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A VERSION OF THE STRONG LAW OF LARGE NUMBERS UNIVERSAL UNDER MAPPINGS

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ABSTRACT. Let $(\Omega, \mathcal{A}, P)$ stand for some probability space, Θ for a separable topological space, and $(Y, \mathcal{Y})$ for a measurable space. Furthermore, $f\colon Y \times \Theta \to \mathbb{R}$ is some function such that f_ϑ is $\mathcal{Y}$-measurable for all $\vartheta \in \Theta$ and $\{f_y : y \in \mathcal{Y}\}$ is pointwise equicontinuous. It is proved that for any sequence $X_1, X_2, \dots$ of Y-valued random variables, which is i.i.d. relative to P such that $E(|f(X_1, \vartheta)|) < \infty$ is valid for any $\vartheta \in \Theta$, there exists some P-zero set N satisfying $\frac{1}{n} \sum_{i=1}^{n} f(X_i(\omega), \vartheta) \to E(f(X_1, \vartheta))$, $\omega \in \Omega \setminus N$, for all $\vartheta \in \Theta$. This result is illustrated by examples and compared with known uniform versions of the SLLN.

1. Introduction and main result

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(Y, \mathcal{Y})$ a measurable space, and $f\colon Y \times \Theta \to \mathbb{R}$ a function such that f_ϑ is $\mathcal{Y}$-measurable for all $\vartheta \in \Theta$, where Θ stands for some non-empty and not necessarily countable set. Then it seems quite interesting to inquire, whether the following uniform version of the strong law of large numbers (SLLN) holds true: Does there exists for any (w.r.t. P) independent and identically distributed (i.i.d.) sequence of Y-valued random variables $X_1, X_2, \dots$ satisfying $E(|f(X_1, \vartheta)|) < \infty$, $\vartheta \in \Theta$, some P-zero set $N \in \mathcal{A}$ such that $\lim\limits_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i(\omega), \vartheta) \to E(f(X_1, \vartheta))$ holds true for all $\omega \in \Omega \setminus N$ and any $\vartheta \in \Theta$?

Now it will be shown that the following conditions are sufficient:

1. Θ is some separable topological space.
2. $f\colon Y \times \Theta \to \mathbb{R}$ has the property that $\{f_y : y \in Y\}$ is pointwise equicontinuous.

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The pointwise equicontinuity of $\{f_y : y \in Y\}$ implies for any $\vartheta_0 \in \Theta$ and $\varepsilon > 0$ the existence of some neighbourhood $U_\varepsilon(\vartheta_0)$ satisfying $f(y,\vartheta_0) - \varepsilon \leq f(y,\vartheta) \leq f(y,\vartheta_0) + \varepsilon$, $\vartheta \in U_\varepsilon(\vartheta_0)$, $y \in Y$, from which the inequalities $\frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta_0) - \varepsilon \leq \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta) \leq \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta_0) + \varepsilon$, $\omega \in \Omega$, $\vartheta \in U_\varepsilon(\vartheta_0)$, follow. Therefore, the inequalities

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta_0) - \varepsilon \leq \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta) \leq \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta_0) + \varepsilon$$

are valid for all $\omega \in \Omega$ and any $\vartheta \in U_\varepsilon(\vartheta_0)$, i.e. $\left|\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta) - \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta_0)\right| \leq \varepsilon$, $\omega \in \Omega$, $\vartheta \in U_\varepsilon(\vartheta_0)$, holds true, which proves that the function defined by $\vartheta \to \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta)$, $\vartheta \in \Theta$, is continuous for all $\omega \in \Omega$. By a similar argument the function introduced by $\vartheta \to \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta)$, $\vartheta \in \Theta$, is continuous for all $\omega \in \Omega$. Furthermore, the function $\vartheta \to E(f(X_1,\vartheta))$, $\vartheta \in \Theta$, is continuous. Now the classical SLLN implies that the set S introduced by $\left\{(\omega,\vartheta) \in \Omega\times\Theta : \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta) < E(f(X_1,\vartheta)) \text{ or } \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta) > E(f(X_1,\vartheta))\right\}$ satisfies $P(S_\vartheta) = 0$ for all $\vartheta \in \Theta$. Furthermore, the continuity of the functions $\vartheta \to \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\vartheta),\vartheta)$, $\vartheta \to \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i\vartheta)$, and $\vartheta \to E(f(X_1,\vartheta))$, $\vartheta \in \Theta$, together with the existence of some countable and dense subset Θ' of Θ yields the universal P-zero set $N \in \mathcal{A}$ defined by $\bigcup_{\vartheta\in\Theta'} S_\vartheta$ of the type described by the following theorem.

