Rudolf Oláh
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NOTE ON THE OSCILLATION
OF DIFFERENTIAL EQUATION WITH ADVANCED
ARGUMENT

RUDOLF OLÁH

We want to consider the oscillatory behaviour of solutions of the nonlinear
differential equation with advanced argument
(1) \[ y^{(n)}(t) + p(t)f(y(g(t))) = 0, \quad n \geq 2, \]
where:

a) \( p(t) \) is continuous and nonnegative on \([t_0, \infty)\);

b) \( g(t) \) is a nondecreasing continuous function on \([t_0, \infty)\) and such that \( t < g(t) \);

c) \( f(u) \) is a continuous function on \((-\infty, \infty)\) such that \( uf(u) > 0 \) for \( u \neq 0 \).

A solution \( y(t) \) of the equation (1) is called oscillatory if it has arbitrarily large
zeros, and it is called nonoscillatory otherwise.

We introduce the notation:
\[ M_I = \max \left\{ \limsup_{y \to -\infty} -y, \limsup_{y \to -\infty} \frac{y}{f(y)} \right\} > 0. \]

We restrict our consideration to those solutions \( y(t) \) of (1) which exist on some
interval \([T_y, \infty)\) and satisfy
\[ \sup \{|y(t)|: t_0 \leq t < \infty\} > 0 \quad \text{for any} \quad t_0 \in [T_y, \infty). \]

Lemma 1 (Kiguradze) [1]. Let \( y(t) \) be a solution of the equation (1) satisfying the condition
\[ y(t) > 0 \quad \text{for} \quad t \in [T_y, \infty) \]
and let \( y^{(n)}(t) \leq 0 \) for \( t \in [t_0, \infty) \).

Then there exist a \( t_i \in [t_0, \infty) \) and an integer \( l \in \{0, 1, \ldots, n\} \) such that \( l + n \) is odd
and
\[ y^{(i)}(t) > 0 \quad \text{for} \quad t \in [t_i, \infty) \quad (i = 0, \ldots, l - 1), \]
\[ (-1)^{i+l} y^{(i)}(t) > 0 \quad \text{for} \quad t \in [t_i, \infty) \quad (i = l, \ldots, n - 1). \]

An analogous statement can be made if \( y(t) < 0 \) and \( y^{(n)}(t) \geq 0 \) for \( t \in [t_0, \infty) \).

The next lemma characterizes the oscillatory behaviour of bounded solutions.
Lemma 2. Suppose that the conditions a)—c) are satisfied and, in addition,
\begin{equation}
\int_{0}^{\infty} t^{n-1} p(t) \, dt = \infty.
\end{equation}

Then every bounded solution of equation (1) is oscillatory if \( n \) is even, and every bounded solution of equation (1) is oscillatory or \( \lim_{t \to \infty} y^{(i)}(t) = 0, \ i = 0, 1, \ldots, n-1, \) if \( n \) is odd.

Proof. Let \( y(t) \) be a bounded and positive solution of equation (1) on \([t_{0}, \infty)\). From the equality
\begin{equation}
y^{(i)}(t) = \sum_{i=j}^{n-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} y^{(j)}(s) +
\end{equation}

\begin{equation}
+ \frac{(-1)^{n-j}}{(n-j-1)!} \int_{s}^{t} (u-t)^{n-j-1} y^{(n)}(u) \, du,
\end{equation}

\( s \geq t \geq t_{0}, \) with regard to equation (1) we get
\begin{equation}
y^{(i)}(t) = \sum_{i=j}^{n-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} y^{(j)}(s) +
\end{equation}

\begin{equation}
+ \frac{(-1)^{n-j+1}}{(n-j-1)!} \int_{s}^{t} (u-t)^{n-j-1} p(u) f(y(g(u))) \, du.
\end{equation}

Let \( n \) be even. Since \( y(t) \) is a positive and bounded solution of equation (1), in view of Lemma 1 we have \( l = 1 \) and for \( j = 1 \), from (4) we get
\begin{equation}
y'(t) \geq \frac{1}{(n-2)!} \int_{t}^{\infty} (u-t)^{n-2} p(u) f(y(g(u))) \, du.
\end{equation}

Integrating the last inequality from \( T \) to \( t, \ t > T \geq t_{0}, \) we obtain
\begin{equation}
y(t) \geq \frac{1}{(n-1)!} \int_{T}^{t} (u-T)^{n-1} p(u) f(y(g(u))) \, du.
\end{equation}

Let \( y(t) \to c > 0 \) as \( t \to \infty. \) Since \( y(t) \) is nondecreasing, \( \frac{c}{2} \leq y(t) < c \) for \( t \geq t_{1} \geq T. \)

Then there exist positive constants \( c_{1}, c_{2} \) such that \( c_{1} \leq f(y(g(t))) \leq c_{2}, \ t \geq t_{1}. \) As \( t \to \infty, \) we have
\begin{equation}
c \geq \frac{c_{1}}{2} \frac{c_{1}}{(n-1)!} \int_{t_{1}}^{\infty} (u-T)^{n-1} p(u) \, du,
\end{equation}

which is a contradiction to (3).

