Gary Chartrand; Ortrud R. Oellermann; Sergio Ruiz

Randomly H graphs

Mathematica Slovaca, Vol. 36 (1986), No. 2, 129--136

Persistent URL: http://dml.cz/dmlcz/130913

Math. Slovaca 36, 1986, No. 2, 129—136

RANDOMLY H GRAPHS

GARY CHARTRAND[1], ORTRUD R. OELLERMANN[2], SERGIO RUIZ

Introduction and historical background

In 1951 Ore [13] defined and then studied graphs that he called arbitrarily traceable but which were later to be referred to as randomly eulerian graphs. A graph G is randomly eulerian from a vertex v of G if every trail of G with initial vertex v can be extended to a v-v eulerian circuit. (See [2] or [12] for basic graph theory terminology.) A graph is randomly eulerian if it is randomly eulerian from each of its vertices. These graphs were studied further by Bäbler [1] and Harary [11]; this concept was later extended by Chartrand and White [6] Erickson [8].

In 1968 Chartrand and Kronk [3] introduced the concept of randomly hamiltonian graphs. A graph G is randomly hamiltonian from a vertex v if every path of G with initial vertex v can be extended to a v-v hamiltonian cycle. A graph is randomly hamiltonian if it is randomly hamiltonian from each of its vertices. Randomly hamiltonian graphs were characterized in [3] as follows:

Theorem A. *A graph G of order p ($\geqslant 3$) is randomly hamiltonian if and only if G is isomorphic to one of K_p, C_p and $K(p/2, p/2)$, the last being possible if and only if p is even.*

In 1973 Dirac and Thomassen [7] determined, for given integers n and p with $3 \leqslant n \leqslant p$, all those graphs G of order p having the property that G contains paths of length $n-1$ and every such path can be extended to an n-cycle. For a positive integer k and a nonempty set S of positive integers, Parsons [14] studied those graphs G containing paths of length k and for which every such path can be extended to an s-cycle of G, for some $s \in S$.

In 1973 Thomassen [16] studied graphs G that are randomly traceable from a vertex v of G, i.e., every path of G with initial vertex v can be extended to a hamiltonian path with initial vertex v. Thomassen [17] also characterized those

[1]Research supported by a Western Michigan University faculty research fellowship.

[2]Research supported in part by a College of Arts and Sciences and Graduate College research assistantship from Western Michigan University.

graphs G of order $p(\geq 4)$ for which every path of length at most $p-2$ can be extended to a $(p-1)$-cycle.

In 1984 Fink [10] extended the concept of randomly traceable grahps to consider graphs he called randomly near-traceable, a graph G being *randomly near-traceable* if, in any depth-first search of that graph, whenever we backtrack to a previously visited vertex, that vertex is adjacent to at least one unvisited vertex.

In 1979 Sumner [15] defined and characterized randomly matchable graphs, namely those graphs in which every set of independent edges can be extended to a l-factor.

Thus far in literature, nearly all subgraphs that have been defined to occur randomly, in some sense, have been spanning subgraphs of the given graph (i.e., factors) and, with the exception of Sumner's randomly matchable graphs, all subgraphs have had some sequential aspect of them. In this article, we generalize these concepts in a manner so that neither of the above restrictions is required.

We note in concluding this section that some of the concepts defined above have analogues in directed graphs (see [4], [5], [8], [9] and [11]).

Randomly H graphs

Let G be a graph containing a subgraph H without isolated vertices. Then G is called a *randomly H graph* if whenever F is a subgraph of G without isolated vertices that is isomorphic to a subgraph of H, then F can be extended to a subgraph H_1 of G such that $H_1 \cong H$. Thus, every nonempty graph is randomly K_2 while every graph G without isolated vertices is a randomly G graph. The graph G of Figure 1 is not randomly $2K_2$ since the subgraph F_1 of G cannot be extended to a subgraph of G isomorphic to $2K_2$. Although this graph G is randomly P_3, it is not randomly P_4 since the subgraph F_2 of G cannot be extended to a subgraph of G isomorphic to P_4.

Before proceeding further, we make a few comments regarding the definition given of randomly H graphs. In the definition it was stipulated that H and F be without isolated vertices. Suppose that H has order m and that $\mathcal{G}$ is class of all randomly H graphs. Let $H' = H \cup nK_1$, where $n \geq 1$. If, in the definition of randomly H graphs, we were to delete the requirement that H be without isolated vertices, then the class $\mathcal{G}'$ of all randomly H' graphs consists of all those elements of $\mathcal{G}$ having order at least $m+n$. Hence it suffices to assume that H is without isolated vertices.

