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SUBGROUPS OF THE BASIC SUBGROUP IN A MODULAR GROUP RING

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ABSTRACT. Suppose R is an unitary commutative ring of prime characteristic p and G is an arbitrary abelian group with p-component G_p. The main results are that $S(RG)$ and $S(RG)/G_p$ are both starred groups, provided G_p is not a divisible group. In the case when G_p is divisible and R is a perfect field, $S(RG)$ and $S(RG)/G_p$ are simply presented, whence a direct sum of a divisible group and a starred group. These claims enlarge statements argued by the author in Math. Bohem. (2004) and also give a new contribution to the old-standing Direct Factor Conjecture for group rings.

1. Introduction

Throughout this paper, let RG be an abelian group ring over a ring R of prime characteristic p and with unity, let $V(RG)$ be the group of all normalized invertible elements in RG and let $S(RG)$ be the normed Sylow p-group in RG with a basic subgroup $B^* = B_{S(RG)}$. For G an abelian group, $B = B_G$ denotes its p-basic subgroup and G_p its p-component. If H is a p-subgroup of G, we let $S(RG; H)$ denote the multiplicative p-group $1 + I(RG; H)$, where $I(RG; H)$ is the relative augmentation ideal of RG with respect to H. In the sequel, the notations and the terminology of the monographs of L. Fuchs [4], [5] will be followed.

In [2], a necessary and sufficient condition is given $S(RG; H)$ to be a basic subgroup of $S(RG)$ provided H is p-torsion. The same theme was considered in [3] as well. Besides, in [1], a criterion is obtained for $S(FH)$ to be basic in $S(FG)$, assuming G is p-primary and F a field of characteristic $p \neq 0$.

In the present article, we will investigate the following major question: If B is a p-basic subgroup of G, then what is the explicit form, that depends on B,

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of the corresponding basic subgroup of $S(RG)$ about B? That is why this note is a supplement of the papers [1], [2], [3].

2. Main results

Foremost we need some elementary, however, useful observations:

(1) *A p-pure subgroup of a p-divisible group is p-divisible.*

(2) *Let C be p-pure in G. Then $C \cdot G_p$ is p-pure in G.*

In order to verify the second point, choose $x \in (C \cdot G_p) \cap G^{p^n}$ for an arbitrary but fixed natural n, i.e. $x = c \cdot g_p = g_n^{p^n}$, where $c \in C$, $g_p \in G_p$ and $g_n \in G$. If $g_p^{p^k} = 1$ for some positive integer k, we conclude $c^{p^k} = g_n^{p^{n+k}} \in C \cap G^{p^{n+k}} = C^{p^{n+k}}$. Henceforth $c^{p^k} = c_1^{p^{n+k}}$ for some $c_1 \in C$ and obviously $c \in c_1^{p^n} G_p$. So, for some $a_p \in G_p$ we derive $x = c_1^{p^n} a_p = g_n^{p^n} = \left(g_n c_1^{-1}\right)^{p^n} \cdot c_1^{p^n} \in G_p^{p^n} \cdot C^{p^n} = (G_p \cdot C)^{p^n}$ since $(g_n c_1^{-1})^{p^n} = a_p \in G_p$, i.e. $g_n c_1^{-1} \in G_p$. The statement is fulfilled.

(3) *If B is a p-basic subgroup of G, then B_p is basic in G_p.*

Indeed, $B_p \subseteq B$ is a direct sum of cyclic p-groups. Moreover, B_p is pure in B, and B is p-pure in G. Therefore B_p is pure in G, i.e. B_p is pure in G_p, because G_p is pure in G. Besides, $G_p/B_p = G_p/(B \cap G_p) \cong G_pB/B$, and G_pB/B is p-pure in G/B by (2). Using (1), G_pB/B is p-divisible, i.e. G_p/B_p is divisible. Finally, B_p is basic in G_p according to the definition in [4] or [5], as claimed.

Thereby, we shall use the notation B_p for a basic subgroup of G_p.

Our conclusions in the proofs of the central results are based on the following excellent Kovács criterion ([7], [4; Theorem 29.5] or [5; p. 167, Theorem 33.4]), which is analogous to the well-known Kulikov's criterion for direct sums of cyclic groups (see, for instance, [5]) and which is also its important generalization.