THEOREM. *Let $(\Omega,\mathcal{A},P)$ denote a probability space, Θ some separable topological space, and $(Y,\mathcal{Y})$ some measurable space. Furthermore, let $f: Y\times\Theta \to \mathbb{R}$ be a function such that f_ϑ is $\mathcal{Y}$-measurable for any $\vartheta \in \Theta$, and $\{f_y : y \in Y\}$ is pointwise equicontinuous. Then for any sequence $X_i\colon \Omega \to Y$, $i = 1,2,\ldots$, of $\mathcal{Y}$-measurable random variables, which are i.i.d. with respect to P and satisfy $E(|f(X_1,\vartheta)|) < \infty$, $\vartheta \in \Theta$, there exists some P-zero set $N \in \mathcal{A}$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i(\omega),\vartheta) = E(f(X_1,\vartheta))$ is valid for all $\omega \in \Omega\backslash N$ and any $\vartheta \in \Theta$.*

2. Examples and comparison to known results

The following first example shows that one cannot drop the assumption that $\{f_y : y \in Y\}$ is pointwise equicontinuous and that the corresponding domain Θ is a separable topological space without introducing some other conditions.

EXAMPLE 1. (Projections of random vectors)
First of all it will be shown that the exceptional zero-set occurring in the SLLN cannot be empty in general. For this purpose let Ω stand for the interval $[0,1)$ with the corresponding Borel σ-algebra $\mathcal{B}([0,1))$ and let $X_k : \Omega \to \mathbb{R}$ stand for the $\mathcal{B}([0,1))$-measurable random variable defined by $X_k(\omega) = \omega_k$, $\omega \in \Omega$, $k \in \mathbb{N}$, where $\omega = \sum_{k=1}^{\infty} \frac{\omega_k}{2^k}$, $\omega_k \in \{0,1\}$, $k \in \mathbb{N}$, is the dyadic expansion of ω, which is unique if there does not exist any $j \in \mathbb{N}$ satisfying $\omega_k = 1$, $k \geq j$. Then the random variables $X_1, X_2, \dots$ are independent and identically distributed with respect to the probability measure P on $\mathcal{B}([0,1))$ introduced as the Lebesgue-measure restricted to $\mathcal{B}([0,1))$. Obviously, the set N defined by $\left\{\omega \in [0,1) : \lim_{n\to\infty} \frac{\omega_1+\cdots+\omega_n}{n} \neq \frac{1}{2}\right\}$ is not empty. Now let the set Y stand for $[0,1]^T$, T being some uncountable set, where the σ-algebra $\mathcal{Y}$ of subsets of Y is introduced as the direct product $\bigotimes_{t\in T} \mathcal{A}_t$. Here the σ-algebra $\mathcal{A}_t$ of subsets of $[0,1]$ coincides with $\mathcal{B}([0,1])$ for all $t \in T$. The σ-algebra $\mathcal{Y}$ has the following property: For any $A \in \mathcal{Y}$ there exists some countable subset S of T such that $(y_t)_{t\in T} \in A$ and $y_s = y'_s$, $s \in S$, for some $(y'_t)_{t\in T} \in Y$ implies $(y'_t)_{t\in T} \in A$, i.e. the countable subset S of T determines A. Now if one introduces Θ by the set consisting of all one-dimensional projections $\pi_t\colon [0,1]^T \to [0,1]$, $t \in T$, and the function $f\colon Y \times \Theta \to \mathbb{R}$ by $f\big((y_t)_{t\in T}, \pi_t\big) = y_t = \pi_t\big((y_t)_{t\in T}\big)$, $t \in T$, then f_{π_t} is $\mathcal{Y}$-measurable for all $t \in T$. Furthermore, in connection with the Y-valued and $\mathcal{Y}$-measurable random vectors $Y_n\colon [0,1)^T \to [0,1]^T$ defined by $Y_n\big((\omega_t)_{t\in T}\big) = \big(X_n(\omega_t)\big)_{t\in T}$, $n \in \mathbb{N}$, where X_n, $n \in \mathbb{N}$, has been introduced at the beginning of this example, one gets that the exceptional zero-sets N_s defined by $\left\{(\omega_t)_{t\in T} \in [0,1)^T : \lim_{n\to\infty} \frac{1}{n}\big(X_1(\omega_s) + \cdots + X_n(\omega_s)\big) \neq \frac{1}{2}\right\}$ is equal to $X_{t\in T} A_t$, $A_t = [0,1)$, $t \in T \setminus \{s\}$, $A_s = N$. Here N has already been defined at the beginning of Example 1 and the underlying probability measure on $\bigotimes_{t\in T} \mathcal{A}_t$, $\mathcal{A}_t = \mathcal{B}([0,1))$, $t \in T$, is the direct product $\bigotimes_{t\in T} P_t$, $P_t = P$, $t \in T$, P being the Lebesgue-measure restricted to $\mathcal{B}([0,1))$. Now it will be shown that for $\bigcup_{t\in T} N_t$ there does not exist any $M \in \bigotimes_{t\in T} \mathcal{A}_t$, $\mathcal{A}_t = \mathcal{B}([0,1))$, $t \in T$, satisfying $\Big(\bigotimes_{t\in T} P_t\Big)(M) = 0$ and $\bigcup_{t\in T} N_t \subset M$. For this purpose one observes

