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Infinite precise objects


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INFINITE PRECISE OBJECTS

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Introduction

This paper was motivated by the following problem (see [2], [3]): Is it possible to construct for each integer $h \geq 2$ two non-isomorphic graphs $G$ and $G'$ with the following properties:

1. $G$ and $G'$ have the same infinite number $c$ of vertices;
2. for each pair $(a, b)$ of different vertices of $G$ (of $G'$, respectively) there exists exactly one path of the length $\leq k$ joining $a$ and $b$;
3. $G$ and $G'$ are not trees.

Such graphs are called infinite tied graphs of type $(c, k)$ (or infinite Moore graphs). Tied graphs with finite number of vertices are just few and they have been almost completely characterized (see [1], [5], [11]). Using a simple construction we shall show that infinite tied graphs are as many as possible. This solves the above problem. Moreover we prove a stronger result.

As usually every ordinal number will be considered as the set of all smaller ordinals (ordinal numbers will be denoted by $\iota$, $\lambda$, ...). A cardinal number (usually denoted by $c$) will be identified with the smallest ordinal number of the same cardinality.

**Theorem A**: For each integer $k \geq 2$ and infinite cardinal number $c$ there exists a family $G_{t} \leq 2^{c}$ of tied graphs with parameters $(c, k)$ such that there is no homomorphism $G_{\iota} \rightarrow G_{\lambda}$ for any $\iota \neq \lambda$.

(This is much stronger than the fact that the graphs $G_{t}$ are non-isomorphic. A homomorphism $f : G = (V, E) \rightarrow G' = (V', R')$ is a mapping $f : V \rightarrow V'$ which satisfies $[f(x), f(y)] \in E'$ for every $[x, y] \in E$.)

This is proved in §2. The method of the proof is general enough to obtain similar results for tactical configurations and friendship graphs. This is contained in §3.

During the final preparations of this paper I learned that a similar technique was used independently by V. Chvátal and A. Kotzig in a special case; their paper will be published in Canad. Math. Bull. (they constructed $2^{c}$ non-isomorphic friendship graphs with $c$ vertices).

Finally, let us remark that the results of this paper may be interpreted negatively: infinite precise objects are not rare and there is such a “jump” between finite and
§2. Proof of Theorem A

A graph $G = (V, E)$ means an undirected loopless graph; $V, E$ may be infinite sets. A digraph $G$ is a couple $(V, R)$ where $R \subseteq V \times V$; both $V$ and $R$ may be infinite. The class of all digraphs will be denoted by $\text{DiGra}$.

Let $t \geq 3$ be a fixed natural number. Denote by $\text{Cyc}(t)$ the class of all graphs without cycles of the length $\leq t$.

The chromatic number of a graph is denoted by $\chi(G)$. It is known by [7] that for every natural number $n$ there exists a graph $G \in \text{Cyc}(t)$ such that $\chi(G) = n$.

Let $k \geq 2$ be fixed. Put $t = 2k$. Obviously every subgraph of a tied graph with parameters $(c, k)$ belongs to the class $\text{Cyc}(t)$. Denote by $\text{Tied}(k)$ the class of all tied graphs with parameters $(c, k)$ for some $c$. Therefore $\text{Tied}(k) \subseteq \text{Cyc}(t)$.

Now we consider the opposite procedure: We start with the class $\text{Cyc}(t)$ and we modify (in a canonical way) each graph from $\text{Cyc}(t)$ into a graph from $\text{Tied}(k)$. The construction is as follows:

Let $G = (V, E) \in \text{Cyc}(t)$. Put $G_0 = (V_0, E_0) = G$.

Let $G_n = (V_n, E_n) \in \text{Cyc}(t)$ be defined.

For each pair $x, y$ of vertices of $G_n$ with $d_{G_n}(x, y) > k$ ($d_{G_n}(x, y)$ is the distance of $x$ and $y$ in the graph $G_n$) adjoin a path $P(x, y)$ from $x$ to $y$ of the length $k$ in such a way that, with a possible exclusion of endvertices, these paths are mutually disjoint and have no vertices common with $V_n$.

Do this simultaneously for all pairs $(x, y) \subseteq V_n$ with $d_{G_n}(x, y) > k$. Call the resulting graph $G_{n+1}$. Clearly $G_n \subseteq G_{n+1}$. Put $\lim G = \cup (G_n; n \in \mathbb{N})$. Let $i_G: G \rightarrow \lim G$ be the naturally defined inclusion.

The construction of $\lim G$ has first appeared in [3] and has been studied in [2].

The following two propositions give a basic information on the homomorphism properties of the class $\text{Tied}(k)$:

**Proposition 1:** Let $G, G' \in \text{Tied}(k)$. Then every homomorphism $f: G \rightarrow G'$ is a distance preserving monomorphism (i.e. $f$ is $1 - 1$ and $d_{G'}(f(x), f(y)) = d_G(x, y)$ for all $x, y$).

