Mathematica Slovaca

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Mathematica Slovaca, Vol. 30 (1980), No. 2, 157--162

Persistent URL: http://dml.cz/dmlcz/130779

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SOME PROPERTIES OF OSCILLATION

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In this article we shall deal with some properties of oscillation. This function was introduced in connection with the notion of continuity of functions (see [1], p. 120). The set $M(X)$ of all real locally bounded functions defined on a metric space (X, ϱ) will be investigated. The set $M(X)$ can be considered as a metric space with the metric $\sigma(f, g) = \min\{\sup_{x \in X} |f(x) - g(x)|, 1\}$. Further the map o: $M(X) \to M(X)$ given by the formula $o(f, x) = \inf_{\delta > 0} d(f(K(x, \delta)))$, where $K(x, \delta) = \{y: \varrho(x, y) < \delta\}$, $d(A) = \sup_{x, y \in A} \{|x - y|\}$ and $o(f, x)$ is the value of $o(f)$ at $x \in X$, will be called oscillation. We shall deal with the sets $C = \{f \in M(X): o(f)$ is continuous$\}$ and $S = \{f \in M(X): o(f) = f\}$. It will be shown that the oscillation determines a natural decomposition of $M(X)$. Under the assumption that X is a Baire space the characterization of classes of this decomposition will be given. For the Baire space X we shall also give a characterization of the set $o(M(X))$.

Let $f \in M(X)$. It is known that if for each cluster point x of X ($x \in X^d$) we put $f^-(x) = \max\{f(x), \limsup_{t \to x} f(t)\}$, $f_-(x) = \min\{f(x), \liminf_{t \to x} f(t)\}$, and $f^-(x) = f_-(x) = f(x)$ for each isolated point x of X, then the function $f^-: M(X) \to M(X)$ ($f_-: M(X) \to M(X)$) is upper (lower) semicontinuous and $o(f) = f^- - f_-$ (see [1], pp. 128 and 131).

Theorem 1. *Let $f \in M(X)$.*

(i) *Then $o(f)$ is continuous if and only if f^- and f_- are continuous;*

(ii) *Then the following statements are equivalent:*

a) $o(f) = f$;

b) *f is upper semicontinuous* (usc), *$\liminf_{t \to x} f(t) = 0$ for each $x \in X^d$ and*

$$f(x) = 0 \text{ for each } x \notin X^d;$$

c) *$f^- = f$ and $f_- = 0$;*

(iii) *Let X be a Baire space. Then $o(f) = f$ if and only if f is* usc *and there exists a residual set $R \subset X$ ($R = R_f$) such that R contains every isolated point of X and $f(x) = 0$ for each x in R.*

Proof. (i): The sufficiency of the above mentioned condition for the continuity of $o(f)$ is obvious. If $o(f)$ is continuous, then the usc function $f^{-}=o(f)+f$ as the sum of lower semi continuous (lsc) functions is also lsc, hence it is continuous. The continuity of f_{-} can be shown analogously.

(ii): a)$\rightarrow$b). Let $o(f)=f$. Since $f=f^{-}-f_{-}$, the function f is usc. The necessity of the condition $\liminf_{t\rightarrow x} f(t)=0$ for each $x\in X^d$ can be proved by contradiction. Since $o(f,t)\geqslant 0$, we have $f(t)\geqslant 0$ for each $t\in X$, hence $\liminf_{t\rightarrow x} f(t)\geqslant 0$. Let us suppose $\liminf_{t\rightarrow x} f(t)=\alpha>0$. The usc of f at x implies $f(x)\geqslant\liminf_{t\rightarrow x} f(t)$, consequently $o(f,x)=f^{-}(x)-f_{-}(x)=f(x)-\alpha<f(x)$, a contradiction. The equality $f(x)=0$ in each isolated point of X is an immediate consequence of the continuity of f in x.

b)$\rightarrow$c). The usc of f implies $f^{-}=f$. Conditions $\liminf_{t\rightarrow x} f(t)=0$ for $x\in X^d$ and $f(x)=0$ for $x\notin X^d$ imply $f_{-}=0$.

c)$\rightarrow$a). $o(f)=f^{-}-f_{-}=f-0=f$.

(iii): Let $o(f)=f$. The necessity of the usc of f follows from (ii). Since f is Baire 1 function the set D_f of all its discontinuity points is of first category (see [1], p. 182). The complement R of D_f is a residual set which obviously contains all isolated points of X. From the continuity of f on R it follows that $f(x)=o(f,x)=0$ for each $x\in R$. The mentioned conditions are also sufficient. The residual set R in the Baire space X is dense. Since f is usc and $f(x)=0$ for $x\in R$, we have $f(x)\geqslant 0$ for each $x\in X$. Hence $\liminf_{t\rightarrow x} f(t)=0$ holds for each $x\in X^d$. The equality $f(x)=0$ for $x\notin X^d$ is warranted by the inclusion $X-X^d\subset R$. Since the conditions of part (ii) b) are fulfilled, part (iii) is also proved.

