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GMV-ALGEBRAS AND MEET-SEMILATTICES WITH SECTIONALLY ANTITONE PERMUTATIONS

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ABSTRACT. GMV-algebras (pseudo MV-algebras) are a non-commutative extension of known MV-algebras. We show that any GMV-algebra is a (meet-semi)lattice with sectionally antitone permutations, an SAP-(semi)lattice, and hence SAP-semilattices can be viewed as a generalization of GMV-algebras.

1. Semilattices with antitone permutations

Let $\langle S; \wedge, 0\rangle$ be a meet-semilattice with the least element 0. For any $a \in S$, the principal ideal $(a] = \{x \in S : x \leq a\}$ is called a *section* in S. An *antiautomorphism* on $(a]$ is a one-to-one mapping f from $(a]$ onto $(a]$ such that $x \leq y$ iff $f(x) \geq f(y)$ for all $x, y \in (a]$. Obviously, f is an antiautomorphism on $(a]$ if and only if both f and its inverse mapping f^{-1} are antitone permutations. We say that a semilattice $\langle S; \wedge, 0\rangle$ has *sectionally antitone permutations* if there exists an antiautomorphism f_a on each section $(a]$. Accordingly, a *semilattice with sectionally antitone permutations* (an SAP-*semilattice* for short) is a structure $\left\langle S; \wedge, 0, (f_a)_{a\in S}\right\rangle$, where $\langle S; \wedge, 0\rangle$ is a meet-semilattice with a least element and for any $a \in S$, f_a is an antitone permutation on $(a]$.

Given an SAP-semilattice $\left\langle S; \wedge, 0, (f_a)_{a\in S}\right\rangle$, we can define two total binary operations on S by

$$x * y := f_x(x \wedge y) \qquad \text{and} \qquad x \circ y := f_x^{-1}(x \wedge y)\,.$$

It is evident that $x * 0 = x = x \circ 0$, $0 * x = 0 = 0 \circ x$ and $x * x = 0 = x \circ x$ for all $x \in S$.

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EXAMPLE 1.1. Let $\langle G; +, 0, \vee, \wedge\rangle$ be a lattice-ordered group (an ℓ-group), that is, a group endowed with a compatible lattice order, and let $G^+ = \{x \in G : x \geq 0\}$ be its positive cone. Then $\left\langle G^+; \wedge, 0, (f_a)_{a\in G^+}\right\rangle$ is an SAP-semilattice, where for any $a \in G^+$, the antitone permutation f_a is defined by $f_a(x) := a - x$. The operations $*$ and $\circ$ are then given by $x * y := x - (x \wedge y) = (x - y) \vee 0$ and $x \circ y := -(x \wedge y) + x = (-y + x) \vee 0$. It is easily seen that $(x * y) \circ z = (x \circ z) * y$ for all $x, y, z \in G^+$.

More generally, let X be a convex subset of G^+ containing 0. Then $\left\langle X; \wedge, 0, (f_a)_{a\in X}\right\rangle$ is an SAP-semilattice in which $x * y = (x - y) \vee 0$ and $x \circ y = (-y + x) \vee 0$ for all $x, y \in X$.

THEOREM 1.2.

(i) *Let $\left\langle S; \wedge, 0, (f_a)_{a\in S}\right\rangle$ be an SAP-semilattice. Then for any $a \in S$, $f_a(x) = a * x$ and $f_a^{-1}(x) = a \circ x$, and the structure $\Phi(S) = \langle S; \wedge, 0, *, \circ\rangle$ satisfies the identities*

$$x \wedge y = x * (x \circ y) = x \circ (x * y)\,, \tag{1.1}$$

$$x * y = (x * y) \wedge \big(x * (y \wedge z)\big)\,, \quad x \circ y = (x \circ y) \wedge \big(x \circ (y \wedge z)\big)\,. \tag{1.2}$$

(ii) *Let $\langle S; \wedge, 0, *, \circ\rangle$ be an algebra of type $(2, 0, 2, 2)$ such that $\langle S; \wedge, 0\rangle$ is a meet-semilattice with a least element. For any $a \in S$ define the mapping*

$$f_a \colon x \mapsto a * x\,, \qquad x \in (a]\,.$$

If S satisfies the identities (1.1) and (1.2), then $\Psi(S) = \left\langle S; \wedge, 0, (f_a)_{a\in S}\right\rangle$ is an SAP-semilattice and we have $x*y = f_x(x\wedge y)$ and $x\circ y = f_x^{-1}(x\wedge y)$ for all $x, y \in S$.

(iii) *The above mappings Φ and Ψ are mutually inverse bijections.*

Proof.

(i) It is easily seen that $f_a(x) = a * x$ and $f_a^{-1}(x) = a \circ x$. We have

$$x * (x \circ y) = f_x\big(x \wedge f_x^{-1}(x \wedge y)\big) = f_x\big(f_x^{-1}(x \wedge y)\big) = x \wedge y$$

and analogously $x \circ (x * y) = x \wedge y$, which is (1.1), and from $x \wedge y \geq x \wedge y \wedge z$ it follows that $x * y = f_x(x \wedge y) \leq f_x(x \wedge y \wedge z) = x * (y \wedge z)$ and $x \circ y = f_x^{-1}(x \wedge y) \leq f_x^{-1}(x \wedge y \wedge z) = x \circ (y \wedge z)$ proving (1.2).

