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ON $|\mathbf{T}|_k$ SUMMABILITY AND ABSOLUTE NÖRLUND SUMMABILITY

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ABSTRACT. This paper gives the necessary and sufficient conditions in order that a series $\sum a_n$ should be summable $|\mathbf{T}|_k$, $k \geq 1$, whenever $\sum |a_n| < \infty$, and so extends the known results of [2] and [3] to the case $k > 1$.

1. Definitions and notations

Let $\sum a_n$ be an infinite series with the sequence of its partial sums (s_n) and let $\mathbf{T} = (a_{nv})$ be an infinite matrix. Suppose that

$$T_n = \sum_{v=0}^{\infty} a_{nv} s_v \,, \qquad (v = 0, 1, 2, \dots) \tag{1}$$

exists (i.e., the series on the right-hand side converges for each n). If $(T_n) \in \mathrm{bv}$, i.e.,

$$\sum_{n=0}^{\infty} |T_n - T_{n-1}| < \infty \,, \qquad (T_{-1} = 0) \tag{2}$$

the series $\sum a_n$ is said to be *absolutely summable by the matrix* $\mathbf{T}$ or simple $|\mathbf{T}|$. As known, the series $\sum a_n$ is said to be $|\mathbf{N}, p_n|$ *summable* if (2) holds whenever $\mathbf{T}$ is a Nörlund matrix, [2]. By a *Nörlund matrix*, we mean one that

$$a_{nv} = \frac{p_{n-v}}{P_n} \quad \text{for} \quad 0 \leq v \leq n \,, \quad \text{and} \quad a_{nv} = 0 \quad \text{for} \quad n > v \,,$$

where (p_n) is a sequence of real or complex numbers for which

$$P_n = p_0 + p_1 + \dots + p_n \neq 0 \,, \quad P_{-1} = 0 \,.$$

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Let (T_n) be given by (1). If

$$\sum_{n=1}^{\infty} n^{k-1}|T_n - T_{n-1}|^k < \infty\,, \tag{3}$$

then $\sum a_n$ is said to be $|\mathbf{T}|_k$ *summable*, $k > 0$, [5], and for $k = 1$ this is the usual definition of $|\mathbf{T}|$ summability. Moreover, when $\mathbf{T}$ is a Nörlund matrix, this definition reduces to the customary definition of absolute summability $|\mathbf{N}, p_n|_k$, as given by Borwein and Cass [1], for example.

Mears [2] established the necessary and sufficient conditions in order that $\sum a_n$ should be summable $|\mathbf{T}|$ whenever $\sum |a_n| < \infty$. Also Mc Fadden [3] obtained some comparison theorems between the summabilities $|\mathbf{N}, p_n|$ and $|\mathbf{N}, q_n|$, using Mears's result. But, since $|\mathbf{T}|_k$ summability includes the $|\mathbf{T}|$ summability, this also raises the problem: what are the necessary and sufficient conditions in order that $\sum a_n$ should be $|\mathbf{T}|_k$ summable whenever $\sum |a_n| < \infty$, which enables us to extend Mears's and McFadden's results to the case $k > 0$. We give an affirmative answer to the problem for $k \geq 1$.

Let $(\mathbf{N}, p_n)$ and $(\mathbf{N}, q_n)$ be regular Nörlund means, and let t_n and u_n denote $(\mathbf{N}, p_n)$ and $(\mathbf{N}, q_n)$ means of $\sum a_n$, i.e., for $n = 0, 1, 2, \dots$,

$$t_n = \sum_{v=0}^{n} \frac{p_{n-v}}{P_n} s_v \tag{4}$$

and

$$u_n = \sum_{v=0}^{n} \frac{q_{n-v}}{Q_n} s_v\,. \tag{5}$$

Then

$$t_n = \sum_{v=0}^{n} \frac{R_{n-v} Q_v}{P_n} u_v\,, \tag{6}$$

where R_k is determined such that

$$p_0 = q_0 R_0\,, \quad p_1 = q_1 R_0 + q_0 R_1, \dots, p_k = q_k R_0 + \dots + q_0 R_k\,. \tag{7}$$

2. Main results

We now prove the following theorems:

THEOREM 2.1. *The necessary and sufficient conditions in order that $\sum a_v$ should be $|\mathbf{T}|_k$ summable, $k \geq 1$, are, whenever $\sum |a_v| < \infty$,*

(i) $\sum_{v=0}^{\infty} a_{nv}$ *converges for all* n,

(ii) $\sum_{n=1}^{\infty} n^{k-1} \left| \sum_{i=v}^{\infty} (a_{ni} - a_{n-1,i}) \right|^k \leq M < \infty$ *for all* v.

The case $k = 1$ of this Theorem was proved by Mears.