Theorem 2. *Let (X,ϱ) be a metric space, which has at least one cluster point. Then the sets $C=\{f\in M(X): o(f)$ is continuous$\}$ and $S=\{f\in M(X): o(f)=f\}$ are perfect and nowhere dense in $(M(X),\sigma)$.*

Proof. Let $f\notin C$. Then there exists a discontinuity point x of $o(f)$. Hence $x\in X^d$ and there is a sequence $\{x_n\}_1^{\infty}, x_n\rightarrow x, x_n\neq x$ such that $\lim_{n\rightarrow\infty} o(f,x_n)=\beta<o(f,x)$. If we put $6\varepsilon\leqslant o(f,x)-\beta$, $0<\varepsilon<1$, then $\limsup_{t\rightarrow x} o(g,x_n)<o(g,x)$ holds for each function $g\in K(f,\varepsilon)$ and the point x is a discontinuity point of $o(g)$. Consequently, $K(f,\varepsilon)\subset M(X)-C$ and the set C is closed in $M(X)$. Let c be any real number. Since $f\in C$ if and only if $f+c\in C$, the set C is dense in itself.

Let $x\in X^d$, $f\in C$ and $K(f,\varepsilon)$ be any sphere. From the definition of $o(f,x)$ it follows that for each natural number n, y_n and z_n exist such that $y_n, z_n\in K(x,n^{-1})$

and $f(z_n)-f(y_n)>o(f,x)-n^{-1}$. The sequence $\{z_n\}_1^\infty$ can be found such that $z_n\neq z_m$ whenever $n\neq m$. Let us define a function g in the following way: $g(z_{2n})=f(z_{2n})+\varepsilon 2^{-1}$ for $n=1,2,\ldots$ and $g(t)=f(t)$ for $t\neq z_{2n}$. We have $g(z_{2n})-g(y_{2n})=f(z_{2n})+\varepsilon 2^{-1}-f(y_{2n})>\varepsilon 2^{-1}+o(f,x)-(2n)^{-1}$, $d(g(K(x,(2n)^{-1})))>\varepsilon 2^{-1}+o(f,x)-(2n)^{-1}$, $o(g,x)\geq\varepsilon 2^{-1}+o(f,x)$. Since $z_{2n-1}\to x$ and $\lim_{n\to\infty}o(g,z_{2n-1})=\lim_{n\to\infty}o(f,z_{2n-1})=o(f,x)<o(g,x)$, the point x is a discontinuity point of $o(g)$, $g\notin C$. Hence the interior of the closed set C is void, i.e. C is a nowhere dense set.

Let $f\notin S$. Then according to Theorem 1 (ii) f is not usc, or $x\in X^d$ exists such that $\liminf_{t\to x}f(t)\neq 0$, or there is a point $x\notin X^d$ such that $f(x)\neq 0$. If f is not usc, then there is $x\in X^d$ with the property $f(x)<\limsup_{t\to x}f(t)=\beta$. If we put $3\varepsilon\leq\beta-f(x)$, $0<\varepsilon<1$, then obviously each function $g\in K(f,\varepsilon)$ is not usc. If $x\in X^d$ exists such that $\liminf_{t\to x}f(t)=\alpha\neq 0$, then $\liminf_{t\to x}g(t)\neq 0$ holds for each $g\in K(f,\varepsilon)$ ($\varepsilon\leq|\alpha|2^{-1}$, $0<\varepsilon<1$) and $g\notin S$. If $f(x)\neq 0$ for a point $x\notin X^d$, then also $g(x)\neq 0$ for each $g\in K(f,\varepsilon)$, $\varepsilon\leq|f(x)|2^{-1}$, $0<\varepsilon<1$, and again $g\notin S$. It follows from the above mentioned considerations that S is closed in $M(X)$. Let $x\in X^d$ and c be a positive real number. If $f\in S$, then also $f_c\in S$, where f_c is defined in the following way: $f_c(x)=f(x)+c$ and $f_c(t)=f(t)$ for $t\neq x$. Hence the set S is dense in itself.

Let $f\in S$ and $K(f,\varepsilon)$ be any sphere. The function $g(g(t)=f(t)+\varepsilon 2^{-1}$, $t\in X)$ belongs to $K(f,\varepsilon)$, but $g\notin S$. Consequently, S is a nowhere dense set in $M(X)$.