that the inclusion $\bigcup_{t\in T} N_t \subset M$ together with some $(\omega_t)_{t\in T} \in [0,1)^T$ results in $(\omega_t)_{t\in T} \in M$, i.e. $M = [0,1)^T$, which might be seen as follows: Let S denote some countable subset of T, which determines M and let $(\omega_t')_{t\in T}$ be any element of $[0,1)^T$ satisfying $\omega_t' = \omega_t$, $t \in T \setminus \{t_0\}$, and $\omega_{t_0}' \in N$, where t_0 is some element of $T \setminus S$. Hence $(\omega_t')_{t\in T} \in N_{t_0}$ together with $N_{t_0} \subset M$ implies $(\omega_t)_{t\in T} \in M$. Finally, $\bigcup_{t\in T} N_t$ and $X_{t\in T} B_t$, where B_t stands for N^c, $t \in T$, are disjoint, i.e. $\bigcup_{t\in T} N_t \notin \bigotimes_{t\in T} \mathcal{A}_t$, $\mathcal{A}_t = \mathcal{B}([0,1))$, $t \in T$, holds true.

The second example results in some application of the preceding theorem.

EXAMPLE 2. (Power series with random coefficients)
Let Y and Θ stand for second countable topological spaces and let $f\colon Y\times\Theta \to \mathbb{R}$ be some continuous function with respect to the corresponding product topology of $Y \times \Theta$. Then there exists for any $y \in Y$ some neighborhood $U(y)$ such that $\{f_{y'} : y' \subset U(y)\}$ is pointwise equicontinuous (since otherwise there would exist $\vartheta_0 \in \Theta$, $y_0 \in Y$, and $\varepsilon_0 > 0$ satisfying $|f(y_n, \vartheta_n) - f(y_n, \vartheta_0)| \geq \varepsilon_0$, $n \in \mathbb{N}$, where $(y_n)_{n\in\mathbb{N}}$, $y_n \in Y$, $n \in \mathbb{N}$, and $(\vartheta_n)_{n\in\mathbb{N}}$, $\vartheta_n \in \Theta$, $n \in \mathbb{N}$, are sequences with $\lim_{n\to\infty} y_n = y_0$ and $\lim_{n\to\infty} \vartheta_n = \vartheta_0$, which is a contradiction to the property of f to be continuous) and a theorem of Lindelöf (cf. [2; I.4.13, p. 12]) yields the existence of some countable collection $U(y_k)$, $k = 1, 2, \ldots$, satisfying $\bigcup_{k=1}^{\infty} U(y_k) = \bigcup_{y\in Y} U(y) = Y$. Now the theorem above results in the existence of some universal zero set with respect to $\{f_y : y \in Y\}$ in connection with the SLLN, if the σ-algebra $\mathcal{Y}$ of subsets of Y is chosen as the corresponding Borel σ-algebra $\mathcal{B}(Y)$. In particular, in connection with $\sum_{n=1}^{\infty} |a_n| \frac{|\vartheta|^n}{n!} < \infty$, $|\vartheta| < \vartheta_0$ for some $\vartheta_0 > 0$ and some $(a_n)_{n\in\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, one might introduce the continuous function $f\colon Y \times \Theta \to \mathbb{R}$ with $Y = \{(y_n)_{n\in\mathbb{N}} \in \mathbb{R}^N : |y_n| \leq |a_n|,\ n \in \mathbb{N}\}$, and $\Theta = (-\vartheta_0, \vartheta_0)$ defined by $f((y_n)_{n\in\mathbb{N}}, \vartheta) = \sum_{n=1}^{\infty} y_n \frac{\vartheta^n}{n!}$, $(y_n)_{n\in\mathbb{N}} \in Y$, $\vartheta \in \Theta$, where $\mathbb{R}^{\mathbb{N}}$ is equipped with the product topology and Θ with the relative topology of $\mathbb{R}$.

Remark. (Comparison with known uniform strong laws of large numbers)
In [3; p. 107–111] and [5; p. 854] one might find the following uniform version of the Strong law of large numbers:

$$P\Big\{ \lim_{n\to\infty} \sup_{\vartheta\in\Theta} \Big| \frac{1}{n} \sum_{i=1}^{n} f(X_i, \vartheta) - E\big(f(X_1, \vartheta)\big) \Big| = 0 \Big\} = 1$$

under the assumption that Θ is some compact and metric space (tacit assumption, cf. [3; p. 110]), $\vartheta \to f(y, \vartheta)$, $\vartheta \in \Theta$, is continuous for all $y \in Y$, and there exists some $\mathcal{Y}$-measurable function $g\colon Y \to \mathbb{R}$ such that $g \circ X_1$ is P-integrable and $|f(y,\vartheta)| \le g(y)$, $y \in Y$, $\vartheta \in \Theta$. This result might also be derived easily by the theorem above together with a version of the theorem of Arzela-Ascoli, which might be found in [6; p. 369]. However, there appears the stronger pointwise equicontinuity assumption for $\{f_y : y \in Y\}$. Finally, one might consult [1; p. 4], and [4; p. 1308], for a version concerning the existence of some universal P-zero set in connection with the SLLN.

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