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Mathematica Slovaca, Vol. 55 (2005), No. 4, 399--407

Persistent URL: http://dml.cz/dmlcz/130795

Math. Slovaca, **55** (2005), No. 4, 399–407

Mathematica Slovaca

DIRECT PRODUCT FACTORS IN GMV-ALGEBRAS

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(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. GMV-algebras are non-commutative generalizations of MV-algebras and by A. Dvurečenskij they can be represented as intervals of unital lattice ordered groups. Moreover, they are polynomially equivalent to dually residuated ℓ-monoids ($DR\ell$-monoids) from a certain variety of $DR\ell$-monoids. In the paper, using these correspondences, direct product factors in GMV-algebras are introduced and studied and the lattices of direct factors are described. Further, the polars of projectable GMV-algebras are described.

1. Introduction

The Łukasiewicz infinite valued propositional logic is one of the most important logics behind the theory of fuzzy sets. It is well known that MV-algebras introduced by C. C. Chang in [2] are an algebraic counterpart of the Łukasiewicz logic. Recently the first author in [14] and, independently, G. Georgescu and A. Iorgulescu in [7], have introduced non-commutative generalizations of MV-algebras (non-commutative MV-algebras in [14] and pseudo MV-algebras in [7]) which are equivalent. Here, we will use for these algebras the name *generalized MV-algebras*, briefly *GMV-algebras.*

By A. Dvurečenskij [4], GMV-algebras can be considered as intervals in unital lattice ordered groups (ℓ-groups). Moreover, by [14], there is a mutual correspondence between GMV-algebras and dually residuated lattice ordered monoids ($DR\ell$-monoids) belonging to a certain variety of $DR\ell$-monoids. At the same time, the ideals of GMV-algebras correspond to the convex ℓ-subgroups of the corresponding unital ℓ-groups and also to the ideals of the induced

2000 Mathematics Subject Classification: Primary 06D35; Secondary 03G25, 06F05, 06F15.
Keywords: GMV-algebra, $DR\ell$-monoid, ℓ-group, direct factor, ideal, polar.

The first author was supported by the Council of Czech Government, J 14/98: 153100011.
The second author was supported by the Council of Czech Government, J 17/98: 275100015.

$DR\ell$-monoids. These correspondences are used in the paper to studying direct decompositions of GMV-algebras. Further, they make it possible to consider direct factors of GMV-algebras in the form of their ideals, although ideals, in general, are not subalgebras of GMV-algebras. Moreover, projectable GMV-algebras are described here.

The necessary results concerning the theories of MV-algebras and of ℓ-groups can be found in [3], [9], [6] and in [1], [8], respectively.

2. Basic notions, denotations and relations

DEFINITION. Let $A = (A; \oplus, \neg, \sim, 0, 1)$ be an algebra of type $\langle 2, 1, 1, 0, 0\rangle$. Set $x \odot y = \sim(\neg x \oplus \neg y)$ for any $x, y \in A$. Then A is called a *generalized MV-algebra* (briefly: *GMV-algebra*) if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = x = 0 \oplus x$;
(A3) $x \oplus 1 = 1 = 1 \oplus x$;
(A4) $\neg 1 = 0 = \sim 1$;
(A5) $\neg(\sim x \oplus \sim y) = \sim(\neg x \oplus \neg y)$;
(A6) $x \oplus (y \odot \sim x) = y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y = (\neg x \odot y) \oplus x$;
(A7) $(\neg x \oplus y) \odot x = y \odot (x \oplus \sim y)$;
(A8) $\sim \neg x = x$.

If we put $x \leq y$ if and only if $\neg x \oplus y = 1$, then $(A, \leq)$ is a bounded distributive lattice (0 is the least and 1 is the greatest element) with $x \vee y = x \oplus (y \odot \sim x)$ and $x \wedge y = x \odot (y \oplus \sim x)$.

Let $G = (G; +, \vee, \wedge)$ be a lattice ordered group (ℓ-group) and $0 \leq u \in G$. For any $x, y \in [0, u] = \{x \in G : 0 \leq x \leq u\}$ put $x \oplus y = (x + y) \wedge u$, $\neg x = u - x$ and $\sim x = -x + u$. Then $\Gamma(G, u) = \big([0, u]; \oplus, \neg, \sim, 0, u\big)$ is a GMV-algebra.