Criterion. (Kovács, 1958) *A subgroup C of a p-group G can be expanded to a basic subgroup B of G if and only if C is the union of an ascending chain of subgroups $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n \subseteq \cdots$ such that the heights (taken in G) of the elements of C_n are bounded, i.e. $C = \bigcup_{n=1}^{\infty} C_n$, where $C_n \subseteq C_{n+1}$ and $C_n \cap G^{p^n} = 1$ for each $n \in \mathbb{N}$.*

(4) *Suppose $G_1 \subseteq G_2$ are abelian p-torsion groups and $C_1 \subseteq C_2$, $C_1 \subseteq G_1$, $C_2 \subseteq G_2$. Thus, if C_2 can be expanded to a basic subgroup of G_2, then C_1 can be expanded to a basic subgroup of G_1.*

Proof. By the hypothesis and the criterion of Kovács, $C_2 = \bigcup_{n=1}^{\infty} C_2^{(n)}$, where $C_2^{(n)} \subseteq C_2^{(n+1)}$ and $C_2^{(n)} \cap G_2^{p^n} = 1$. Hence, $C_1 = \bigcup_{n=1}^{\infty} (C_1 \cap C_2^{(n)}) = \bigcup_{n=1}^{\infty} C_1^{(n)}$, putting $C_1^{(n)} = C_1 \cap C_2^{(n)}$. Furthermore we have $C_1^{(n)} \subseteq C_1^{(n+1)}$ and $C_1^{(n)} \cap G_1^{p^n} = C_1 \cap C_2^{(n)} \cap G_1^{p^n} \subseteq C_2^{(n)} \cap G_2^{p^n} = 1$. Then we can apply again the Kovács theorem, which yields the result. □

I. $S(RG;H)$ as a subgroup of B^*.

Now we are in a position to formulate and prove our first main affirmation.

THEOREM 1. *Suppose R is a commutative ring with identity of prime characteristic p and G is an abelian group whose subgroup H is p-torsion. Then $S(RG;H)$ can be expanded to B^* if and only if H can be expanded to B_p.*

First we shall prove in details some principal assertions.

LEMMA 1. *For each ordinal σ*

$$S^{p^\sigma}(RG) = S\big(R^{p^\sigma} G^{p^\sigma}\big)$$

is valid.

Proof. Take $\sigma = 1$. Apparently $S^p(RG) \subseteq S(R^pG^p)$. Conversely, given $x \in S(R^pG^p)$, we have $x = \sum_{i=1}^{n} r_i^p g_i^p$, where $\sum_{i=1}^{n} r_i^p = 1$ and $r_i \in R$, $g_i \in G$, $n \in \mathbb{N}$. It is not difficult to see that $x = \Big(1 + \sum_{i=1}^{n} r_i(g_i - 1)\Big)^p \in S^p(RG)$, because $1 - \sum_{i=1}^{n} r_i(1 - g_i) \in S(RG)$. Further the proof goes on a standard way by means of a transfinite induction. This completes the proof. □

LEMMA 2. *Suppose that L is a subring of R with the same unity, and A and C are subgroups of G. Then*

$$\big(1 + I(RG;A)\big) \cap S(LC) \subseteq 1 + I(LC; C \cap A).$$

Proof. Given $x \in \big(1 + I(RG;A)\big) \cap S(LC)$, we have $x = \sum_{c \in C} \alpha_c c$ and $\sum_{c \in \bar{c}A} \alpha_c = \begin{cases} 1, & \bar{c} \in A, \\ 0, & \bar{c} \notin A \end{cases}$ for each $\bar{c} \in C$; $\alpha_c \in L$. Clearly, $\bar{c}A \cap C = \bar{c}(A \cap C)$ because $\bar{c} \in C$. Thus $\sum_{c \in \bar{c}(A \cap C)} \alpha_c = \begin{cases} 1, & \bar{c} \in A \cap C, \\ 0, & \bar{c} \notin A \cap C \end{cases}$. Finally we extract that $x \in 1 + I(LC; C \cap A)$, as required. □