Since a graph G without isolated vertices is a randomly H graph if and only if $G \cup K_1$ is randomly H, we assume that every randomly H graph is free of isolated vertices. In order to avoid a situation where only complete graphs would be randomly H for a variety of graph H, we have required F to be without isolated vertices; otherwise, for example, only complete gıaphs of order at least 2 would be randomly K_2 (which can be seen by taking $F = \bar{F}_2$, where $V(\bar{K}_2) = \{x, y\}$ and x and y are any two vertices of G).

In this section we present characterizations of randomly H graphs for certain graphs H of small size as well as when H is complete or a star. A *double star* is a tree containing exactly two vertices that are not end-vertices.

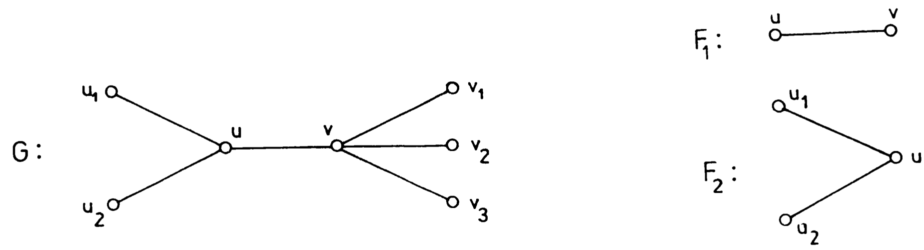

Fig. 1

Proposition 1. (i) *A graph G is randomly $2K_2$ if and only if G is neither a star nor a double star.*

(ii) A graph G is randomly P_3 if and only if no component of G is isomorphic to K_2.

Again, recall that we are considering only randomly H graphs that are free of isolated vertices. We now characterize randomly K_n graphs.

Proposition 2. (i) *A graph G is randomly K_3 if and only if each component of G is a complete graph of order at least 3.*

(ii) *A graph G is randomly K_n ($n \geq 4$) if and only if $G \cong K_p$ for some $p \geq n$.*

Proof. (i) The sufficiency is obvious. Assume that G is randomly K_3. Certainly, then, every component of G has order at least 3. Suppose, to the contrary, that G contains a component G_1 that is not complete. Then G_1 contains nonadjacent vertices u and v having the distance $d(u, v) = 2$. Let u, w, v be a $u - v$ path of length 2 in G_1. Then $\langle \{u, w, v\} \rangle \cong P_3$, which is a subgraph of K_3. This implies that uv is an edge of G_1, producing the desired contradiction.

(ii) Again, the sufficiency follows immediately. For necessity, let G be a randomly K_n graph, where $n \geq 4$. First we show that G is connected. If G were not connected, then selecting an edge from each of two components of G produces a subgraph F of G isomorphic to $2K_2$, which is a subgraph of K_n. However, F cannot be extended to a graph isomorphic to K_n. Thus G is connected, as claimed. Necessarily, G has order at least n. That G is complete follows by the same argument as that used in the proo of (i).

We now present a characterization of randomly H graphs, where H is a star, i.e., where $H \cong K(l, n)$ for some $n \geq 1$. Since we have already considered the cases $n = 1$ and $n = 2$, we assume that $n \geq 3$.

Proposition 3. *A graph G is randomly $K(l, n)$, $n \geqslant 3$, if and only if G contains no component isomorphic to K_2 and every vertex of G has degree l or degree at least n.*

Proof. Let G be a randomly $K(l, n)$ graph, $n \geqslant 3$. Certainly, G contains no component isomorphic to K_2. Let v be a vertex of G such that $\deg v > 1$. Then G contains a subgraph $F \cong K(l, 2)$ such that $v \in V(F)$ and $\deg_F v = 2$. Since F can be extended to a subgraph H_1 of G such that $H_1 \cong K(l, n)$, it follows that $\deg_{H_1} v = n$ so that $\deg_G v \geqslant n$.

Conversely, suppose that G contains no component isomorphic to K_2 and every vertex of G has degree l or degree at least $n(\geqslant 3)$. Since G has no component isomorphic to K_2, the graph G must contain at least one vertex having degree at least n. Therefore, G contains a subgraph isomorphic to $K(l, n)$. Let F be a subgraph of $K(l, n)$ without isolated vertices. Then $F \cong K(l, m)$, where $1 \leqslant m \leqslant n$. Assume first that $2 \leqslant m \leqslant n$. Suppose $v \in V(F)$ such that $\deg_F v = m$. Since $\deg_G v \geqslant n$, we can extend F to a subgraph of G that is isomorphic to $K(l, n)$. Next, assume that $F \cong K(l, l) = K_2$. Since G contains no component isomorphic to K_2, at least one vertex u of F has degree different from l in G. By hypothesis, $\deg_G u \geqslant n$, implying that F can be extended to a subgraph of G that is isomorphic to $K(l, n)$. Therefore, G is randomly $K(l, n)$.