(ii) Assume that $\langle S; \wedge, 0, *, \circ\rangle$ satisfies the equations (1.1) and (1.2). Then $a * x \in (a]$ for any $x \in (a]$ since $a \wedge (a * x) = a * \big(a \circ (a * x)\big) = a * (a \wedge x) = a * x$ by (1.1). Analogously, $a \circ x \in (a]$. If $a * x = a * y$ for $x, y \in (a]$, then $x = a \wedge x = a \circ (a * x) = a \circ (a * y) = a \wedge y = y$ again by (1.1), and in addition, every $y \in (a]$ can be written in the form $y = a * x$, where $x = a \circ y \in (a]$. Thus the mapping f_a is a permutation on $(a]$. Because of (1.1), $f_a^{-1}(x) = a \circ x$ for all $x \in (a]$.

Let $x, y \in (a]$ and $x \leq y$. Then by (1.2),

$$(a * y) \wedge (a * x) = (a * y) \wedge \big(a * (x \wedge y)\big) = a * y\,,$$

so $a*y \leq a*x$. Similarly, $a \circ y \leq a \circ x$ whenever $x \leq y$, and hence $a*y \leq a*x$ implies $x = a \circ (a * x) \leq a \circ (a * y) = y$. Therefore, $\big\langle S; \wedge, 0, (f_a)_{a\in S}\big\rangle$ is an SAP-semilattice.

For the last claim, $f_x(x \wedge y) = x * (x \wedge y) = x * \big(x \circ (x * y)\big) = x \wedge (x * y)$ by (1.1) and

$$\begin{aligned} x * y &= (x * y) \wedge \big(x * (x \wedge y)\big) \\ &= (x * y) \wedge \big(x * \big(x \circ (x * y)\big)\big) \\ &= (x * y) \wedge x \wedge (x * y) \\ &= x \wedge (x * y) \end{aligned}$$

by (1.2), so that $f_x(x \wedge y) = x * y$. The dual assertion follows by symmetry. □

Remark 1.3. In view of (1.1) we obtain

$$x * (x \circ y) = x \circ (x * y) = y * (y \circ x) = y \circ (y * x)\,, \tag{1.1'}$$

and (1.2) can be rewritten in the language $\{*, \circ\}$ as follows:

$$\begin{aligned} x * y &= (x * y) * \big((x * y) \circ \big(x * \big(y * (y \circ z)\big)\big)\big)\,, \\ x \circ y &= (x \circ y) \circ \big((x \circ y) * \big(x \circ \big(y \circ (y * z)\big)\big)\big)\,. \end{aligned} \tag{1.2'}$$

However, if $\langle S; \wedge, 0, *, \circ\rangle$ satisfies (1.1′) and (1.2′), then $x * (x \circ y)$ need not be equal to $x \wedge y$ and the mapping $f_a\colon x \mapsto a * x$ is not necessarily an antitone involution on $[0, a]$:

EXAMPLE 1.4. Let $\langle S; \wedge, 0\rangle$ be the semilattice from Figure 1 and let the operation $*$ be given as follows:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	b	a	d	0

Then $\langle S; \wedge, 0, *, *\rangle$ fulfils the equations (1.1′) and (1.2′), but, for instance, $a * (a * c) = 0$ while $a \wedge c = a$, and $f_c\colon x \mapsto c * x$ is not an antitone involution on $[0, c]$.

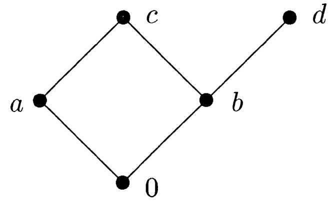

FIGURE 1.

Let us recall e.g. from [1] that a variety $\mathcal{V}$ with a nullary fundamental operation 0 is said to be

(a) *weakly regular* if every congruence Θ on any algebra A in $\mathcal{V}$ is determined by its kernel $[0]_\Theta$, and *regular* if Θ is determined by any single class $[a]_\Theta$;

(b) *distributive at* 0 if $[0]_{(\Theta\vee\Phi)\cap\Psi} = [0]_{(\Theta\cap\Psi)\vee(\Phi\cap\Psi)}$ for all $\Theta, \Phi, \Psi \in \mathrm{Con}(A)$ and $A \in \mathcal{V}$, and *distributive* if the congruence lattice $\mathrm{Con}(A)$ of every $A \in \mathcal{V}$ is distributive;

(c) *permutable at* 0 if $[0]_{\Theta\circ\Phi} = [0]_{\Phi\circ\Theta}$, *permutable* if $\Theta\circ\Phi = \Phi\circ\Theta$ and n*-permutable* if $\Theta\circ\Phi\circ\Theta\circ\cdots = \Phi\circ\Theta\circ\Phi\circ\cdots$ (n-times) for all $\Theta, \Phi \in \mathrm{Con}(A)$ and for each $A \in \mathcal{V}$;

(d) *arithmetical at* 0 if it is both distributive and permutable at 0, and *arithmetical* if $\mathcal{V}$ is both distributive and permutable.

THEOREM 1.5. *The variety of all* SAP*-semilattices is weakly regular,* 3*-permutable, arithmetical at* 0 *and distributive.*

Proof. Let $\mathcal{V}$ be the variety of all SAP-semilattices. It is known (see e.g. [1]) that $\mathcal{V}$ is weakly regular if and only if there exist binary terms $p_1, \dots, p_n$ for some $n \in \mathbb{N}$ such that $p_1(x,y) = \cdots = p_n(x,y) = 0$ iff $x = y$. We can take $n = 2$ and $p_1(x,y) := x * y$, $p_2(x,y) := y * x$. Clearly, $p_1(x,x) = p_2(x,x) = 0$, and conversely, if $p_1(x,y) = p_2(x,y) = 0$, then $x \wedge y = x \circ (x * y) = x$ and $x \wedge y = y \circ (y * x) = y$, so $x = y$.