We are now ready for:

Proof of Theorem 1. Let us assume that $H \subseteq B_p$. Hence from the Kovács criterion, $H = \bigcup_{n=1}^{\infty} H_n$, $H_n \subseteq H_{n+1}$ and $H_n \cap G_p^{p^n} = 1$. Therefore $S(RG;H) = \bigcup_{n=1}^{\infty} S(RG;H_n)$, $S(RG;H_n) \subseteq S(RG;H_{n+1})$ and in view of Lemmas 1 and 2, we establish that $S(RG;H_n) \cap S^{p^n}(RG) = S(RG;H_n) \cap S(R^{p^n}G^{p^n}) = S(R^{p^n}G^{p^n}; G^{p^n} \cap H_n) = 1$ since $H_n \cap G^{p^n} = 1$. So, relabeling and applying the Kovács theorem, $S(RG;H) \subseteq B^*$, as claimed.

The reverse statement holds by application of (4), which gives the result. □

As a direct consequence we obtain the following:

COROLLARY 1. *Under the above conditions from the theorem, $S(RG;B_p) \subseteq B^*$.*

Proof. Follows immediately when $H = B_p$. The proof is completed. □

Remark. If R is perfect and G/G_p is p-divisible, then, in [2], it was documented that $S(RG;B_p) = B^*$.

Next, we come to the section II:

II. $S(RH)$ as a subgroup of B^*.

THEOREM 2. *Let G be an abelian p-group, H be its subgroup and R be a commutative ring with unity of prime characteristic p. Then $S(RH) \subseteq B^*$ if and only if $H \subseteq B$.*

Proof. For the sufficiency, we presume that $H \subseteq B$. Thus $H = \bigcup_{n=1}^{\infty} H_n$, $H_n \subseteq H_{n+1}$ and $H_n \cap G^{p^n} = 1$. Hence, $S(RH) = \bigcup_{n=1}^{\infty} S(RH_n)$ and $S(RH_n) \subseteq S(RH_{n+1})$. By making use of Lemma 1, we compute that $S(RH_n) \cap S^{p^n}(RG) = S(RH_n) \cap S(R^{p^n}G^{p^n}) = S(R^{p^n}(H_n \cap G^{p^n})) = 1$, as desired. By virtue of the Kovács criterion, we are done.

The necessity is trivial by application of (4). This proves the theorem. □

One directly sees that a major consequence is when $H = B$.

COROLLARY 2. *Under the above restrictions from the theorem, $S(RB) \subseteq B^*$.*

The following statement is an important step for the next theorem.

PROPOSITION 1. *If R has a trivial nil-radical, then $S(RG) = S(RG; G_p)$.*

P r o o f. It is easy to see that $S(RG; G_p) \subseteq S(RG)$. For the converse, choose $x \in S(RG)$, i.e. $x = \sum_{i=1}^{n} r_i g_i$; $r_i \in R$, $g_i \in G$; $\sum_{i=1}^{n} r_i = 1$ and $\sum_{i=1}^{n} r_i^{p^k} g_i^{p^k} = 1$ for some $k \in \mathbb{N}$. Thus, $(r_1^{p^k} + \dots + r_t^{p^k}) g_1^{p^k} + (r_{t+1}^{p^k} + \dots + r_m^{p^k}) g_{t+1}^{p^k} + r_u^{p^k} g_u^{p^k} + \dots + r_n^{p^k} g_n^{p^k} = 1$ is an element written in canonical form for $g_1^{p^k} = \dots = g_t^{p^k} \neq g_{t+1}^{p^k} = \dots = g_m^{p^k}$, $g_u^{p^k} \neq \dots \neq g_n^{p^k} \neq g_u^{p^k}$, $u = m+1$.

We will differ two points.

1 case. Let $r_1^{p^k} + \dots + r_t^{p^k} = 1$ and $g_1^{p^k} = 1$.
Consequently $r_1 + \dots + r_t = 1$ and $g_1 \in G_p$. Moreover $r_{t+1} + \dots + r_m = 0$, $r_u = \dots = r_n = 0$. Finally $x = 1 + r_1(-1+g_1) + \dots + r_t(-1+g_t) + r_{t+1} g_m(-1 + g_m^{-1} g_{t+1}) + \dots + r_{m-1} g_m(-1 + g_m^{-1} g_{m-1}) \in 1 + I(RG; G_p) = S(RG; G_p)$, as wanted.