We require the following result of Maddox ([4], Theorem 5, p. 167) for the proof of the Theorem.

THEOREM 2.2. $\mathbf{C} = (c_{nv}) \in (\ell_1, \ell_k)$ *if and only if*

$$\sup_v \sum_n |c_{nv}|^k < \infty, \quad \textit{for the cases} \quad 1 \le k < \infty .$$

Proof of Theorem 2.1.

Sufficiency. Since, by (i), $A_{nv} = \sum_{i=v}^{\infty} a_{ni}$ converges for each n, v, $A_{nv} \to 0$ as $v \to \infty$, and so there exists a sequence (β_n) such that $|A_{nv}| \le \beta_n$ for all v. Therefore $T_n = \sum_{v=0}^{\infty} A_{nv} a_v$ converges for each n, since

$$\sum_{v=0}^{\infty} |A_{nv} a_v| \le \beta_n \sum_{v=0}^{\infty} |a_v| < \infty .$$

On the other hand we have, for $n \ge 0$,

$$T_n - T_{n-1} = \sum_{v=0}^{\infty} (A_{nv} - A_{n-1,v})\, a_v , \qquad (A_{-1,v} = 0). \tag{8}$$

Now, denote $v_n = n^{1-1/k}(T_n - T_{n-1}) = \sum_{v=0}^{\infty} n^{1-1/k}(A_{nv} - A_{n-1,v}) a_v$, $n \ge 1$, and $v_0 = \sum_{v=0}^{\infty} A_{0v} a_v$. Then (v_n) is the $\mathbf{C}$-transform sequence of $(a_v) \in \ell_1$, where, for all $v \ge 0$,

$$c_{nv} = \begin{cases} n^{1-1/k}(A_{nv} - A_{n-1,v}) & \text{if} \quad n \ge 1 \\ A_{0v} & \text{if} \quad n = 0 . \end{cases}$$

Therefore, it follows from Theorem 2.2 and (ii) that $\mathbf{C} \in (\ell_1, \ell_k)$, $k \ge 1$, i.e, $\sum a_n$ is $|\mathbf{T}|_k$-summable, whenever $\sum |a_n| < \infty$.

Necessity. Choosing $s_v = 1$ for all v, we have that $T_n = \sum_{v=0}^{\infty} a_{nv}$ converges. Thus (i) of the Theorem is necessary and A_{nv} is defined for all v, n. Now, by Theorem 2.2 and (8), we complete the proof of the Theorem as the above discussion.

THEOREM 2.3. *The necessary and sufficient conditions in order that* $|\mathbf{N}, q_n| \implies |\mathbf{N}, p_n|_k$, $k \geq 1$, *are*

$$\sum_{n=1}^{\infty} n^{k-1} \left| \sum_{v=i}^{n} \left(\frac{R_{n-v}}{P_n} - \frac{R_{n-1-v}}{P_{n-1}} \right) Q_v \right|^k \leq M < \infty, \qquad (R_{-1} = 0) \tag{9}$$

for all i.

The case $k = 1$ of the theorem is due to McFadden (see [3]).

Proof. If we define the matrix $\mathbf{T} = (a_{nv})$ in the following way:

$$a_{nv} = \begin{cases} \frac{R_{n-v} Q_v}{P_n} & \text{if } 0 \leq v \leq n, \\ 0 & \text{if } v > n, \end{cases}$$

then the conditions of Theorem 2.1 reduce to the conditions of Theorem 2.3. Therefore the Theorem is proved by considering (6).

COROLLARY 2.4. *For* $k > 1$, $|\mathbf{N}, p_n| \not\Rightarrow |\mathbf{N}, p_n|_k$, *and so* $|\mathbf{C}, 1| \not\Rightarrow |\mathbf{C}, 1|_k$, *i.e., there exists a series that is summable* $|\mathbf{N}, p_n|$ *but not summable* $|\mathbf{N}, p_n|_k$.

In this case, since by (7), $R_0 = 1$ and $R_v = 0$ for all $v \geq 1$, condition (9) is reduced to

$$\sum_{n=1}^{i-1} n^{k-1} \left| \sum_{v=i}^{n} \left(\frac{R_{n-v}}{P_n} - \frac{R_{n-1-v}}{P_{n-1}} \right) P_v \right|^k + i^{k-1} \left| \left(\frac{R_0}{P_i} - \frac{R_{-1}}{P_{i-1}} \right) P_i \right|^k$$
$$+ \sum_{n=i+1}^{\infty} n^{k-1} \left| \sum_{v=i}^{n} \left(\frac{R_{n-v}}{P_n} - \frac{R_{n-1-v}}{P_{n-1}} \right) P_v \right|^k = i^{k-1} \leq M \qquad \text{for all} \quad i \geq 2,$$

which is impossible.

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