Remark. From the proof of Theorem 2 it follows that the assumption $X^d\neq\emptyset$ was used to prove that C is a nowhere dense set and S is dense in itself. If $X^d=\emptyset$, then the space (X,ϱ) has a discrete topology and each function f in $M(X)$ is continuous. Hence $C(=M(X))$ is not a nowhere dense set. The unique function f for which the equality $o(f)=f$ is fulfilled is the function $f=0$ and hence the set S is not dense in itself.

It is possible, by using the map o, to assign to each $f\in M(X)$ a sequence of functions $\{f_n\}_{n=1}^\infty$, $f_n\in M(X)$ in the following way: $f_1=f$, $f_n=o(f_{n-1})$ for $n>1$.

Theorem 3. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions which is assigned to $f\in M(X)$ in the above mentioned way. Then $f_n=f_3$ holds for each $n>3$.*

Proof. To prove Theorem 3 it is sufficient to show that $f_3\in S$. We show that the conditions of Theorem 1 (ii) are fulfilled. The functions $f_2=o(f)$ and $f_3=o(f_2)$ are usc and non-negative. If $x\in X^d$, then $f_3(x)=f_2^-(x)-f_{2-}(x)=f_2(x)-\liminf_{t\to x}f_2(t)=f_2(x)-\lim_{k\to\infty}f_2(x_k)$, where $\{x_k\}_{k=1}^\infty$ is a suitably choosen sequence, $x_k\to x$. The equality $\liminf_{t\to x} f_3(t)=0$ will be proved by contradiction. Let $\liminf_{t\to x} f_3(t) = \alpha>0$. Then $\liminf_{k\to\infty} f_3(x_k)\geqslant\alpha$ and we can suppose $x_k\in X^d$ $(k=1, 2, \ldots)$. Hence there is k_0 such that for each $k\geqslant k_0$ we have $f_2(x_k)-\lim_{n\to\infty} f_2(x_k^{(n)}) = f_3(x_k)\geqslant\alpha 2^{-1}$, where $\{x_k^{(n)}\}_{n=1}^\infty$ is a sequence with the properties $x_k^{(n)}\to x_k$ and $\lim_{n\to\infty} f_2(x_k^{(n)})=\liminf_{t\to x_k} f_2(t)$. Consequently, for each $k\geqslant k_0$ there is n_k such that $f_2(x_k)-f_2(x_k^{(n_k)})\geqslant\alpha 4^{-1}$ and $\varrho(x_k, x_k^{(n_k)})<k^{-1}$. Obviously $x_k^{(n_k)}\to x$ and from the ineqauality $f_2(x_k^{(n_k)})+\alpha 4^{-1}\leqslant f_2(x_k)$ it follows that there is a subsequence $\{y_m\}_{m=1}^\infty$ of the sequence $\{x_k^{(n_k)}\}_{k=1}^\infty$ such that $\lim_{m\to\infty} f_2(y_m)+\alpha 4^{-1} \leqslant \liminf_{t\to x} f_2(t)$, a contradiction. If $x\notin X^d$, then obviously $f_3(x)=0$.

Theorem 3 enables us to introduce the following decomposition of $M(X)$.

Definition. *Let $f\in M(X)$ and let $\{f_n\}_{n=1}^\infty$ be the sequence determined by f. It is said that the function f belongs to the class O_i $(i=1, 2, 3)$ if i is the smallest index with the property: $f_n=f_i$ holds for each $n>i$.*

From the definition of S it follows immediately that $O_1=S$. In the following there will be given a characterization of O_2 and O_3 for the case when X is a Baire space.

Lemma. *Let X be a Baire space, $f\in M(X)$ and $D_f\subset X$ be the set of all discontinuity points of f. Then the following statements are equivalent:*

a) $o(f)\in S$;

b) *D_f is of first category in X;*

c) *D_f has the dense complement in X.*

Proof. a)$\to$b). By contradiction. Let D_f be of second category in X. Then $D_f=\bigcup_{n=1}^\infty A_n$, $A_n=\{x\in D_f\colon o(f, x)\geqslant n^{-1}\}$, and some of the sets A_n, e.g. A_m, is of second category in X. Obviously $A_m\subset X^d$. The set A_m is closed (see [1], p. 120) and it has a non—empty interior. Hence there is $x\in X^d$ and $\delta>0$ such that $K(x, \delta)\subset A_m$. Consequently, $\liminf_{t\to x} o(f, t)\geqslant m^{-1}$ and $o(f)\notin S$ according to Theorem 1 (ii).