To show that $\mathcal{V}$ is 3-permutable, we have to find ternary terms t_1, t_2 such that $t_1(x,y,y) = x$, $t_1(x,x,y) = t_2(x,y,y)$ and $t_2(x,x,y) = y$. It is obvious that the terms $t_1(x,y,z) := x * (y \circ z)$ and $t_2(x,y,z) := z * (y \circ x)$ have this property.

$\mathcal{V}$ is arithmetical at 0 if and only if there exists a binary term t with $t(x,x) = t(0,x) = 0$ and $t(x,0) = x$. Obviously, one may take $t(x,y) := x * y$.

Finally, $\mathcal{V}$ is congruence distributive since it is both weakly regular and distributive at 0. □

An SAP-*lattice* is an algebra $\langle L; \vee, \wedge, 0, *, \circ \rangle$, where $\langle L; \vee, \wedge \rangle$ is a lattice and $\langle L; \wedge, 0, *, \circ \rangle$ is an SAP-semilattice. For instance, if X is a lattice ideal of the positive cone G^+ of any ℓ-group G, then $\langle X; \vee, \wedge, 0, *, \circ \rangle$ is an SAP-lattice.

THEOREM 1.6. *The variety of all* SAP*-lattices is regular and arithmetical.*

Proof. Let now $\mathcal{V}$ be the variety of SAP-lattices. It is known that $\mathcal{V}$ is regular if and only if there exist ternary terms $p_1, \dots, p_n$ with $p_1(x,y,z) = \dots = p_n(x,y,z) = z$ iff $x = y$.

Let

$$p_1(x,y,z) := (x * y) \vee (y * x) \vee z\,,$$
$$p_2(x,y,z) := \big(z * (x * y)\big) \wedge \big(z * (y * x)\big)\,.$$

One immediately sees that $p_1(x,x,z) = p_2(x,x,z) = z$. If $p_1(x,y,z) = p_2(x,y,z) = z$, then $z \geq x * y, y * x$ and $z = z * (x * y) = z * (y * x)$ since $z = \big(z * (x * y)\big) \wedge \big(z * (y * x)\big)$ and $z \geq z * (x * y)$, $z * (y * x)$, whence it follows that $0 = z \circ z = z \circ \big(z * (x * y)\big) = z \wedge (x * y) = x * y$ and $0 = z \circ z = z \circ \big(z * (y * x)\big) = z \wedge (y * x) = y * x$, and therefore $x = y$.

Further, $\mathcal{V}$ is arithmetical if and only if there exists a ternary term m such that $m(x,y,y) = m(x,y,x) = m(y,y,x) = x$. It can be easily seen that the term

$$m(x,y,z) := (x \wedge z) \vee \big(x * (y \circ z)\big) \vee \big(z * (y \circ x)\big)$$

satisfies these conditions. □

2. GMV-algebras

In 1958, C. C. Chang introduced the notion of an MV-algebra as an algebraic counterpart of the Łukasiewicz propositional calculus. The research on MV-algebras has burgeoned in the last two decades. Starting from intervals of (not necessarily commutative) lattice-ordered groups, J. Rachůnek established in [9] the concept of a GMV-*algebra* (*generalized* MV-*algebra*). Non-commutative MV-algebras, named *pseudo* MV-*algebras* were independently defined by G. Georgescu and A. Iorgulescu in [7].

A GMV-*algebra* is an algebra $\langle A; \oplus, \neg, \sim, 0, 1 \rangle$ of type $(2,1,1,0,0)$ satisfying the following axioms:

(A1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
(A2) $x \oplus 0 = 0 \oplus x = x$,
(A3) $x \oplus 1 = 1 \oplus x = 1$,
(A4) $\neg 1 = {\sim} 1 = 0$,
(A5) $\neg({\sim} x \oplus {\sim} y) = {\sim}(\neg x \oplus \neg y)$,

(A6) $x \oplus (y \odot {\sim}x) = y \oplus (x \odot {\sim}y) = (\neg x \odot y) \oplus x = (\neg y \odot x) \oplus y$,
(A7) $(\neg x \oplus y) \odot x = y \odot (x \oplus {\sim}y)$,
(A8) ${\sim}\neg x = x$,

where the additional operation $\odot$ is defined via

$$x \odot y := {\sim}(\neg x \oplus \neg y)\,.$$

If $\oplus$ is commutative, then $\sim$ coincides with $\neg$ and $\langle A; \oplus, \neg, 0, 1\rangle$ becomes an MV-algebra. For basic properties of MV- and GMV-algebras we refer to [4] and [7], respectively.

The prototypical example of a GMV-algebra arises from lattice-ordered groups. Let G be any ℓ-group and $u \in G^+ \setminus \{0\}$. Define $\Gamma(G,u) := \langle [0,u]; \oplus, \neg, \sim, 0, u\rangle$ by $x \oplus y := (x+y) \wedge u$, $\neg x := u - x$ and ${\sim}x := -x + u$. It is straightforward to verify that the structure $\Gamma(G,u)$ is a GMV-algebra. A. Dvurečenskij generalized D. Mundici's fundamental result on categorical equivalence of MV-algebras and Abelian ℓ-groups with strong order unit[1] (see [8]) and proved that every GMV-algebra is isomorphic with $\Gamma(G,u)$ for an appropriate ℓ-group G with a strong order unit $u \in G^+$ (see [5]).