Let $G, G'$ be graphs. A homomorphism $f: G \rightarrow G'$ is called $k$-distance preserving if $d_G(x, y) = d_{G'}(f(x), f(y))$ whenever $d_G(x, y) \leq k$ and $d_{G'}(f(x), f(y)) > k$ whenever $d_G(x, y) > k$.

**Proposition 2:** Let $2k = t$, $k \geq 2$. Let $G \in \text{Cyc}(t)$, $G' \in \text{Cyc}(t)$. Then for every $k$-distance preserving homomorphism $f: G \rightarrow G'$ there exists exactly one homo-
morphism \( f' : \text{lim } G \rightarrow \text{lim } G' \) such that \( f' \circ \iota_G = \iota_{G'} \circ f \). The homomorphism \( f' \) will be denoted by \( \text{lim } f \).

\[
\begin{array}{ccc}
G & \overset{f}{\rightarrow} & G' \\
\iota_G \downarrow & & \downarrow \iota_{G'} \\
\text{lim } G & \overset{f}{\rightarrow} & \text{lim } G'
\end{array}
\]

Proof of Proposition 1: Let \( G, G', f : G \rightarrow G' \) be given. Assume that \( d_G(f(x), f(y)) \neq d_G(x, y) \), consequently \( d_G(f(x), f(y)) < d_G(x, y) \). But then the unique path \( P(x, y) \) of the length \( \leq k \) which joins \( x \) and \( y \) in \( G \) has to be mapped into the unique path \( P(f(x), f(y)) \) of the length \( \leq k \) which joins \( f(x) \) and \( f(y) \) in \( G' \). As \( f \) is not distance preserving this implies that \( f \) is not \( 1 - 1 \). Assume that \( f(x) = f(y) \) for the vertices \( x \neq y \). It can be easily seen that \( x \) and \( y \) are contained in a cycle \( C \) of the length \( 2k + 1 \). As the homomorphic image of an odd cycle contains an odd cycle and as \( G' \in \text{Cyc}(2k) \) we get a contradiction.

Proof of Proposition 2: Let \( f : G \rightarrow G' \) be a \( k \)-distance preserving monomorphism. Define \( \lim f = f' \) by: \( f'|_{V_{0}} = f \); let \( f'|_{V_{1} - V_{0}} \) be defined by the condition \( f'(P_{G}(x, y)) = P_{G'}(f(x), f(y)) \) for every pair \( x, y \) satisfying \( d_{G}(x, y) > k \) (as \( f \) is \( k \)-distance preserving it is also \( d_{G}(f(x), f(y)) > k \)), where \( P_{G}(x, y) \) the unique path of the length \( k \) in \( G \), which connects \( x \) and \( y \) and analogously \( P_{G'}(f(x), f(y)) \). It can be verified that \( f'|_{V_{1}} : G_{1} \rightarrow G_{1}' \) is again \( k \)-distance preserving and consequently this procedure may be carried on by induction. This gives then definition of \( f' : \lim G \rightarrow \lim G' \). It is easy to verify that \( f' \) is uniquely determined by \( f \).

In the language of the category theory Propositions 1 and 2 mean that the category \( \text{Tied}(k) \) (with all homomorphisms as its morphisms) is a reflexive subcategory of the category \( \text{Cyc}(k) \) (considered with all \( k \)-distance preserving homomorphisms as its morphisms).

It is very important that the statement of Proposition 2 can be often reversed. In many cases for every homomorphism \( g : \lim G \rightarrow \lim G' \), \( G, G' \in \text{Cyc}(2k) \) there exists a homomorphism \( f : G \rightarrow G' \) such that \( g = \text{lim } f \). (Of course this is not always the case. It suffices to consider \( G = G' = \) a discrete graph with at least 5 vertices.)

This reversed procedure may be established as follows:

1. Let \( H = (W, F) \in \text{Cyc}(t) \) be a finite graph with the following properties:
   (i) \( \chi(H) \geq 4 \);
   (ii) \( H \) is a vertex critical graph;
   (iii) there are \( a, b \in W \) such that \( d_{H}(a, b) \geq k \).

By [7] such a graph exists.

2. Put \( |W| = n \). We prove that for each infinite cardinal number \( \kappa \) there exists
a family $H_t, t < 2^c$, of digraphs with the following properties:

(i) $|H_t| = c, t < 2^c$;

(ii) there exists no homomorphism $H_t \to H_{\lambda}$ for all $t \neq \lambda, t, \lambda < 2^c$;

(iii) the undirected graphs which arise from $H_t, t < 2^c$ (by destroying orientations of edges) belong to $\text{Cyc}(n)$.