2 case. Let $r_u^{p^k} = 1$ and $g_u^{p^k} = 1$.
So $r_u = 1$ and $g_u \in G_p$. Besides $r_1 + \dots + r_t = 0$, $r_{t+1} + \dots + r_m = 0$, etc. and $r_n = 0$. Therefore $x = 1 + r_u(-1+g_u) + r_1 g_t(-1 + g_t^{-1} g_1) + \dots + r_{t-1} g_t(-1 + g_t^{-1} g_{t-1}) + r_{t+1} g_m(-1 + g_m^{-1} g_{t+1}) + \dots + r_{m-1} g_m(-1 + g_m^{-1} g_{m-1}) \in S(RG; G_p)$, as promised. The proof is finished. □

Now we can attack the following.

THEOREM 3. *Let G be an abelian group, H be its subgroup and R be a commutative ring with unity of prime characteristic p without nilpotent elements. Then $S(RH) \subseteq B^*$ if and only if $H_p \subseteq B_p$.*

P r o o f. The necessity holds by (4). To treat the reverse, suppose that $H_p \subseteq B_p$. Owing to the result of K o v á c s, $H_p = \bigcup_{n=1}^{\infty} H_p^{(n)}$, where $H_p^{(n)} \subseteq H_p^{(n+1)}$ and $H_p^{(n)} \cap G_p^{p^n} = 1$. By virtue of the proposition we derive, $S(RH) = S(RH; H_p) = \bigcup_{n=1}^{\infty} S(RH; H_p^{(n)})$ and $S(RH; H_p^{(n)}) \subseteq S(RH; H_p^{(n+1)})$. Moreover, employing Lemma 1 and Lemma 2, we calculate $S(RH; H_p^{(n)}) \cap S^{p^n}(RG) \subseteq S(RG; H_p^{(n)}) \cap S(R^{p^n} G^{p^n}) = S(R^{p^n} G^{p^n}; G^{p^n} \cap H_p^{(n)}) = 1$ since $H_p^{(n)} \cap G^{p^n} = 1$, which finishes the proof according to the K o v á c s theorem again. □

One immediately sees that an important special case is when $H = B$.

COROLLARY 3. *By the above conditions in the theorem, $S(RB) \subseteq B^*$.*

PROBLEM. Let R be with nontrivial nil-radical. Then what is the necessary and sufficient condition $S(RH)$ to be a subgroup of B^*?

We continue with the section III.

III. $S(RG;H)G_p/G_p$ **as a subgroup of** $B_{S(RG)/G_p}$.

We start with the following lemma.

Lemma 3. *Let* $1 \in L \le R$, $A \le G$, $C \le G$. *Then*

$$[G_p(1+I(RG;A))] \cap S(LC) \subseteq G_p(1+I(LC;C\cap A)).$$

Proof. Choose x from the left-hand side. Furthermore, $x = \sum_{c\in C} \alpha_c c = g_p \sum_{g\in G} r_g g$ where $g_p \in G_p$ and $\sum_{c\in \bar{c}A} \alpha_c = \begin{cases} 1, & \bar{c}\in A, \\ 0, & \bar{c}\notin A \end{cases}$ for every $\bar{c}\in C$. Certainly, without loss of generality, we may presume that the second sum contains some $g' \in G_p$. Write $x = g_p g' \sum_{g\in G} r_g g g'^{-1}$ where we elementary observe that $gg'^{-1} \in C$ and $\alpha_c = r_g$. But then $\sum_{c\in \bar{c}A\cap C} \alpha_c = \sum_{c\in\bar{c}(A\cap C)} \alpha_c = \begin{cases} 1, & \bar{c}\in A\cap C, \\ 0, & \bar{c}\notin A\cap C \end{cases}$ for all $\bar{c}\in C$. The inclusion is verified. □

After this, we concentrate on the following theorem:

Theorem 4. *Suppose* R *is a commutative unitary ring of prime characteristic* p *and* G *is an abelian group whose subgroup* H *is* p*-torsion. Then* $S(RG;H)G_p/G_p$ *can be expanded to a basic subgroup of* $S(RG)/G_p$ *if* H *can be expanded to* B_p.