GMV-algebras are another source of SAP-(semi)lattices: If we define $x \le y$ iff $\neg x \oplus y = 1$, the *natural order* on A, then by [7; Corollary 1.19], $\langle A; \le\rangle$ is a bounded distributive lattice with

$$x \vee y = x \oplus {\sim}(\neg y \oplus x) = \neg(x \oplus {\sim}y) \oplus x$$

and

$$x \wedge y = x \odot {\sim}(\neg y \odot x) = \neg(x \odot {\sim}y) \odot x\,.$$

Moreover, $\oplus$ as well as $\odot$ distributes over both $\vee$ and $\wedge$ (which implies that $\oplus$ and $\odot$ respect $\le$), and we have $x \le y$ iff $\neg y \le \neg x$ iff ${\sim}y \le {\sim}x$. Consequently, for any $a \in A$, the mapping $f_a\colon x \mapsto \neg x \odot a$ is an antitone permutation on $[0,a]$; the inverse mapping is given by $f_a^{-1}\colon x \mapsto a \odot {\sim}x$.

THEOREM 2.1. *Let $\langle A; \oplus, \neg, \sim, 0, 1\rangle$ be a* GMV-*algebra. Then upon defining $x \wedge y := (\neg x \oplus y) \odot x$, $x * y := \neg y \odot x$ and $x \circ y := x \odot {\sim}y$, the structure $\langle A; \wedge, 0, *, \circ\rangle$ is an* SAP-*semilattice satisfying the equation*

$$(x * y) \circ z = (x \circ z) * y. \tag{2.1}$$

Proof. In view of the previous remarks, it is obvious that $\langle A; \wedge, 0, *, \circ\rangle$ is an SAP-semilattice. For the identity (2.1) calculate $(x*y)\circ z = (\neg y \odot x) \odot {\sim}z = \neg y \odot (x \odot {\sim}z) = (x \circ z) * y$. □

In what follows, we concentrate on SAP-semilattices satisfying the identity (2.1).

[1] We call $u \in G^+$ a *strong order unit* if for every $x \in G$ there exists $n \in \mathbb{N}$ such that $x \le nu$.

THEOREM 2.2. *Let* $\langle S; \wedge, 0, *, \circ\rangle$ *be an* SAP*-semilattice satisfying* (2.1). *Let* $a \in S\setminus\{0\}$ *and define* $x \oplus_a y := a * \big((a\circ x)\circ y\big)$, $\neg_a x := a * x$ *and* $\sim_a x := a\circ x$. *Then* $\big\langle [0,a]; \oplus_a, \neg_a, \sim_a, 0, a\big\rangle$ *is a* GMV*-algebra.*

Before proving the theorem we need two lemmata.

LEMMA 2.3. *Let* $\langle S; \wedge, 0, *, \circ\rangle$ *be an* SAP*-semilattice satisfying* (2.1). *Then for any* $a \in S\setminus\{0\}$, *the section* $[0,a]$ *is a lattice in which*

$$x \vee_a y = a * \big((a\circ x)\circ(y\circ x)\big) = a\circ\big((a*y)*(x*y)\big).$$

P r o o f . Since the mappings $f_a\colon x \mapsto a*x$ and $f_a^{-1}\colon x\mapsto a\circ x$ are antitone permutations on $[0,a]$, it should be obvious that $x\vee_a y := a*\big((a\circ x)\wedge(a\circ y)\big)$ is the supremum of $\{x,y\}$ and we have

$$\begin{aligned} x\vee_a y &= a*\big((a\circ x)\wedge(a\circ y)\big)\\ &= a*\big((a\circ x)\circ\big((a\circ x)*(a\circ y)\big)\big)\\ &= a*\big((a\circ x)\circ\big((a*(a\circ y))\circ x\big)\big)\\ &= a*\big((a\circ x)\circ(y\circ x)\big). \end{aligned}$$

The other equality follows for symmetric reasons. □

LEMMA 2.4. *Let* $\langle S; \wedge, 0, *, \circ\rangle$ *be an* SAP*-semilattice with* (2.1), $a\in S\setminus\{0\}$. *Then for all* $x,y,z\in[0,a]$,

(i) $a*\big((a\circ x)\circ y\big) = a\circ\big((a*y)*x\big)$,
(ii) $a*\big(((a\circ x)\circ y)\circ z\big) = a\circ\big(((a*z)*y)*x\big)$.

P r o o f .

(i) Put $\alpha = a*\big((a\circ x)\circ y\big)$ and $\beta = a\circ\big((a*y)*x\big)$. Then clearly $\alpha,\beta\in[0,a]$ and we have $a\circ\alpha = a\circ\big(a*\big((a\circ x)\circ y\big)\big) = a\wedge\big((a\circ x)\circ y\big) = (a\circ x)\circ y$, whence $(\alpha\circ x)\circ y = \big((a*(a\circ\alpha))\circ x\big)\circ y = \big((a\circ x)*(a\circ\alpha)\big)\circ y = \big((a\circ x)\circ y\big)*(a\circ\alpha) = 0$, so $\alpha\circ x\le y$. But $\alpha\circ x\le y$ is equivalent to $\alpha*y\le x$ since $(\alpha\circ x)*y = (\alpha*y)\circ x$. Hence we obtain $(a*y)\circ(a*\alpha) = \big(a\circ(a*\alpha)\big)*y = \alpha*y\le x$, which yields $(a*y)*x\le a*\alpha$, and therefore $\beta = a\circ\big((a*y)*x\big)\ge a\circ(a*\alpha) = \alpha$. The proof of the converse inequality can be achieved analogously.