Let \( n \) be odd. In view of the fact that \( y(t) \) is bounded, \( l = 0 \) and from the equality (4) for \( j = 0 \) we get

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Let \( y(t) \to L > 0 \) as \( t \to \infty \). Since \( y(t) \) is a nonincreasing solution of the equation (1), then \( L < y(t) \leq 2L \) for \( t \geq t_1 \geq T \). Then there exist positive constants \( L_1, L_2 \) such that \( L_1 \leq f(y(g(t))) \leq L_2, \ t \geq t_1 \). As \( t \to \infty \), we get

\[
y(T) > y(T) - L \geq \frac{L_1}{(n-1)!} \int_t^T (u - T)^{n-1} p(u) \, du,
\]

which is a contradiction to (4), so \( \lim_{t \to \infty} y(t) = 0 \). The proof of Lemma 2 is complete.

In this paper the theorems have specific character for differential equations with advanced argument. The assertions of these theorems are not true for the corresponding ordinary differential equations.

**Theorem 1.** Suppose that the conditions a)—c) are satisfied, \( M_f < \infty \) and in addition

\[
\lim_{t \to \infty} \sup \int_t^{a(t)} (s - t)^{n-1} p(s) \, ds > M_f(n-1)!.
\]

Then every solution of equation (1) is oscillatory if \( n \) is even, and every solution of equation (1) is oscillatory or \( \lim_{t \to \infty} y^{(i)}(t) = 0, \ i = 0, 1, \ldots, n-1 \), if \( n \) is odd.

**Proof.** Let \( y(t) \) be a nonoscillatory solution of the equation (1). Without loss of generality we may suppose that \( y(t) \) is eventually positive on \( [t_0, \infty) \).

Suppose that \( n \) is even and \( l = 1 \). From (4) with regard to Lemma 1 for \( j = 1 \) we obtain

\[
y'(t) \leq \frac{1}{(n-2)!} \int_t^{\sigma(t)} (u - t)^{n-2} p(u) f(y(g(u))) \, du, \quad t \geq t_0.
\]

Integration of the last inequality from \( t \) to \( g(t), \ t > t_0 \), yields

\[
y(g(t)) \geq \frac{1}{(n-1)!} \int_t^{a(t)} (u - t)^{n-1} p(u) f(y(g(u))) \, du.
\]

We remind that the condition (5) implies (3). If now \( y(t) \) increases to a finite limit as \( t \to \infty \), then similarly as in the proof of Lemma 2 we get a contradiction to (3).

Let \( y(t) \) increase to infinity as \( t \to \infty \). From (6) we get

\[
y(g(t)) \geq \frac{y(g(t))}{(n-1)!} \int_t^{a(t)} (u - t)^{n-1} p(u) \frac{f(y(g(u)))}{y(g(u))} \, du,
\]

\[
(n-1)! \geq \inf_{\sigma(t) \geq u \geq t} \frac{f(y(g(u)))}{y(g(u))} \int_t^{a(t)} (u - t)^{n-1} p(u) \, du,
\]

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\[ (n - 1)! \sup_{y(g(t)) \neq y(g(u))} \frac{z}{f(z)} \geq \int_{t}^{y(g(t))} (u - t)^{n-1} p(u) \, du, \]
\[ \lim_{t \to \infty} \sup_{y(g(t)) \neq y(g(u))} \frac{z}{f(z)} \geq \lim_{t \to \infty} \sup_{y(g(t)) \neq y(g(u))} \int_{t}^{y(g(t))} (u - t)^{n-1} p(u) \, du, \]
which is a contradiction to the condition \( (5) \).

Let \( n \) be odd and \( l = 0 \). In view of Lemma 1, from (4) for \( j = 0, t > t_0 \), we have
\[ y(t_0) - y(t) \geq \frac{1}{(n-1)!} \int_{t_0}^{t} (u - t_0)^{n-1} p(u) f(y(g(u))) \, du. \]
Since \( y'(t) \leq 0 \) for \( t > t_0 \), \( y(t) \) decreases to limit \( L \geq 0 \) as \( t \to \infty \). Let \( L > 0 \). Then similarly as in the proof of Lemma 2 we get a contradiction to (3), so \( \lim_{t \to \infty} y(t) = 0 \).