By a *unital ℓ-group* we will mean a pair (G, u), where G is an ℓ-group and u is a strong order unit in G. (Recall that $0 < u \in G$ is a *strong order unit* in G if for any $a \in G$ there is $n \in \mathbb{N}$ such that $-nu \leq a \leq nu$, i.e., the convex ℓ-subgroup of G generated by u is equal to G.) Unital ℓ-groups and GMV-algebras are in a very close connection because A. Dvurečenskij in [4] proved that for any GMV-algebra A there is a unital ℓ-group (G, u) such that A is isomorphic to $\Gamma(G, u)$.

Definition. An algebra $M = (M; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of type $\langle 2,0,2,2,2,2\rangle$ is called a *$DR\ell$-monoid* if $(M; +, 0, \vee, \wedge)$ is a lattice ordered monoid satisfying the conditions $(x, y, r, s \in M)$:

$$s + y \geq x \iff x \rightharpoonup y \leq s \quad \text{and} \quad y + r \geq x \iff x \leftharpoondown y \leq r\,;$$
$$\big((x \rightharpoonup y) \vee 0\big) + y \leq x \vee y\,, \qquad y + \big((x \leftharpoondown y) \vee 0\big) \leq x \vee y\,,$$
$$x \rightharpoonup x \geq 0\,, \qquad x \leftharpoondown x \geq 0\,.$$

GMV-algebras and $DR\ell$-monoids are also in a close connection. Indeed, if $A = (A; \oplus, \neg, \sim, 0, 1)$ is a GMV-algebra and if we put $x \rightharpoonup y = \neg y \odot x$ and $x \leftharpoondown y = x \odot {\sim} y$ for any $x, y \in A$, then by [14], $M(A) = (A; \oplus, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a bounded $DR\ell$-monoid (with 1 the greatest element and 0 the least) satisfying the identities

(i) $(\forall x \in A)\big(1 \leftharpoondown (1 \rightharpoonup x) = x = 1 \rightharpoonup (1 \leftharpoondown x)\big)$,

(ii) $(\forall x \in A)(\forall y \in A)\Big(1 \rightharpoonup \big((1 \leftharpoondown x) + (1 \leftharpoondown y)\big) = 1 \leftharpoondown \big((1 \rightharpoonup x) + (1 \rightharpoonup y)\big)\Big)$.

Conversely, if $M = (M; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a bounded $DR\ell$-monoid with a greatest element 1 satisfying (i) and (ii) and if we put $\neg x = 1 \rightharpoonup x$ and ${\sim} x = 1 \leftharpoondown x$ for $x \in M$, then by [14], $A(M) = (M; +, \neg, \sim, 0, 1)$ is a GMV-algebra.

Recall that if A is a GMV-algebra and $\emptyset \neq H \subseteq A$, then H is called an *ideal* of A if H is closed under the operation $\oplus$ and $y \leq x$ implies $y \in H$ for any $x \in H$ and $y \in A$. An ideal is called *normal* if $\neg x \odot y \in H$ if and only if $y \odot {\sim} x \in H$ for each $x, y \in A$. The normal ideals are exactly the kernels of GMV-homomorphisms.

For any $\emptyset \neq H \subseteq A$ we have that H is an ideal of A if and only if H is a convex *sub-$DR\ell$-monoid* of $M(A)$. (Convex sub-$DR\ell$-monoids of a $DR\ell$-monoid M are also called *ideals* of M.) Further, if M is a $DR\ell$-monoid and I is a convex sub-$DR\ell$-monoid of M, then I is called *normal* if and only if $x + I = I + x$ for any $x \in M$. One can prove that for a GMV-algebra A, an ideal H of A is normal if and only if H is a normal convex sub-$DR\ell$-monoid of $M(A)$. (See [12].) We will use these relations when studying direct decompositions of GMV-algebras, because ideals of GMV-algebras, in contrast to convex sub-$DR\ell$-monoids of $DR\ell$-monoids, need not be subalgebras of GMV-algebras.