(ii) Let $\alpha = a*\big(((a\circ x)\circ y)\circ z\big)$ and $\beta = a\circ\big(((a*z)*y)*x\big)$. Then $a\circ\alpha = \big((a\circ x)\circ y\big)\circ z$, which yields $\big(((a\circ x)\circ y)*(a\circ\alpha)\big)\circ z = \big(((a\circ x)\circ y)\circ z\big)*(a\circ\alpha) = 0$, i.e. $\big((a\circ x)\circ y\big)*(a\circ\alpha)\le z$. Further, $(\alpha\circ x)\circ y = \big((a*(a\circ\alpha))\circ x\big)\circ y = \big((a\circ x)*(a\circ\alpha)\big)\circ y = \big((a\circ x)\circ y\big)*(a\circ\alpha)\le z$, which is equivalent to $(\alpha*z)\circ x = (\alpha\circ x)*z\le y$, and consequently to $(\alpha*z)*y\le x$. But $(\alpha*z)*y = \big((a\circ(a*\alpha))*z\big)*y = \big((a*z)\circ(a*\alpha)\big)*y = \big((a*z)*y\big)\circ(a*\alpha)$, so that $\big((a*z)*y\big)\circ(a*\alpha)\le x$, whence it follows $\big((a*z)*y\big)*x\le a*\alpha$ and finally $\beta = a \circ \big(((a * z) * y) * x\big) \geq a \circ (a * \alpha) = \alpha$. The same argument shows $\alpha \geq \beta$. □

Proof of Theorem 2.2. Note that by Lemma 2.4(i) we have

$$x \oplus_a y = a * \big((a \circ x) \circ y\big) = a \circ \big((a * y) * x\big).$$

(A1) follows from Lemma 2.4(ii):

$$\begin{aligned}(x \oplus_a y) \oplus_a z &= a * \Big(\big(a \circ \big(a * ((a \circ x) \circ y)\big)\big) \circ z\Big) = a * \Big(\big((a \circ x) \circ y\big) \circ z\Big) \\ &= a \circ \Big(\big((a * z) * y\big) * x\Big) = a \circ \Big(\big(a * \big(a \circ ((a * z) * y)\big)\big) * x\Big) \\ &= x \oplus_a (y \oplus_a z).\end{aligned}$$

For (A2), $x \oplus_a 0 = a * \big((a \circ x) \circ 0\big) = a * (a \circ x) = x$ and similarly $0 \oplus_a x = x$. Analogously, $x \oplus_a a = a \circ \big((a * a) * x\big) = a \circ (0 * x) = a$ and likewise $a \oplus_a x = a$, which is (A3). The axiom (A4) obviously holds as $\neg_a a = a * a = 0 = a \circ a = \sim_a a$. To see (A5), calculate

$$\begin{aligned}\neg_a(\sim_a x \oplus_a \sim_a y) &= a * \Big(a \circ \big((a * (a \circ y)) * (a \circ x)\big)\Big) \\ &= y * (a \circ x) = \big(a \circ (a * y)\big) * (a \circ x) \\ &= \big(a * (a \circ x)\big) \circ (a * y) = x \circ (a * y) \\ &= a \circ \Big(a * \big((a \circ (a * x)) \circ (a * y)\big)\Big) \\ &= \sim_a(\neg_a x \oplus_a \neg_a y).\end{aligned}$$

For (A6), observe that $y \odot_a \sim_a x = \sim_a(\neg_a y \oplus_a x) = a \circ \Big(a * \big((a \circ (a * y)) \circ x\big)\Big) = y \circ x$, whence $x \oplus_a (y \odot_a \sim_a x) = a * \big((a \circ x) \circ (y \circ x)\big) = x \vee_a y$ by Lemma 2.3. Similarly $\neg_a y \odot_a x = x * y$, and hence $(\neg_a y \odot_a x) \oplus_a y = x \vee_a y$. Furthermore, $(\neg_a x \oplus_a y) \odot_a x = \neg_a(x \odot_a \sim_a y) \odot_a x = x * (x \circ y) = x \wedge y$ and analogously we obtain $y \odot_a (x \oplus_a \sim_a y) = y \circ (y * x) = x \wedge y$, which verifies (A7). Finally, (A8) is clear: $\sim_a \neg_a x = a \circ (a * x) = a \wedge x = x$. □

COROLLARY 2.5. *Let $\langle S; \wedge, 0, *, \circ\rangle$ be an* SAP*-semilattice with the greatest element $1 \neq 0$, satisfying* (2.1). *Let $x \oplus y := 1 * \big((1 \circ x) \circ y\big)$, $\neg x := 1 * x$ and $\sim x := 1 \circ x$. Then $\langle S; \oplus, \neg, \sim, 0, 1\rangle$ is a* GMV*-algebra.*

COROLLARY 2.6. *If $\langle S; \wedge, 0, *, \circ\rangle$ is an* SAP*-semilattice satisfying* (2.1), *then every section $[0, a]$ is a distributive lattice.*

Proof. Since $\big\langle [0, a]; \oplus_a, \neg_a, \sim_a, 0, a\big\rangle$ is a GMV-algebra, it follows that $\big\langle [0, a]; \vee_a, \wedge\big\rangle$ is a distributive lattice. □

Combining Theorem 2.1 and Theorem 2.2, we get:

