n_1(T) \geq n_1(T_1) + n_1(T_2) - 2$ we obtain

$$k \cdot n_1(T) \geq k \cdot \big(n_1(T_1) + n_1(T_2)\big) - 2k \geq n(T) + 2k - (2k+1)\gamma_k(T).$$

As $k \cdot n_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$, we conclude that

$$k \cdot n_1(T) = k \cdot \big(n_1(T_1) + n_1(T_2)\big) - 2k = n(T) + 2k - (2k+1)\gamma_k(T),$$

which implies that in (1) and (2) we have equalities and $n_1(T) = n_1(T_1) + n_1(T_2) - 2$. Thus, by induction, $T_1, T_2 \in \mathcal{R}$. Moreover, if $e = uv$ was the edge we removed from T to obtain T_1 and T_2, then $d_{T_1}(u) = d_{T_2}(v) = 1$. It follows that $d_{T_1}(u,x) = 2k \mod (2k+1)$ for any $x \in \Omega(T_1)$ and $d_{T_2}(v,y) = 2k \mod (2k+1)$ for any $y \in \Omega(T_2)$. Hence $d_T(x,y) = 2k \mod (2k+1)$ for all $x, y \in \Omega(T)$ and thus $T \in \mathcal{R}$. □

By Lemmas 15 and 17 we have the following:

THEOREM 18. *If T is a tree, then*

$$k \cdot n_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$$

if and only if T belongs to the family $\mathcal{R}$.

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