We now present a characterization of randomly C_n graphs ($n \geqslant 3$). Our proof will use the following result of Dirac and Thomassen [7].

Theorem B. *Let G be a connected graph of order p containing an n-cycle (so that $3 \leqslant n \leqslant p$). Then G has the property that every path of length $n-1$ can be extended to an n-cycle if and only if G is isomorphic to one of the following:*

(i) K_p, (ii) $C_n (p = n)$, (iii) $K(r, s)$ *is n is even, where $r + s = p$ and $r, s \geqslant n/2$.*

Proposition 4. *The randomly C_n graphs are*

(i) $\bigcup_{i=1}^{k} K_{n_i}$, *where $k \geqslant 1$ and $n_i \geqslant 3$ $(1 \leqslant i \leqslant k)$ if $n - 3$;*

(ii) K_p, *where $p \geqslant 4$, and $K(r, s)$, where $2 \leqslant r \leqslant s$ if $n - 4$;*

(iii) K_p, *where $p \geqslant n$, and C_n if $n \geqslant 5$ and n is odd; and*

(iv) K_p, *where $p \geqslant n$, and $K(n/2, n/2)$ if $n \geqslant 6$ and n is even.*

Proof. First, randomly C_3 graphs are synonymous with randomly K_3 graphs which were characterized in Proposition 2 (i). Thus we consider randomly C_n graphs, where $n \geqslant 4$. Since $2K_2$ is a subgraph of C_n for $n \geqslant 4$, these graphs are necessarily connected. Let G be a randomly C_n graph, where $n \geqslant 4$. Then G is connected and every path of length $n-1$ in G can be extended to an n-cycle. Therefore, by Theorem B, the graph G is isomorphic to K_p, C_n $(p = n)$, or $K(r, s)$ for even n, where $r + s = p$ and $r, s \geqslant n/2$. That $K_p (4 \leqslant n \leqslant p)$ and C_n (if $p = n$) are randomly C_n follows immediately. Also, it is not difficult to verify that for $2 \leqslant r \leqslant s$ the graph $K(r, s)$ is randomly C_4.

We now show that for even $n \geq 6$ the graph $K(r, s)$, where $r \geq n/2$ and $s > n/2$, is *not* randomly C_n. To see this, consider $K(r, s)$ with partite sets U and W, where $|U| = r \geq n/2 \geq 3$ and $|W| = s > n/2 \geq 3$. The graph $P_3 \cup p_{n-3}$ is a subgraph of C_n. Let $F \cong P_3 \cup P_{n-3}$ be a subgraph of $K(r, s)$ with the property that the four end-vertices of F belong to W. Then W contains $(n+2)/2$ vertices of F. (Note that the cardinality of W guarantees the existence of such a subgraph F in $K(r, s)$.) Since no two vertices of W are adjacent, F cannot be extended to an n-cycle.

It remains only to show that for $n = 2r \geq 6$ the graph $K(r, r)$ is randomly C_n. Suppose, to the contrary, that $K(r, r)$ is not randomly C_n. Among the subgraphs of $K(r, r)$ that are subgraphs of n-cycles and that cannot be extended to an n-cycle, let F be one of maximum size q. Since any path of length at most $n-1$ can be extended to an n-cycle in G, the subgraph F must be a linear forest with at least two components (nontrivial paths). Let U and W be the partite sets of $K(r, r)$.

Since

$$\sum_{v \in U} \deg_F v = \sum_{v \in W} \deg_F v = q\,,$$

if one of U and W, say U, contains a vertex u that does not belong to F, then W must contain a vertex w that either does not belong to F or is an end-vertex of F. Then $F + uv$ is a subgraph of an n-cycle. Since the size of $F + uv$ is $q + 1$, the graph $F + uv$ can be extended to an n-cycle in $K(r, r)$, which implies that F can be extended to an n-cycle in $K(r, r)$, contrary to the hypothesis. Therefore, F is a spanning subgraph of $K(r, r)$. Then U and W contain end-vertices u and w, respectively, of F belonging to different paths of F. However, $F + uv$ is a subgraph of an n-cycle, implying, as above, that F can be extended to an n-cycle, which produces a contradiction.

A consequence of this result is the following.

Corollary. *A graph G of order $n(\geq 3)$ is randomly C_n if and only if G is randomly hamiltonian.*

Graphs that are randomly *H* for every subgraph *H*

In this section we characterize those graphs G (without isolated vertices) that are randomly H for every subgraph H of G (without isolated vertices).