The implication b)$\to$c) is an immediate consequence of the assumption that X is a Baire space.

c)$\to$a). The function $o(f)$ is obviously usc and $o(f, t)\geqslant 0$ holds for each $t\in X$. If $x\in X^d$, then $\liminf_{t\to x} o(f, t)=0$, because $o(f, t)=0$ for each t of a dense set $X-D_f$. If $x\notin X^d$, then obviously $x\notin D_f$ and $o(f, x)=0$. Consequently, $o(f)\in S$ according to Theorem 1 (ii).

Theorem 4. *Let X be a Baire space and let O_2 and O_3 be introduced classes of functions. Then*

(i) *$f \in O_2$ if and only if D_f is of first category, or it has a dense complement in X, and $f \notin O_1$;*

(ii) *$f \in O_3$ if and only if D_f is of second category, or its complement is not dense in X.*

Since $o(f) \in S$ if and only if $f \in O_1 \cup O_2$, the statement of Theorem 4 is an immediate consequence of Lemma.

In the following a characterization of the set $o(M(X))$ for a Baire space X will be given.

Theorem 5. *Let X be a Baire space. Then*

(i) *for a function $g \in M(X)$ there is a function $f \in M(X)$ such that $o(f) = g$ if and only if*
   a) *g is non-negative,*
   b) *g is* usc, *and*
   c) *$g(x) = 0$ for each $x \notin X^d$;*

(ii) *the set $o(M(X))$ is perfect and nowhere dense in $(M(X), \sigma)$, whenever $X^d \neq \emptyset$.*

Proof. (i): The necessity of the conditions a), b) and c) is obvious. They are also sufficient. Really, since the function g is Baire 1, the set of all its discontinuity points D_g is of first category and according to the assumptions of Theorem 5 the set $C_g = X - D_g$ is dense in X. The set C_g is dense in each open subset of X, hence also in the set $Y = X - (X - X^d)^-$ (Z^- means the closure of Z). Since $Y \subset X^d$, there are disjoint sets C_1 and C_2, both dense in Y such that $Y \cap C_g = C_1 \cup C_2$ (see [2]). Consequently, the set X can be expressed as the following sum of mutually disjoint summands $X = C_1 \cup C_2 \cup (Y - C_g) \cup (X - X^d)^-$.

Let f be a real function defined on X in the following way: $f(x) = 0$ for $x \in C_1$ and $f(x) = g(x)$ for $x \in X - C_1$. Obviously $f \in M(X)$. We show $o(f) = g$. First $\liminf_{t \to x} f(t) = 0$ for each $x \in X^d$, since $C_1 \cup (X - X^d)$ is dense in X and $f(t) = 0$ holds for each $t \in C_1 \cup (X - X^d)$. If $x \in Y$, then $g(x) = \lim_{n \to \infty} f(t_n)$, where $\{t_n\}_{n=1}^{\infty}$, $t_n \to x$, is a suitable chosen sequence of elements of $Y - C_1$. Hence $o(f, x) \geq g(x)$. Since g is usc in x for each $\varepsilon > 0$, there is a sphere $K(x, \delta)$ such that $g(y) < g(x) + \varepsilon$ holds for every $y \in K(x, \delta)$. The ineqalities $0 \leq f(t) \leq g(t)$ hold for each $t \in X$ and so $d(f(K(x, \delta))) \leq g(x) + \varepsilon$, hence $o(f, x) = g(x)$. If $x \in (X - X^d)^- \cap X^d$, then $f(x) = g(x) \geq \limsup_{t \to x} g(t) \geq \limsup_{t \to x} f(t) \geq \liminf_{t \to x} f(t) = 0$ and again $o(f, x) = g(x)$.

(ii): This part of statement of Theorem 5 can be proved by using a characterization of the set $o(M(X))$ given in part (i), by the method used in the proof of Theorem 2.

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Received March 2, 1978

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О НЕКОТОРЫХ СВОЙСТВАХ ОСЦИЛЯЦИИ

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Резюме

В работе рассматривается множество $M(X)$ всех вещественных локально ограниченных функций определенных в метрическом пространстве X (X не является дискретным топологическим пространством). Множество $M(X)$ снабжено метрикой равномерной сходимости. Рассматривается отображение $o: M(X) \to M(X)$, которое каждой функции f из $M(X)$ ставит в соответствие ее осциляцию $o(f)$. Показано, что множества $\{f: o(f)$ непрерывна$\}$ и $\{f: o(f) = f\}$ являются вообще совершенными и нигде не плотными в $M(X)$. Дана также характеризация множества $o(M(X))$, в случае, когда X является пространством Бэра.