Proof. As above, we write down $H = \bigcup_{n<\omega} H_n$, where $H_n \subseteq H_{n+1}$ and $H_n \cap G_p^{p^n} = 1$. Therefore, $S(RG;H)G_p/G_p = \bigcup_{n<\omega} [S(RG;H_n)G_p/G_p]$. Moreover, adapting Lemma 1, Lemma 3 and the modular law in [5], we compute

$$\begin{aligned}
&[S(RG;H_n)G_p/G_p] \cap [S(RG)/G_p]^{p^n} \\
&\qquad = \big([S(RG;H_n)G_p] \cap [S(R^{p^n}G^{p^n})G_p]\big)/G_p \\
&\qquad = G_p[G_pS(R^{p^n}G^{p^n};H_n\cap G^{p^n})]/G_p = 1,
\end{aligned}$$

as desired. Bearing in mind the Kovács attainment, the proof is completed. □

Corollary 4. *Under the assumptions from Theorem 4,* $S(RG;B_p)G_p/G_p$ *is contained in the basic subgroup of* $S(RG)/G_p$.

IV. $S(RH)G_p/G_p$ **as a subgroup of** $B_{S(RG)/G_p}$.

Lemma 4. *Let* $1 \in L \leq R$, $A \leq G$, $C \leq G$. *Then*

$$[G_p S(RA)] \cap S(LC) = C_p S\big(L(A \cap C)\big).$$

Proof. Evidently, the left hand-side contains the right hand-side. For the opposite relation, we take an arbitrary element x from the left hand-side. Therefore we can write, $x = \alpha_1 c_1 + \dots + \alpha_k c_k = g_p(r_1 a_1 + \dots + r_k a_k)$. The canonical forms yield $\alpha_1 = r_1, \dots, \alpha_k = r_k$; $c_1 = g_p a_1, \dots, c_k = g_p a_k$. Because $r_1 a_1 + \dots + r_k a_k \in S(RA)$, there is a member that belongs to A_p, say $a_1 \in A_p$. Hence $x = g_p a_1 (r_1 + r_2 a_2 a_1^{-1} + \dots + r_k a_k a_1^{-1}) \in C_p S\big(L(A \cap C)\big)$, as well. The proof is over. □

Theorem 5. *Suppose* G *is an abelian* p*-group,* $H \leq G$ *and* R *is a commutative ring with* 1 *of prime characteristic* p*; or* G *is an abelian group* $H \leq G$ *and* R *is a commutative ring with* 1 *of prime characteristic* p *with no nilpotents. If* H *can be expanded to* B*; or* H_p *can be expanded to* B_p*, then* $S(RH)G_p/G_p$ *can be expanded to a basic subgroup of* $S(RG)/G_p$.

Proof. For the first situation,

$$H = \bigcup_{n=1}^{\infty} H_n, \qquad H_n \subseteq H_{n+1} \quad \text{and} \quad H_n \cap G^{p^n} = 1.$$

Thus, $S(RH)G_p/G_p = \bigcup_{n<\omega} [S(RH_n)G_p/G_p]$. Invoking Lemma 4, we calculate

$$\begin{aligned}[S(RH_n)G_p] \cap [S^{p^n}(RG)G_p] &= G_p\big[(G_p S(RH_n)) \cap S(R^{p^n} G^{p^n})\big] \\ &= G_p S\big(R^{p^n}(H_n \cap G^{p^n})\big) = G_p.\end{aligned}$$

For the second half, $H_p = \bigcup_{n<\omega} \Gamma_n$, $\Gamma_n \subseteq \Gamma_{n+1}$ and $\Gamma_n \cap G_p^{p^n} = 1$. Consequently, utilizing Proposition 1,

$$S(RH)G_p/G_p = S(RH; H_p)G_p/G_p = \bigcup_{n<\omega} [S(RH;\Gamma_n)G_p/G_p].$$