Let \( l \in \{2, \ldots, n - 1 \} \). With regard to Lemma 1 from (4) for \( j = l, t > t_0 \), we have
\[ y^{(l)}(t) \geq \frac{1}{(n-l-1)!} \int_{t}^{\infty} (u - t)^{n-l-1} p(u) f(y(g(u))) \, du. \]
By integrating the last inequality from \( t_0 \) to \( t, t > t_0 \), we obtain
\[ y^{(l-1)}(t) \geq \frac{(t - t_0)^{n-l}}{(n-l)!} \int_{t}^{\infty} p(u) f(y(g(u))) \, du. \]
Repeating this procedure we get
\[ y'(t) \geq \frac{(t - t_0)^{n-2}}{(n-2)!} \int_{t}^{\infty} p(u) f(y(g(u))) \, du. \]
We integrate last inequality from \( t \) to \( g(t), t > t_0 \),
\[ y(g(t)) \geq \frac{1}{(n-2)!} \int_{t}^{\infty} p(u) f(y(g(u))) \int_{u}^{\infty} (s - t_0)^{n-2} \, ds \, du, \]
\[ y(g(t)) \geq \frac{1}{(n-1)!} \int_{t}^{\infty} (u - t)^{n-1} p(u) f(y(g(u))) \, du, \]
which is the inequality (6). The proof now proceeds as above, when \( y(t) \) increases to infinity. This completes the proof.

**Corollary 1.** We consider the differential equation
\[ (7) \quad y^{(n)}(t) + p(t) y(g(t)) = 0. \]
Suppose that the conditions a), b) are satisfied and in addition
\[ (8) \quad \lim_{t \to \infty} \sup_{y(g(t)) \neq y(g(u))} \int_{t}^{y(g(t))} (s - t)^{n-1} p(s) \, ds > (n - 1)!. \]
Then every solution of the equation (7) is oscillatory if \( n \) is even, and every solution of the equation (7) is oscillatory or \( \lim_{t\to\infty} y^{(i)}(t) = 0, \ i = 0, 1, \ldots, n - 1, \) if \( n \) is odd.

It can occur that the ordinary differential equation has a nonoscillatory solution, but if the corresponding differential equation with advanced argument has a solution, then this solution is oscillatory.

**Example 1.** The ordinary differential equation

\[
y''(t) + \frac{4}{t^3} y(t) = 0, \quad t > 0,
\]

has a nonoscillatory solution \( y(t) = t^4 \), but the corresponding differential equation with advanced argument

\[
y''(t) + \frac{1}{4t^2} y(149t) = 0, \quad t > 0,
\]

in view of the condition (8), has every solution oscillatory.

**Theorem 2.** Suppose that the conditions a)—c) are satisfied, \( M_r < \infty \) and in addition

\[
\lim_{t\to\infty} \sup \int_t^\infty \int_s^\infty (u-s)^{n-2} p(u) du \ ds > M_r(n-2)!
\]

Then the equation (1) has no solution satisfying (2i), \( \lim_{t\to\infty} y^{(i)}(t) = 0, \ i = 0, 1, \ldots, n - 1, \) for every solution of the equation (1) which satisfies (2ii).

**Proof.** Let \( y(t) \) be a positive solution of the equation (1) on \([t_0, \infty)\). Let \( l = 1 \). Then \( n \) is even and from (4) for \( j = 1 \) we get

\[
y'(t) \geq \frac{1}{(n-2)!} \int_t^\infty (u-t)^{n-2} p(u) f(y(g(u))) du.
\]

Integrating from \( t \) to \( g(t), \ t > t_0, \) we obtain

\[
y(g(t)) \geq \frac{1}{(n-2)!} \int_t^{g(t)} \int_s^\infty (u-s)^{n-2} p(u) f(y(g(u))) du \ ds.
\]

We remind that the condition (9) implies (3). Otherwise if

\[
\int t^{n-1} p(t) \ dt < \infty,
\]

then

\[
0 < \lim_{t\to\infty} \sup \int_t^{g(t)} \int_s^\infty (u-s)^{n-2} p(u) du \ ds \leq
\]

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\[
\limsup_{t \to +\infty} \int_0^t \int_s^\infty (u-s)^{n-2} p(u) \, du \, ds = \limsup_{t \to +\infty} \frac{1}{n-1} \int_0^t (u-t)^{n-1} p(u) \, du \leq \\
\limsup_{t \to +\infty} \frac{1}{n-1} \int_0^t (u-t_0)^{n-1} p(u) \, du = 0,
\]

which is a contradiction.

Let \( y(t) \) increase to a finite limit as \( t \to \infty \). We integrate (10) from \( t_0 \) to \( t \),
\[
y(t) \geq \frac{1}{(n-1)!} \int_{t_0}^t (u-t_0)^{n-1} p(u)f(y(g(u))) \, du.
\]

Similarly as in the proof of Lemma 2 we get a contradiction to (3).