Theorem 1. *A graph G is randomly H for every subgraph H of G if and only if G is isomorphic to one of the following: $nK(l, m)$ $(n, m \geq 1)$, nK_3 $(n \geq 1)$, K_p $(p \geq 4)$, C_4, C_5, $K(3, 3)$.*

Proof. We note that it is not difficult to show that each of the graphs listed in the statement of the theorem has the desired proprty. For the converse then, we assume that G is a graph (without isolated vertices) that is randomly H for every subgraph H (without isolated vertices) of G.

Assume first that G is disconnected. Then no component of G contains a subgraph isomorphic to P_4; for otherwise, we could select an edge from each of two components of G, producing a subgraph F isomorphic to $2K_2$. Since F is isomorphic to a subgraph of P_4, it follows that F can be extended to a path of length 3, but this is impossible. We conslude therefore that

$$G \cong nK(l, m),\ n \geq 2, m \geq 1, \text{ or } G \cong nK_3, n \geq 2.$$

We henceforth assume that G is connected. If all vertices of G have degree at most 2, then $G \cong P_n$ for some $n \geq 2$ or $G \cong C_n$ for some $n \geq 3$. We show that $G \not\cong P_4$. Suppose, to the contrary, that $G \cong P_4$. Since G contains a subgraph isomorphic to $2K_2$ and, consequently, is randomly $2K_2$, by Proposition 1 G is not a double star. However, P_4 is a double star and a contradiction has been reached. That $G \not\cong P_n$, $n \geq 5$, and $G \not\cong C_n$, $n \geq 6$, follows by applying an indirect proof and using the fact that G is randomly P_4 and that every subgraph of G isomorphic to $2K_2$ can be extended to a subgraph isomorphic to P_4.

We may now assume that G has some vertices with degrees at least 3. Suppose that G has maximum degree $r(\geq 3)$. Then, by Proposition 3, every vertex of G has degree r or degree l.

Next we show that the diameter of G is at most 3; for suppose, to the contrary, that diam $G \geq 4$. Then G contains vertices u and v with distance $d(u, v) = 4$. Let P be a $u - v$ path of length 4, and let F be the subgraph of G induced by the terminal edges of P. Since G is randomly P_4 and F is isomorphic to a subgraph of P_4, it follows that F can be extended to a subgraph isomorphic to P_4, implying that $d(u, v) \leq 3$, thereby producing a contradiction. If diam $G = 1$, then $G \cong K_p$ for some $p \geq 4$. Hence we assume that diam $G = 2$ or diam $G = 3$.

Suppose that G contains a vertex of degree l. If diam $G = 2$, then $G \cong K(l, r)$; while if diam $G = 3$, then G is a double star. In the latter case, G is randomly $2K_2$, contrary to Proposition 1.

Henceforth we assume that G is r-regular for some $r \geq 3$. Suppose that diam $G = 3$. Then G contains vertices u and v such that $d(u, v) = 3$. Let u, u_1, w, v be a $u - v$ path of length 3 in G. Suppose further that the neighborhood of u is $\{u_1, u_2, \ldots, u_r\}$, and let $e_i = uui$ $(i = 1, 2, \ldots, r)$ and $f = vw$ (see Figure 2). For $i = 2, 3, \ldots, r)$, let F_i be the subgraph induced by the two edges e_i and f, so that $F_i \cong 2K_2$. Since G is randomly P_4, and P_4 contains a subgraph isomorphic to each such F_i, we conclude that F_i can be extended to a subgraph of G that is isomorphic to P_4. This implies that w is adjacent to u_i $(i = 1, 2, \ldots, r)$ as well as to v, but then deg $w \geq r + 1$, which is a contradiction.

Finally, then, suppose that diam $G = 2$. Let v be a vertex of G and let V_i $(i = 1, 2)$ be the set of those vertices of G at distance i from v. Then $V(G) = \{v\} \cup V_1 \cup V_2$ and $|V_1| = r$. We note that G contains no triangle; for otherwise G is randomly K_3, which implies by Proposition 2 that G is complete, contradicting the fact that diam $G=2$. In particular, this shows that no two vertices of V_1 are adjacent.

Suppose two vertices u_2 and v_2 of V_2 are adjacent. Then G contains a 5-cycle v, u_1, u_2, v_2, v_1, v (see Figure 3). Therefore, G is randomly C_5, which, by Proposition 4, implies that $G \cong K_p$, for some $p \geq 5$, or $G \cong C_5$, neither of which is possible.