Conforming with Lemma 3 and the modular law from [5], we have

$$\begin{aligned}[S(RH;\Gamma_n)G_p] \cap [S^{p^n}(RG)G_p] &= G_p\big[(G_p S(RH;\Gamma_n)) \cap S^p(R^{p^n} G^{p^n})\big] \\ &= G_p S\big(R^{p^n}(H \cap G^{p^n}); \Gamma_n \cap G^{p^n}\big) = G_p.\end{aligned}$$

Finally, in both cases, we can apply the Kovács criterion to complete the claim. □

Corollary 5. *Under the assumptions from Theorem* 5, $S(RB)G_p/G_p$ *is contained in the basic subgroup of* $S(RG)/G_p$.

3. Applications

An abelian p-group is called *starred* if it has the same power as its basic subgroup (see, for instance, [6]). In particular, all finite groups, or, more generally, the direct sums of reduced countable torsion abelian groups, are starred. In this aspect, we begin with certain interesting characterizations of groups, starting with one of the announcements from [0; Theorem 11].

THEOREM 6. *Let G be an abelian group whose G_p is not divisible and let R be a commutative ring with 1 of prime characteristic p. Then $S(RG)$ is a starred group.*

Proof. We study only the infinite case for G or R because the other is self-evident. Since G_p is not divisible, i.e. $B_p \neq \{1\}$, by Theorem 1 and in particular by Corollary 1, constructing the elements $1 + rg(1 - b_p)$ where $0 \neq r \in R$ is arbitrary, $1 \neq g \in G \setminus h\langle b_p \rangle$ for every h with $g \neq h \in G$ when G is infinite or $g = 1$ when G is finite, and $1 \neq b_p \in B_p$ is fixed, we yield $|B^*| \geq |S(RG; B_p)| = \max(|R|, |G|)$ that is equivalent to $|B^*| = |S(RG)|$. Finally, in the spirit of the definition, we finish the proof. □

Next, we proceed by proving the other part of the announcement [0; Theorem 11].

THEOREM 7. *Let G be an abelian group for which G_p is not divisible and let R be a commutative ring with 1 of prime characteristic p. Then $S(RG)/G_p$ is a starred group.*

Proof. We elementarily observe that only the infinite case for R or G is necessary to consider. Complying with Corollary 4, we detect that $|B_{S(RG)/G_p}| \geq |S(RG; B_p)G_p/G_p| = |S(RG; B_p)/B_p|$ because of the fact that $S(RG; B_p)G_p/G_p \cong S(RG; B_p)/B_p$. Besides, we extract

$$|S(RG; B_p)/B_p| = \max(|R|, |G|) .$$

Indeed, we treat only $|G| \geq \aleph_0$ since otherwise B_p must be finite and thus $|S(RG; B_p)/B_p| = |S(RG; B_p)| = |R| \geq \aleph_0$ by appealing to the proof of Theorem 6, so we are done. Thereby, we examine the elements $[1 + rg(1 - b_p)]B_p$, where $0 \neq r \in R$, $g \in G \setminus \langle b_p \rangle$, $1 \neq b_p \in B_p$, such that the following conditions are satisfied: r and g vary in R and G respectively so that $g \notin a\langle b_p \rangle$ for each a with $g \neq a \in G$, and $b_p \neq 1$ is a fixed but arbitrary element.

Assume now, $[1 + rg - rgb_p]B_p = [1 + \alpha h - \alpha hb_p]B_p$ for some such elements $0 \neq \alpha \in R$, $h \in G \setminus \langle b_p \rangle$ with $h \notin a\langle b_p \rangle$ for any a with $h \neq a \in G$ and $1 \neq b_p \in B_p$. Therefore $1 + rg - rgb_p = [1 + \alpha h - \alpha hb_p]c_p = c_p + \alpha hc_p - \alpha hb_pc_p$ for some $c_p \in B_p$. These two group ring elements are in canonical forms, whence we have the following combinations: $c_p = 1$, $g = hb_p$, $gb_p = h$; or $g = hb_pc_p$, $gb_p = c_p$, $hc_p = 1$; or $g = c_p$, $gb_p = hb_pc_p$, $hc_p = 1$; or $g = hc_p$, $gb_p = c_p$, $hb_pc_p = 1$; or $g = c_p$, $gb_p = hc_p$, $hb_pc_p = 1$, but these last equalities are impossible, hence as a final conclusion $r = \alpha$ and $g = h$, thus confirming our claim.