If A is a GMV-algebra, denote by $\mathcal{C}(A)$ and $\mathcal{N}(A)$ the set of ideals and of normal ideals of A, respectively. Analogously, if M is a $DR\ell$-monoid, then $\mathcal{C}(M)$ and $\mathcal{N}(M)$ will denote the set of convex sub-$DR\ell$-monoids and of normal convex sub-$DR\ell$-monoids, respectively. It is obvious that $(\mathcal{C}(A), \subseteq)$, $(\mathcal{N}(A), \subseteq)$, $(\mathcal{C}(M), \subseteq)$ and $(\mathcal{N}(M), \subseteq)$ are complete lattices.

Let $A = \Gamma(G, u)$ be a GMV-algebra and let $(\mathcal{C}(G), \subseteq)$ and $(\mathcal{N}(G), \subseteq)$ be the complete lattices of convex ℓ-subgroups and of ℓ-ideals of G, respectively.

Let us consider the mapping $\varphi\colon \mathcal{C}(A) \to \mathcal{C}(G)$ such that $\varphi(H) = \{x \in G : |x| \wedge u \in H\}$ for any $H \in \mathcal{C}(A)$. By [15; Theorem 2], φ is an isomorphism of $\mathcal{C}(A)$ onto $\mathcal{C}(G)$ and the inverse isomorphism to φ is the mapping ψ such that $\psi(K) = K \cap [0, u]$ for each $K \in \mathcal{C}(G)$. Moreover, by [5; Theorem 6.1], the restriction of φ on $\mathcal{N}(A)$ is an isomorphism between $\mathcal{N}(A)$ and $\mathcal{N}(G)$.

3. Direct factors of GMV-algebras

In this part we will deal with direct decompositions of GMV-algebras which we will introduce by means of direct decompositions of the induced $DR\ell$-monoids.

DEFINITION. We will say that a $DR\ell$-monoid M is an *inner direct product* of its convex sub-$DR\ell$-monoids (i.e. ideals) M_1 and M_2 if there is an isomorphism φ of M onto the (external) direct product $M_1 \times M_2$ of $DR\ell$-monoids M_1 and M_2 such that for each $x \in M_1$ and each $y \in M_2$ the relations $\varphi(x) = (x, 0)$ and $\varphi(y) = (0, y)$ are valid.

In such a case, we will also write $M = M_1 \times M_2$ and say that M is a *direct product* of its sub-$DR\ell$-monoids M_1 and M_2.

DEFINITION. If A is a GMV-algebra and $H_1, H_2 \in \mathcal{C}(A)$, then A will be called a *direct product of the ideals* H_1 and H_2 if $M = M(A) = M(H_1) \times M(H_2)$, where $M(H_i)$ is the convex sub-$DR\ell$-monoid of M induced by H_i, $i = 1, 2$.

We will write $A = H_1 \times H_2$ and say that H_1 and H_2 are *direct factors* of the GMV-algebra A.

Remark.

a) By [16; Theorem 6], if $M_1, M_2 \in \mathcal{C}(M)$, then $M = M_1 \times M_2$ if and only if

1. $M_1 + M_2 = M$, $M_1 \cap M_2 = \{0\}$;
2. $(\forall x_1, y_1 \in M_1)(\forall x_2, y_2 \in M_2)$
   $\big(x_1 + x_2 = y_1 + y_2 \implies (x_1 = y_1 \ \&\ x_2 = y_2)\big)$.

Moreover, if $M = M_1 \times M_2$, then $M_1, M_2 \in \mathcal{N}(M)$ and $M_1 = M_2^{\perp}$ and $M_2 = M_1^{\perp}$, where $M_2^{\perp}$ and $M_1^{\perp}$ are the polars of M_2 and M_1, respectively.

That means, if $M = M(A)$ for a GMV-algebra A, then M is bounded (with the least element 0), and hence, for instance, $M_1 = M_2^{\perp} = \{x \in M : (\forall b \in M_2)(b \wedge x = 0)\}$.

b) If A is a GMV-algebra and $H \in \mathcal{C}(A)$, then we will not distinguish H and $M(H)$.