Corollary 2.7. *Let* $\langle A; \oplus, \neg, \sim, 0, 1\rangle$ *be a* GMV*-algebra,* $a \in A\setminus\{0\}$. *Define* $x \oplus_a y := (x \oplus y) \wedge a$, $\neg_a x := \neg x \odot a$ *and* $\sim_a x := a \odot \sim x$ *for* $x, y \in [0, a]$. *Then* $\big\langle [0, a]; \oplus_a, \neg_a, \sim_a, 0, a\big\rangle$ *is a* GMV*-algebra.*

P r o o f . Calculate

$$\begin{aligned} x \oplus_a y &= (x \oplus y) \wedge a = \neg\big(a \odot \sim(x \oplus y)\big) \odot a \\ &= \neg(a \odot \sim x \odot \sim y) \odot a = a * \big((a \circ x) \circ y\big) . \end{aligned}$$

□

Corollary 2.8. *Let* $\langle S; \wedge, 0, *, \circ\rangle$ *be an* SAP*-semilattice satisfying the identity* (2.1). *If every section* $[0, a]$ *is finite, then* S *is commutative, i.e.,* $x * y = x \circ y$ *for all* $x, y \in S$.

P r o o f .

If $[0, a]$ is a finite set, then by [6; Theorem 3.2], $\big\langle [0, a]; \oplus_a, \neg_a, \sim_a, 0, a\big\rangle$ is an MV-algebra, that is, $\neg_a x = \sim_a x$ for all $x \in [0, a]$. Hence $f_a \colon x \mapsto \neg_a x$ is an antitone involution on $[0, a]$. Consequently, we have $x * y = f_x(x \wedge y) = f_x^{-1}(x \wedge y) = x \circ y$ for all $x, y \in S$. □

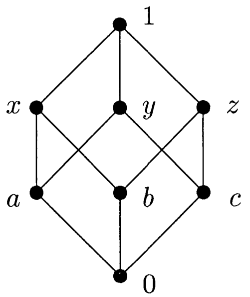

Figure 2.

Remark 2.9. Due to Corollary 2.6, every bounded SAP-lattice satisfying the equation (2.1) is distributive. In addition, by Corollary 2.8, every finite SAP-lattice with (2.1) is commutative in the sense that the operations $*$ and $\circ$ coincide. We now give an example of a finite non-commutative distributive SAP-lattice in which (2.1) fails to be true:

Example 2.10. Let $\langle L; \vee, \wedge\rangle$ denote the lattice whose Hasse diagram is shown in Figure 2. Let the antitone permutation f_1 on $L = [0, 1]$ be defined by $0 \mapsto 1$, $a \mapsto y$, $y \mapsto c$, $c \mapsto z$, $z \mapsto b$, $b \mapsto x$, $x \mapsto a$ and $1 \mapsto 0$; the antitone permutations on the other sections assign to an element its relative complement in the section. The SAP-lattice is not commutative since e.g. $1 * a = f_1(a) = y \neq x = f_1^{-1}(a) = 1 \circ a$. Moreover, it is straightforward to verify that e.g. $(1 * a) \circ b = y \circ b = y$ while $(1 \circ b) * a = z * a = z$.

THEOREM 2.11. *Let $\langle S; \wedge, 0\rangle$ be a meet-semilattice with 0 such that every section $[0,a]$, $a \in S\setminus\{0\}$, is a carrier of a* GMV*-algebra $\langle [0,a]; \oplus_a, \neg_a, \sim_a, 0, a\rangle$ whose natural order coincides with that induced by $\wedge$. Assume that the following compatibility condition is satisfied:*

If $x \le a \le b$, then $\neg_a x = \neg_b x \odot_b a$ and $\sim_a x = a \odot_b \sim_b x$.

Define

$$x * y := \begin{cases} \neg_x (x \wedge y) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and

$$x \circ y := \begin{cases} \sim_x (x \wedge y) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $\langle S; \wedge, 0, *, \circ\rangle$ is an SAP*-semilattice satisfying the identity* (2.1).

Proof. If $x = 0$, then all the identities (1.1), (1.2) and (2.1) obviously hold, so let $x \neq 0$. Thus $x * (x \circ y) = \neg_x\big(x \wedge \sim_x(x \wedge y)\big) = \neg_x \sim_x (x \wedge y) = x \wedge y$ and similarly $x \circ (x * y) = x \wedge y$, which verifies (1.1). The identities (1.2) are also almost evident since $x \wedge y \ge x \wedge y \wedge z$ implies $x * y = \neg_x(x \wedge y) \le \neg_x(x \wedge y \wedge z) = x * (y \wedge z)$ and $x \circ y = \sim_x(x \wedge y) \le \sim_x(x \wedge y \wedge z) = x \circ (y \wedge z)$.