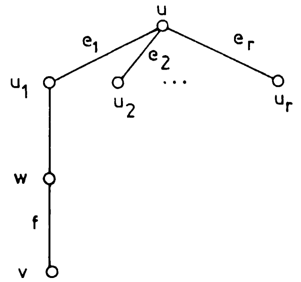

Fig. 2

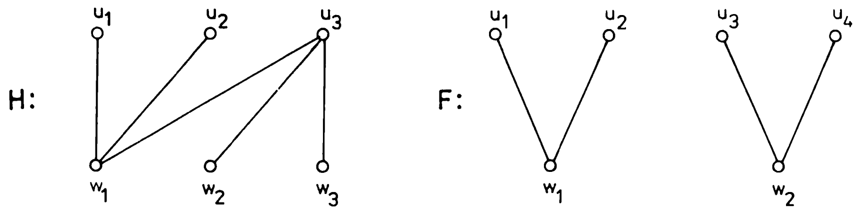

Fig. 3

Hence we conclude that no two vertices of V_2 are adjacent, implying that $G \cong K(r, r)$, $r \geq 3$. We now show that $G \cong K(3, 3)$, for suppose that $G \cong K(r, r)$, $r \geq 4$, where G has partite sets $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_r\}$. Let H be the subgraph of G shown in Figure 4. Since the subgraph F of G shown in Figure 4 is isomorphic to a subgraph of H and G is randomly H, it follows that F can be extended to a subgraph of G isomorphic to H, which is not the case (since $w_1 w_2 \notin E(G)$).

This completes the proof.

REFERENCES

[1] BÄBLER, F.: Über eine spezielle Klasse Eulerscher Graphen. Comment. Math. Helv. 27, 1953, 81—100.

[2] BEHZAD, M.—CHARTRAND, G.—LESNIAK-FOSTER, L.: Graphs ∀ Digraphs. Wadsworth International, Belmont, CA, 1979.

[3] CHARTRAND, G.—HKRONK, H. V.: Randomly traceable graphs. SIAM J. Appl. Math. 16, 1968, 696—700.
[4] CHARTRAND, G.—KRONK, H. V.—LICK, D. R.: Randomly hamiltonian digraphs. Fund. Math. 65, 1969, 223—226.
[5] CHARTRAND, G.—LICK, D. R.: Randomly eulerian digraphs. Czech. Math. J. 21, 1971, 424—430.
[6] CHARTRAND, G.—WHITE, A. T.: Randomly traversable graphs. Elem. Math. 25, 1970, 101—107.
[7] DIRAC, G. A.—THOMASSEN, C.: Graphs in which every finite path is contained in a circuit. Math. Ann. 203, 1973, 65—75.
[8] ERICKSON, D. B.: Arbitrarily traceable graphs and digraphs. J. Combinatorial Theory Ser. B 19, 1975, 5—23.
[9] FINK, J. F.: Randomly antitraceable digraphs. J. Graph Theory 6, 1982, 481—488.
[10] FINK, J. F.: Randomly near-tranceable graphs. (Submitted for publication)
[11] HARARY, F.: On arbitrarily traceable graphs and directed graphs. Scripta Math. 23, 1957, 37—41.
[12] HARARY, F.: Graph Theory. Addison-Wesley, Reading, MA, 1969.
[13] ORE, O.: A problem regarding the tracing of graphs. Elem. Math. 6, 1951, 49-53.
[14] PARSONS, T. D.: Paths extendable to cycle. J. Graph Theory 2, 1978, 337—339.
[15] SUMNER, D. P.: Randomly matchable graphs. J. Graph Theory 3, 1979, 183—186.
[16] THOMASSEN, C.: On randomly Hamiltonian graphs. Math. Ann. 200, 1973, 195—208.
[17] THOMASSEN, C.: Graphs in which every path is contained in a Hamiltonian path. J. Reine Angew. Math. 268/269, 1974, 271—282.

Received May 31, 1984

Gary Chartrand—Ortrud R. Oellermann
Western Michigan University
Kalamazoo, Michigan 49008 — 3899

Sergio Ruiz
Universidad Católica de Valparaíso

СЛУЧАЙНО *H* ГРАФЫ

Gary Chartrand—Ortrud R. Oellermann—Sergio Ruiz

Резюме

Граф G содержающий субграф H без изолированных вершин называется случайно H графом, если из того, что F является подграфом графа G без изолированных вершин изоморфным подграфу графа H, вытекает, что F может быть расширен на некоторый подграф графа G изоморфный графу H. Случайно H графы охарактеризованные для всех циклов и для несколько малых графов H. Найдены все графы, которые являются случайно H графами для всех своих подграфов H без изолированных вершин.