Let \( y(t) \) increase to infinity as \( t \to \infty \). From (11) we get
\[
(n-2)! \sup_{z \geq y(g(t))} \frac{z}{f(z)} \geq \limsup_{t \to +\infty} \int_0^t \int_s^\infty (u-s)^{n-2} p(u) \, du \, ds,
\]
\[
(n-2)! \limsup_{t \to +\infty} \frac{z}{f(z)} \geq \limsup_{t \to +\infty} \int_0^t \int_s^\infty (u-s)^{n-2} p(u) \, du \, ds,
\]

which is a contradiction to condition (9).

Let \( l = 0 \). Then \( n \) is odd and from (4) for \( j = 0 \) we have
\[
y(t_0) - y(t) \geq \frac{1}{(n-1)!} \int_{t_0}^t (u-t_0)^{n-1} p(u)f(y(g(u))) \, du.
\]

Let \( \lim_{t \to +\infty} y(t) = L > 0 \). In view of the fact that the condition (9) implies (3), similarly as in the proof of Lemma 2 we get a contradiction to (3). So \( \lim_{t \to +\infty} y(t) = 0 \).

**Corollary 2.** We consider the differential equation
\[
y''(t) + p(t)f(y(g(t))) = 0.
\]

Suppose that the conditions a)---c) are satisfied, \( M_f < \infty \) and in addition
\[
\limsup_{t \to +\infty} \int_0^t \int_s^\infty p(u) \, du \, ds > M_f.
\]

Then every solution of this equation is oscillatory.

**Example 2.** We cannot decide about the oscillatory character of solutions of the differential equation with advanced argument
\[
y''(t) + \frac{1}{4t} y(81t) = 0, \quad t > 0,
\]

with regard to the condition (8). But in view of condition (12) every solution of this equation is oscillatory.

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Theorem 3. Suppose that the conditions a)—c) are satisfied, $M_f<\infty$ and in addition

\begin{equation}
\limsup_{t \to \infty} \int_t^{\varphi(t)} (s-t)s^{\alpha-2} p(s) \, ds > M_f(n-1)!. \tag{13}
\end{equation}

Then the equation (1) has no solution satisfying (2i), $l \in \{2, \ldots, n-1\}$.

Proof. Let $y(t)$ be a positive solution of the equation (1) on $[t_0, \infty)$ which satisfies (2i), $l \in \{2, \ldots, n-1\}$. Similarly as in the proof of Theorem 1 we get

$$y'(t) \geq \frac{(t-t_0)^{n-2}}{(n-2)!} \int_t^\infty p(u) f(y(g(u))) \, du.$$ \hspace{5cm} (1)

We integrate the last inequality from $t$ to $g(t)$, $t>t_0$,

$$y(g(t)) \geq \frac{1}{(n-2)!} \int_t^{\varphi(t)} p(u) f(y(g(u))) \int_u^{\varphi(u)} (s-t_0)^{n-2} \, ds \, du,$$

$$y(g(t)) \geq \frac{1}{(n-1)!} \int_t^{\varphi(t)} (u-t)(u-t_0)^{n-2} p(u) f(y(g(u))) \, du.$$

From the last inequality we have

$$(n-1)! \limsup_{z \to \infty} \frac{z}{f(z)} \geq \limsup_{t \to \infty} \int_t^{\varphi(t)} (u-t)(u-t_0)^{n-2} p(u) \, du,$$

which is a contradiction to (13). The proof is complete.

Theorem 4. Suppose that the conditions a)—c), (9), (13), $M_f<\infty$ are satisfied. Then every solution of the equation (1) is oscillatory if $n$ is even, and every solution of the equation (1) is oscillatory or $\lim_{t \to \infty} y^{(i)}(t) = 0$, $i=0, 1, \ldots, n-1$, if $n$ is odd.

The proof follows from the Theorems 2, 3.

The above results are new. The sufficient condition [2, Th. 8.4] which guarantees that every solution of the equation from the example 2 is oscillatory

$$\int_0^\infty \beta_0^{i-\varepsilon} p(t) \, dt = \infty, \quad \varepsilon > 0, \quad \beta_0(t) = \min \{ t, g(t) \},$$

is not satisfied. But the condition (12) is satisfied.

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Katedra matematiky
Strojno-elektrotechnickej fakulty VŠDS
Marxa-Engelsa 25
010 88 Žilina

ЗАМЕТКА О КОЛЕБЛЕМОСТИ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ОПЕРЕЖАЮЩИМ АРГУМЕНТОМ

Rudolf Oláh

Резюме

В работе приведены достаточные условия для того, чтобы каждое решение уравнения (1) при четном $n$ являлось колеблющимся, а при нечетном $n$, либо колеблющимся, либо удовлетворяло условию

$$\lim_{t \to \infty} y^{(i)}(t) = 0, \quad i = 0, ..., n-1.$$