In proving (2.1) we make use of the following claim:

CLAIM. *In any* GMV*-algebra we have the identity $\neg x \odot \sim(\neg x \wedge y) = \neg(\sim y \wedge x) \odot \sim y$.*

Calculate

$$\begin{aligned} \neg x \odot \sim(\neg x \wedge y) &= \neg x \odot \sim\big(\neg(\neg x \odot \sim y) \odot \neg x\big) \\ &= \neg x \wedge (\neg x \odot \sim y) \\ &= \neg x \odot \sim y \\ &= \sim y \wedge (\neg x \odot \sim y) \\ &= \neg\big(\sim y \odot \sim(\neg x \odot \sim y)\big) \odot \sim y \\ &= \neg(\sim y \wedge x) \odot \sim y \,. \end{aligned}$$

Assume that $x * y \neq 0 \neq x \circ z$. We have

$$\begin{aligned} (x * y) \circ z &= \sim_{\neg_x (x \wedge y)}\big(\neg_x (x \wedge y) \wedge z\big) \\ &= \sim_{\neg_x (x \wedge y)}\big(\neg_x (x \wedge y) \wedge x \wedge z\big) \\ &= \neg_x (x \wedge y) \odot_x \sim_x\big(\neg_x (x \wedge y) \wedge x \wedge z\big) \end{aligned}$$

by the compatibility condition for $\neg_x(x\wedge y)\wedge x\wedge z \le \neg_x(x\wedge y) \le x$ and similarly

$$\begin{aligned}(x\circ z)*y &= \neg_{\sim_x(x\wedge z)}\big(\sim_x(x\wedge z)\wedge y\big)\\ &= \neg_{\sim_x(x\wedge z)}\big(\sim_x(x\wedge z)\wedge x\wedge y\big)\\ &= \neg_x\big(\sim_x(x\wedge z)\wedge x\wedge y\big)\odot_x \sim_x(x\wedge z)\end{aligned}$$

by the compatibility condition for $\sim_x(x\wedge z)\wedge x\wedge y \le \sim_x(x\wedge z)\le x$. Now by the claim, for $x\wedge y,\ x\wedge z\in[0,x]$ we obtain $(x*y)\circ z=(x\circ z)*y$.

If $x*y=0$, then $(x*y)\circ z=0$ and $x\le y$ since $\neg_x(x\wedge y)=x*y=0$ implies $x\wedge y=\sim_x 0=x$. This along with $x\circ z\le x$ yields $x\circ z\le y$, whence $(x\circ z)*y=\neg_{x\circ z}\big((x\circ z)\wedge y\big)=\neg_{x\circ z}(x\circ z)=0$ if $x\circ z\ne 0$. Analogously, if $x\circ z=0$, then $(x*y)\circ z=(x\circ z)*y=0$. □

Remark 2.12. Observe that the compatibility condition can be captured by the identities

$$\neg_{y\wedge z}(x\wedge y\wedge z)=\neg_z(x\wedge y\wedge z)\odot_z(y\wedge z)\,,$$
$$\sim_{y\wedge z}(x\wedge y\wedge z)=(y\wedge z)\odot_z\sim_z(x\wedge y\wedge z)\,.$$

3. Interval GMV-algebras

In [3] we proved that if $\langle A;\oplus,\neg,0,1\rangle$ is an MV-algebra and $a\in A\setminus\{1\}$, then the structure $\big\langle[a,1];\oplus_a,\neg_a,a,1\big\rangle$ is an MV-algebra, where $x\oplus_a y=\neg(a\oplus\neg x)\oplus y$ and $\neg_a x=\neg x\oplus a$. This leads to the following analogue of Corollary 2.7:

PROPOSITION 3.1. *Let $\langle A;\oplus,\neg,\sim,0,1\rangle$ be a GMV-algebra and $a\in A\setminus\{1\}$. Then upon defining $x\oplus_a y:=\neg(a\oplus\sim x)\oplus y=x\oplus\sim(\neg y\oplus a)$, $\neg_a x:=\neg x\oplus a$ and $\sim_a x:=a\oplus\sim x$, $\big\langle[a,1];\oplus_a,\neg_a,\sim_a,a,1\big\rangle$ is a GMV-algebra.*

P r o o f. We first show that $\neg(a\oplus\sim x)\oplus y=x\oplus\sim(\neg y\oplus a)$. For calculate

$$\begin{aligned}\neg(a\oplus\sim x)\oplus y &= \neg(a\oplus\sim x)\oplus(a\vee y)\\ &= \neg(a\oplus\sim x)\oplus a\oplus\sim(\neg y\oplus a)\\ &= (a\vee x)\oplus\sim(\neg y\oplus a)\\ &= x\oplus\sim(\neg y\oplus a)\,.\end{aligned}$$

Now we have

$$\begin{aligned}(x\oplus_a y)\oplus_a z &= \big(\neg(a\oplus\sim x)\oplus y\big)\oplus_a z\\ &= \neg(a\oplus\sim x)\oplus y\oplus\sim(\neg z\oplus a)\\ &= \neg(a\oplus\sim x)\oplus(y\oplus_a z)\\ &= x\oplus_a(y\oplus_a z)\,,\end{aligned}$$

which is (A1).

One readily sees (A2)–(A4): $x \oplus_a a = x \oplus \sim(\neg a \oplus a) = x \oplus \sim 1 = x \oplus 0 = x$ and similarly $a \oplus_a x = x$; $x \oplus_a 1 = \neg(a \oplus \sim x) \oplus 1 = 1 = 1 \oplus_a x$, and finally, $\neg_a 1 = \neg 1 \oplus a = 0 \oplus a = a = \sim_a 1$.