THEOREM 1. *Let A be a GMV-algebra and $H_1, H_2 \in \mathcal{C}(A)$. Then $A = H_1 \times H_2$ if and only if H_1 and H_2 satisfy condition* 1.

P r o o f . Let $A = \Gamma(G,u)$ be a GMV-algebra and let $H_1, H_2 \in \mathcal{C}(A)$ satisfy condition 1. If $K_1 = \varphi(H_1)$ and $K_2 = \varphi(H_2)$, then (since $H_1, H_2 \in \mathcal{N}(M) = \mathcal{N}(A)$) we get $K_1, K_2 \in \mathcal{N}(G)$. By [16; Proposition 7], $H_1 \oplus H_2 = H_1 \vee H_2$ in $\mathcal{C}(M(A)) = \mathcal{C}(A)$.

Hence we get:

$$G = \varphi(A) = \varphi(H_1 \oplus H_2) = \varphi(H_1 \vee H_2) = \varphi(H_1) \vee \varphi(H_2),$$

and since $\varphi(H_1), \varphi(H_2) \in \mathcal{N}(G)$, we have

$$G = \varphi(H_1) + \varphi(H_2) = K_1 + K_2 .$$

Moreover, from $H_1 \cap H_2 = \{0\}$ it follows that $K_1 \cap K_2 = \{0\}$, thus $G = K_1 \times K_2$ (and so also $K_1 = K_2^{\perp}$ and $K_2 = K_1^{\perp}$).

Let $x_1, y_1 \in H_1$, $x_2, y_2 \in H_2$ and $x_1 \oplus x_2 = y_1 \oplus y_2$. Since $x_1 \wedge x_2 = 0 = y_1 \wedge y_2$, $x_1 \oplus x_2 = x_1 \vee x_2 = x_1 + x_2$ and $y_1 \oplus y_2 = y_1 \vee y_2 = y_1 + y_2$, therefore $x_1 + x_2 = y_1 + y_2$. Hence from $G = K_1 \times K_2$ we obtain $x_1 = y_1$ and $x_2 = y_2$, i.e., H_1 and H_2 satisfy also condition 2 for direct factors in M, and therefore in A, too.

The converse implication is trivial. □

THEOREM 2. *Let $A = \Gamma(G,u)$ be a GMV-algebra and let $G = K_1 \times K_2$ be a direct decomposition of the ℓ-group G. If $H_1 = \psi(K_1)$ and $H_2 = \psi(K_2)$, then $A = H_1 \times H_2$.*

P r o o f . Let $a \in A$. Then there exist $a_1 \in K_1^+$ and $a_2 \in K_2^+$ such that $a = a_1 + a_2$. Since $0 \le a_1, a_2 \le a \le u$, we have $a_1 \oplus a_2 = a_1 + a_2$, and so $a = a_1 \oplus a_2$. Hence $A = H_1 \oplus H_2$. Condition $H_1 \cap H_2 = \{0\}$ is satisfied too, and therefore, by Theorem 1, $A = H_1 \times H_2$. □

The following theorem is now an immediate consequence.

THEOREM 3. *If $A = \Gamma(G,u)$ is a GMV-algebra, then $H \in \mathcal{C}(A)$ is a direct factor of A (and also of the $DR\ell$-monoid $M(A)$) if and only if $\varphi(H)$ is a direct factor of the ℓ-group G.*

Remark. Let $A = \Gamma(G,u)$ be a GMV-algebra, $H_1, H_2 \in \mathcal{C}(A)$ and let A be the direct product of H_1 and H_2. If $u = u_1 + u_2 = u_1 \oplus u_2$, where $u_1 \in H_1$ and $u_2 \in H_2$, then u_i is the greatest element in H_i, $i = 1, 2$, and thus u_1 and u_2 are additively idempotent elements in A, i.e. $H_i = C(u_i) = [0, u_i]$, $i = 1, 2$.

For any GMV-algebra A, the $DR\ell$-monoid $M(A)$ induced by A satisfies the condition

(MV) $\quad x \rightharpoonup (x \leftharpoondown y) = x \wedge y = x \leftharpoondown (x \rightharpoonup y)$.