This follows from the calculus of rigid graphs: By [13] there exists a directed graph $(X, R), R \subseteq X \times X$, which satisfies $|X| = c$ and for which there exists no non-identical homomorphism of $(X, R)$ into itself — such graphs are called rigid. By [8], [9] there exists an undirected graph $H$ with $c$ vertices which is rigid and it may be further assumed that $\tilde{H} \in \text{Cyc}(n)$. We may put $\tilde{H} = (X, R)^{\ast} \tilde{H}$, where $\ast$ is the šip-construction (see step 3 below) and $\tilde{H} \in \text{Cyc}(n)$ is the rigid graph depicted in Fig. 1 (see [8], [10], [12]).

Now it suffices to consider all orientations $H_t, t < 2^c$, of the graph $\tilde{H}$. It is easy to see that this family of digraphs satisfies the above conditions.

3. Define the following construction (a concrete modification of the "šip product" [8]): For a directed loopless graph $G' = (V', E')$ and an undirected graph $H = (W, F)$ with two fixed vertices $a, b$ denote by $G^{\ast}(H, a, b)$ the following graph:

![Fig. 1](image-url)
Put $\tilde{V} = E' \times W$ and let $\sim$ be equivalence on $\tilde{V}$ defined by the condition:

\[((x, y), z) \sim ((x', y'), z')\] if and only if one of the following conditions holds:

i. $(x, y) = (x', y')$, $z = z'$;

ii. $y = x'$, $z = b$, $z' = a$;

iii. $x = y'$, $z = a$, $z' = b$;

iv. $x = x'$, $z = z' = a$;

v. $y = y'$, $z = z' = b$.

Denote by $V$ the set of all equivalence classes of $\tilde{V}$ and define $E$ by $E = \{[[x], [y]] : x = (e, x'), y = (e, y'), [x, y] \in F\}$, where $[x]$ is the equivalence class of $\sim$ containing $x$.

$(V, E)$ is the graph $G^*(H, a, b)$.

As is explained in [8], [9], the graph $G^*(H, a, b)$ is the graph $G$ with each arrow replaced by a copy of $H$.

4. Put $G_t = H^*(H, a, b)$ for each $t <2^c$, where $H_t$, $t <2^c$, is given in step 2 and $(H, a, b)$ is given in step 1.

Claim: For each $i$, $\lambda <2^c$, $i \neq \lambda$, there exists no 1–1 homomorphism $G_i \rightarrow G_\lambda$.

Proof: Let $f : G_i \rightarrow G_\lambda$. as $G_i = H^*(H, a, b)$ and $H_i$ is an orientation of the graph $H \in CyC(n)$, $n = |V(H)| = |W|$ it follows that each cycle of the length $\leq n$ is contained in a “copy” the graph $H$ in $G_i$. It follows that $f$ has to map copies of $H$ in $G_i$ into copies of $H$ in $G_\lambda$. Consequently $f$ induces a monomorphism $f' : H_i \rightarrow H_\lambda$, which is a contradiction with step 2.

5. Now consider the family $\lim G_i$, $t <2^c$. Obviously $\lim G_i$ has $c$ vertices for each $t <2^c$. In order to finish the proof of Theorem A it suffices to prove:

Claim: Let $i \neq \lambda$, $i, \lambda <2^c$. Then there exists no homomorphism $f : \lim G_i \rightarrow \lim G_\lambda$.

Proof: Let $f : \lim G_i \rightarrow \lim G_i$ be a homomorphism, by Proposition 1 $f$ is 1–1. One can easily prove $\chi(\lim G_i) = \chi(G_i)$ and for each subgraph $G^* \subseteq \lim G_i$, $\chi(G^*) = 4$ there holds $\chi(G^*) = \chi(G^* \cap G)$. But $H$ (see step 1) was a colour critical graph; consequently $f$ maps each copy of $H$ which has to belong to $G_i$ into a copy of $H$ in $G_\lambda$. Thus $f$ induces a monomorphism $\tilde{f} : G_i \rightarrow G_\lambda$, which is a contradiction with step 4. This proved Theorem A.

Remark: The assumption $\chi(H) = 4$ in 1 is not necessary. One can lower it to $\chi(H) = 3$ (which of course is the best possible one) by a more elaborate construction of rigid graphs. The aim of this paper is to apply well-known constructions.