Furthermore,

$$\begin{aligned}\neg_a(\sim_a x \oplus_a \sim_a y) &= \neg\big(a \oplus \sim x \oplus \sim\big(\neg(a \oplus \sim y) \oplus a)\big) \oplus a \\ &= \neg\big(a \oplus \sim x \oplus \sim(a \vee y)\big) \oplus a \\ &= \neg\big(a \oplus \sim x \oplus \sim y\big) \oplus a \\ &= \neg\big(a \oplus \sim(x \odot y)\big) \oplus a \\ &= (x \odot y) \vee a\end{aligned}$$

and analogously $\sim_a(\neg_a x \oplus_a \neg_a y) = (x \odot y) \vee a$ proving the identity (A5). To see (A6), compute

$$\begin{aligned}x \oplus_a (y \odot_a \sim_a x) &= x \oplus_a \sim_a(\neg_a y \oplus_a x) \\ &= x \oplus_a \big(a \oplus \sim\big((\neg y \oplus a) \oplus \sim(\neg x \oplus a)\big)\big) \\ &= x \oplus_a \big(a \oplus \sim\big(\neg y \oplus (a \vee x)\big)\big) \\ &= x \oplus_a \big(a \oplus \sim(\neg y \oplus x)\big) \\ &= \neg(a \oplus \sim x) \oplus \big(a \oplus \sim(\neg y \oplus x)\big) \\ &= (a \vee x) \oplus \sim(\neg y \oplus x) \\ &= x \oplus \sim(\neg y \oplus x) \\ &= x \vee y\,.\end{aligned}$$

The parallel argument shows that $(\neg_a x \odot_a y) \oplus_a x = x \vee y$ and by replacing x and y we obtain the remaining equations in (A6).

Note that we have shown that $x \odot_a y = (x \odot y) \vee a$ for any $x, y \in [a, 1]$. Hence

$$\begin{aligned}(\neg_a x \oplus_a y) \odot_a x &= \big((\neg_a x \oplus_a y) \odot x\big) \vee a \\ &= \big(\big((\neg x \oplus a) \oplus \sim(\neg y \oplus a)\big) \odot x\big) \vee a \\ &= \big(\big(\neg x \oplus (a \vee y)\big) \odot x\big) \vee a \\ &= \big((\neg x \oplus y) \odot x\big) \vee a \\ &= (x \wedge y) \vee a = x \wedge y\end{aligned}$$

and similarly $y \odot_a (x \oplus_a \sim_a y) = x \wedge y$, which verifies (A7).

Finally, (A8) is obvious since $\sim_a \neg_a x = a \oplus \sim(\neg x \oplus a) = a \vee x = x$. □

Let $\langle A;\oplus,\neg,\sim,0,1\rangle$ be a GMV-algebra and let $a,b\in A$, $a<b$. By the previous proposition, $\langle [a,1];\oplus_a,\neg_a,\sim_a,a,1\rangle$ is a GMV-algebra again. By Corollary 2.7 we get that $\langle [a,b];\oplus_{ab},\neg_{ab},\sim_{ab},a,b\rangle$ is a GMV-algebra, where

$$\begin{aligned}
x\oplus_{ab} y &= (x\oplus_a y)\wedge b\\
&= \big(x\oplus(y\odot\sim a)\big)\wedge b\\
&= \big((\neg a\odot x)\oplus y\big)\wedge b\,,
\end{aligned}$$

$$\begin{aligned}
\neg_{ab}x &= \neg_a x\odot_a b = \big((\neg x\oplus a)\odot b\big)\vee a\\
&= \big(\neg a\odot(\neg x\oplus a)\odot b\big)\oplus a\\
&= \big(\neg\big(a\oplus\sim(\neg x\oplus a)\big)\odot b\big)\oplus a\\
&= \big(\neg(a\vee x)\odot b\big)\oplus a\\
&= (\neg x\odot b)\oplus a
\end{aligned}$$

and similarly

$$\sim_{ab}x = b\odot_a \sim_a x = a\oplus(b\odot\sim x)\,.$$

We have obtained:

THEOREM 3.2. *Let $\langle A;\oplus,\neg,\sim,0,1\rangle$ be a* GMV*-algebra and let $a,b\in A$ be such that $a<b$. Define $x\oplus_{ab} y := \big(x\oplus(y\odot\sim a)\big)\wedge b = \big((\neg a\odot x)\oplus y\big)\wedge b$, $\neg_{ab}x := (\neg x\odot b)\oplus a$ and $\sim_{ab}x := a\oplus(b\odot\sim x)$ for $x,y\in[a,b]$. Then $\langle [a,b];\oplus_{ab},\neg_{ab},\sim_{ab},a,b\rangle$ is a* GMV*-algebra.*

We call an element a of a GMV-algebra A *Boolean* if it possesses the complement a' in the underlying lattice of A; the set of all Boolean elements of A is denoted by $B(A)$. By [7; Propositions 4.2, 4.3] (cf. also [9; Theorem 9]), $a\in B(A)$ if and only if $a\oplus a=a$ if and only if $a\odot a=a$, and if $a\in B(A)$, then $a\oplus x=x\oplus a=a\vee x$ and likewise $a\odot x=x\odot a=a\wedge x$ for all $x\in A$. Of course, $a'=\neg a=\sim a$ for any $a\in B(A)$.

COROLLARY 3.3. *Let $\langle [a,b];\oplus_{ab},\neg_{ab},\sim_{ab},a,b\rangle$ be that from Theorem* 3.2. *If $a,b\in B(A)$, then $x\oplus_{ab}y=x\oplus y$, $\neg_{ab}x=(\neg x\wedge b)\vee a$ and $\sim_{ab}x=(\sim x\wedge b)\vee a$.*

Proof. We have

$$x\oplus_{ab} y = \big(x\oplus(y\wedge\sim a)\big)\wedge b = (x\oplus y)\wedge(x\oplus\sim a)\wedge b = x\oplus y$$

since $x\oplus y\le b\oplus b=b$ and $x\oplus\sim a\ge a\oplus\sim a=1$. The rest is evident. □

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