Hence the sub-$DR\ell$-monoid in $M(A)$ induced by any ideal in A satisfies condition (MV), too. By [13], the bounded $DR\ell$-monoids satisfying (MV) are just those induced by GMV-algebras. Therefore A is isomorphic to the direct product of the GMV-algebras with underlying sets H_1 and H_2.

(The fact that if a is an idempotent element in a GMV-algebra A, then the interval $[0,a]$ can be considered as a GMV-algebra was proved in [10], and that the operations in the GMV-algebra $[0,a]$ can be expressed explicitly as $x \oplus_a y = x \oplus y$, $\neg_a x = \neg x \wedge a$ and $\sim_a x = \sim x \wedge a$ $(x, y \in [0,a])$ was proved in [14].)

Therefore we now get as a consequence the following theorem, which was proved by different methods in [10; Sections 4, 5].

THEOREM 4. *Let A, A_1 and A_2 be GMV-algebras. Then A is isomorphic to the direct product $A_1 \times A_2$ if and only if there is an idempotent element $a \in A$ such that $A_1 \cong C(a)$ and $A_2 \cong C(\neg a) = C(\sim a)$.*

Moreover, the remark after Theorem 3 together with the fact that the idempotent elements in A form a subalgebra $B(A)$ of A which is a Boolean algebra and in which $\neg a = \sim a = a'$ for each $a \in B(A)$ (see [14]) imply:

THEOREM 5. *The direct factors of a GMV-algebra A form a Boolean sublattice of the lattice $\mathcal{C}(A)$ and also of the lattice of polars in A, which is isomorphic to the Boolean lattice of idempotent elements in A.*

Now, we will describe even more exactly the connections between the direct factors of a GMV-algebra $A = \Gamma(G,u)$ and of those of the corresponding unital ℓ-group (G,u).

PROPOSITION 6. *Let $A = \Gamma(G,u)$ be a GMV-algebra, let $A = H_1 \times H_2$ be a direct decomposition of A and let $K_i = \varphi(H_i)$, $i = 1,2$. If $a \in A$ and $a = a_1 \oplus a_2$, where $a_i \in H_i$, $i = 1,2$, then*

$$a_2/H_1 = (a_2/K_1) \cap A \qquad \textit{and} \qquad a_1/H_2 = (a_1/K_2) \cap A .$$

P r o o f. Let $x \in A$. Then $x \in a_2/H_1$ if and only if $(x \rightharpoonup a_2) \oplus (a_2 \rightharpoonup x) \in H_1$, which holds if and only if

$$\big((x - a_2) \vee 0\big) + \big((a_2 - x) \vee 0)\big) \wedge u \in H_1 ,$$

hence if and only if

$$\big((x - a_2) \vee 0\big) + \big((a_2 - x) \vee 0\big) \in K_1 ,$$

and this is equivalent to

$$\big((x - a_2) + (a_2 - x)\big) \vee (a_2 - x) \vee (x - a_2) \vee 0 \in K_1 .$$

Therefore $x \in a_2/H_1$ if and only if $|a_2 - x| \in K_1$, which is equivalent to $a_2 - x \in K_1$, that means, to $x \in a_2 + K_1$.

The second equality is analogous. □

The direct factors of a GMV-algebra are its normal ideals, hence we can construct corresponding factor GMV-algebras.

Using Proposition 6, now we will easily prove the following theorem.

THEOREM 7. *If A is a GMV-algebra and if $A = H_1 \times H_2$ is a direct decomposition of A, then $H_1 \cong A/H_2$ and $H_2 \cong A/H_1$.*

Proof. Let $A = \Gamma(G, u)$ and let K_1 and K_2 be as in Proposition 6. Let $\overline{f}\colon K_2 \to G/K_1$ be the isomorphism of ℓ-groups such that $\overline{f}(c) = c/K_1 = c+K_1$ for each $c \in K_2$. Let $\widetilde{f} = \overline{f}|_{H_2}$. Let us denote by $f\colon H_2 \to A/H_1$ the mapping such that $f(x) = x/H_1$ for each $x \in H_2$. By Proposition 6, $\widetilde{f}(x) = \widetilde{f}(y)$ if and only if $f(x) = f(y)$ for any $x, y \in H_2$. Thus f is a bijection of H_2 onto A/H_1. At the same time, f is a restriction of the natural homomorphism $\nu\colon A \to A/H_1$ of GMV-algebras, hence f is an isomorphism of H_2 onto A/H_1. Therefore $H_2 \cong A/H_1$.