It is also possible to prove that for every group $\mathcal{G}$ there exists a graph $G \in \text{Tied}(k)$ such that the group $\text{Aut} G$ of all automorphisms of $G$ is isomorphic to $\mathcal{G}$ and other similar results (e.g. for the monoid of endomorphisms).
§3. Other structures

3.1. A friendship graph $G = (V, E)$ is defined by the following condition: for every pair $x, y$ of different vertices of $G$ there exists exactly one vertex $z$ such that $[x, z] \in E$ and $[y, z] \in E$. A complete description of finite friendship graphs is given in [6]: on each set with an odd number of vertices there exists exactly one such graph and there is no such graph with an even number of vertices. In the infinite case the situation is entirely different:

**Theorem B.** On each infinite set of cardinality $c$ there exists a family $G_i, i < 2^c$, of friendship graphs such that given $i \neq \lambda < 2^c$ there exists no monomorphism $f : G_i \to G_\lambda$.

Outline of a proof of Theorem B: The proof is very similar to the above proof of Theorem A, therefore, we only stress the differences between these two proofs.

As regards step 1, it suffices to take a rectangle free graph which contains each edge in a cycle and which does not contain a vertex of degree 2 (this may be established, e.g., again via a chromatic number). Steps 2, 3, and 4 remain unchanged.

As regard step 5, we have to define the construction of $\lim G$. As can be expected this is the “friendship” version of the above procedure: Put $G = G_0 = (V_0, E_0)$. Given $G_n = (V_n, E_n)$ we define $G_{n+1}$ by adjoining a path of the length 2 a to any pair of vertices of $G_n$ which was not in $G_n$ connected by a path of the length 2. One can check that $\lim G = \bigcup (G_n ; n \in \mathbb{N})$ is a friendship graph. Using this we prove Theorem B quite analogously to the proof above.

3.2. Let $k, p, \lambda$ be positive integer, $k \geq p \geq 2$. Let $c$ be a cardinal number.

A $k$-graph is a couple $(V, E)$ where $e \subseteq V$, $|e| = k$ for any $e \in E$. The elements of $E$ are called edges.

A tactical configuration with parameters $(c, k, \lambda, p)$ is a $k$-graph $(V, E)$, $|V| = c$, with the property that each $p$-element subset $P$ of $V$ belongs to exactly $\lambda$ edges thus $(p \leq k)$.

It is difficult to construct a finite tactical configuration with given parameters (this is the basic question of the theory of block designs and related subjects). We prove:

**Theorem C.** Let $c$ be an infinite cardinal. For every $k, \lambda$, and $p$ there exists $2^c$ tactical configurations of the type $(c, k, \lambda, p)$ such that there is no monomorphism between then.

The proof is again very similar to the above one. The crucial point is the following:

Let $G = (V, E)$ be a $k$-graph which satisfies $|e \cap e'| \leq 1$ whenever $e \neq e' \in E$. Let $|V| = c$, and let $k, \lambda, p$ be fixed. Define a tactical configuration $\lim G$ with the parameters $c, k, \lambda, p$ as follows: Put $G_0 = (V_0, E_0) = G$. Let $G_n = (V_n, E_n)$ be
given. For any $p$-element subset $P$ of $V_n$ determine the number $A(P)$ of edges of $E_n$ which contain $P$. If $\lambda(P)<\lambda$, then adjoin to $G_n\lambda-\lambda(P)$ edges which contain $P$ and otherwise are disjoint. Do this simultaneously for all $p$-element subsets of $V_n$ and call the resulting $k$-graph $G_{n+1}$. It may be verified that $\lim G = \bigcup (G_n; n \in \mathbb{N})$ is a $(c, k, \lambda, p)$ tactical configuration.

Now let $c, k,$ and $p$ be fixed. It suffices to start from a tactical configuration $H$ with the parameters $(n, k, 1, 2)$ for a convenient finite $n > k$ (thus $H$ is a nontrivial block design; it is known that such an $n$ exists). As in the proof of Theorem A we obtain oriented graphs $H_t, t < 2^c$, and by the šip-construction we define the $k$-graphs $G_t = H_t*(H, a, b)$, where $a, b$ are arbitrary different vertices of $H$. The šip-construction for the $k$-graphs is defined quite analogously to that for graphs. It can be proved (and this is simpler than in Theorem A) that the $(c, k, \lambda, p)$ tactical configurations $\lim G_t, t < 2^c$, are non-isomorphic and that there is no monomorphism between them. We leave out the details.

Remark: The method of this paper may be further applied to "precise objects" which are defined by a local condition.

A further example is provided by "infinite strongly regular graphs" [4], graphs where each two vertices have exactly $k$ common neighbours.

Finally, let us remark that Theorems B, C may be proved with "homomorphisms" instead of "monomorphisms" (see the above statements) by using a refinement of the above proofs.

REFERENCES


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Резюме

В частности, в работе доказывается, что для каждой бесконечной мощности $\alpha$ существует $2^\alpha$ взаимно не изоморфных
(1) графов Мура с параметрами $(\alpha, k)$, $k \geq 2$;
(2) тактических конфигураций с параметрами $(\alpha, k, \lambda, p)$.
Предлагается метод для доказательства аналогичных теорем.