On the other hand $|S(RG)/G_p| \leq |S(RG)| \leq |RG| = \max(|R|, |G|)$. Finally, we derive $|B_{S(RG)/G_p}| = |S(RG)/G_p|$, as desired. Thus, exploiting the definition, we deduce the proof in general after all. □

THEOREM 8. *Suppose G is an abelian p-group and F is a field of characteristic $p \neq 0$. Then $V(FG)/G$ is totally projective for all abelian p-groups G if and only if $V(FG)/G$ is totally projective for all starred p-groups G.*

P r o o f. Let G be an arbitrary p-primary group. Thus, invoking [6; p. 537, Corollary 8], $A = G \times T$ for some group T and a starred p-group A. Therefore $V(FA)/A \cong V(FG)/G \times [1 + I(FA;T)]/T$. So, our claim follows referring to [4]. □

In the case of primary groups, we find the following.

PROPOSITION 2. *Let G be an abelian p-group and R a perfect commutative ring with 1 and prime characteristic p. Then $V(RG)/G$ is a direct factor of a starred p-group.*

P r o o f. As above, $V(RG)/G$ is formally a direct factor of $V(RA)/A$ for some starred p-group A. Using [3], $V(RA)/A$ is starred, thus completing the proof. □

We close the paper with the following paragraph.

4. Concluding discussion

In the present study, we have established that $S(RG)$ and $S(RG)/G_p$ are both starred groups whenever G_p is not divisible. Our attainments in this way are improvements of results of such a type, argued in [3]. They shed a positive light on the long-stating conjecture that $S(RG)/G_p$ is totally projective whenever R is perfect and G_p is reduced. So, a question of some importance is what are the structures of these groups when G_p is divisible. Does it follow that they are simply presented, provided that R is eventually perfect? The problem may be equivalently restated in a larger form thus: If A is an abelian divisible p-group, then what are the structures of $S(RA)$ and $S(RA)/A$ respectively, provided R is imperfect? Nevertheless, the first weaker problem can be plainly settled when R is a perfect field. In fact, if G_p is divisible, then one immediately sees that G_p is a direct factor of G, hence its balanced subgroup. Therefore each nice subgroup of G_p is nice in G, whence by the routine back-and-forth argument of appropriate nice subgroups in $S(RG)$, described in [8], it follows at once that $S(RG)$ and $S(RG)/G_p$ are both simply presented. But all reduced simply presented groups, often called totally projective groups, are known to be starred. Thus these two groups are direct sums of divisible groups and starred groups.

As a final remark, it is worthwhile noticing that if G_p is divisible and R is perfect, then $S(RG)$ and $S(RG)/G_p$ have isomorphic basic subgroups. In order to check this, write down $G = G_d \times G_r$, the decomposition into divisible part and reduced part. Thus $(G_r)_p = \{1\}$ and $S(RG) = S(RG_d) \times [1 + I_p(RG; G_r)]$, hence $S(RG)/G_p \cong S(RG_d)/(G_d)_p \times [1 + I_p(RG; G_r)]$. Since $S(RG_d)$ and $S(RG_d)/(G_d)_p$ are both divisible, employing [5; p. 164, Exercise 3(a)], we are done.

J. M. Irwin and S. A. Khabbaz (in: On generating subgroups of Abelian groups, Proc. Colloq. on Abelian Groups (Tihany), Budapest, 1964, pp. 87 97) defined the more general conception of a p-group G being *strongly starred* if $|G^{p^n}| = |B^{p^n}|$ for every $n \geq 0$. They have proved also that G is strongly starred precisely when $G = \langle B, B' \rangle$, that is G is generated via two its basic subgroups B and B'. In lieu of the methods used here, in a subsequent research investigation, we shall utilize new ideas, so a successful exploration of such groups in group rings can be realized.

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