The second assertion is analogous. □

4. Projectable GMV-algebras

Projectable ℓ-groups form an important class of ℓ-groups. Recall that an ℓ-group G is called *projectable* if the polar $a^{\perp}$ is a direct factor in G for each $a \in G$. Now we will introduce an analogous notion also for GMV-algebras.

DEFINITION. A GMV-algebra A is called *projectable* if $A = a^{\perp} \times a^{\perp\perp}$ for each $a \in A$.

Remark.

a) By Theorem 1, A is projectable if and only if $A = a^{\perp} \oplus a^{\perp\perp}$ for each $a \in A$.

b) If a GMV-algebra A is projectable, then every polar in A is a normal ideal in A. Hence by [5], every projectable GMV-algebra, similarly as in the case of ℓ-groups, is representable.

In the next theorem, we will show connections between principal ideals and polars in projectable GMV-algebras.

THEOREM 8. *Let A be a projectable GMV-algebra. Then every polar in A is an intersection of principal (normal) ideals of A generated by elements from the set $B(A)$ of all idempotent elements of A.*

Proof. If A is a projectable GMV-algebra, then for any element $a \in A$ there is an element $b \in B(A)$ such that $a^{\perp} = C(b)$ and $a^{\perp\perp} = C(\neg b)$. Let $C \subseteq A$ be a polar in A. Then

$$C = \bigcap_{d \in a^{\perp}} d^{\perp} = \bigcap_{d \in a^{\perp}} C(c_d),$$

where c_d is an element in $B(A)$ such that $d^{\perp} = C(c_d)$. Thus every polar in A is an intersection of principal (normal) ideals generated by elements of $B(A)$ (i.e., an intersection of intervals in the form $[0,x]$ where $x \in B(A)$). □

LEMMA 9. *Let $A = \Gamma(G,u)$ be a GMV-algebra and let $H \in \mathcal{C}(A)$. Then H is the principal ideal $C_A(a)$ in A generated by an element $a \in A = [0,u]$ if and only if $\varphi(H)$ is the principal convex ℓ-subgroup $C_G(a)$ in G generated by a.*

Proof. Let $a \in A$, $J \in \mathcal{C}(A)$ and $a \in J$. Then obviously $a \in \varphi(J) \in \mathcal{C}(G)$.

Conversely, if $L \in \mathcal{C}(G)$ and $a \in L$ (thus $C_G(a) \subseteq L$), then $a \in L \cap [0,u] = \psi(L)$, that means $C_A(a) \subseteq \psi(L) = \varphi^{-1}(L)$.

Therefore $\varphi(C_A(a)) = C_G(a)$. □

PROPOSITION 10. *Let $A = \Gamma(G,u)$ be a GMV-algebra. Then A is a projectable GMV-algebra if and only if G is a projectable ℓ-group.*

Proof. Let $a \in A$. Then $a^{\perp_A}$ is the pseudo-complement of the ideal $C_A(a)$ in the lattice $\mathcal{C}(A)$, and hence, $\varphi(a^{\perp_A})$ is by Lemma 9 the pseudo-complement of the convex ℓ-subgroup $C_G(a)$ in the lattice $\mathcal{C}(G)$. Therefore $\varphi(a^{\perp_A}) = a^{\perp_G}$.

The assertion now follows from Theorem 3. □

The following theorem is a consequence of Theorem 8, Lemma 9 and Proposition 10.

THEOREM 11. *Let (G,u) be a projectable unital ℓ-group. Then every polar in G is an intersection of principal convex ℓ-subgroups (which are ℓ-ideals) of G generated by elements $x \in G^+$ satisfying the condition $(x+x) \wedge u = x$.*

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Received November 27, 2003
Revised June 21, 2004

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