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Dedicated to Valter Šeda
on the occasion of his 70th birthday

ON THE LEFSCHETZ FIXED POINT THEOREM

LECH GÓRNIEWICZ

(*Communicated by Milan Medveď*)

ABSTRACT. The aim of this paper is to present current results concerning the Lefschetz Fixed Point Theorem for metric spaces. Some new results are included. In particular, an abstract version and also the Lefschetz Fixed Point Theorem for condensing mappings are proved.

0. Introduction

In 1923 S. Lefschetz formulated the famous fixed point theorem so which is now known as the Lefschetz fixed point theorem. Later, in 1928 H. Hopf gave a new proof of the Lefschetz fixed point theorem for self-mappings of polyhedra. Let us remark that Lefschetz formulated his theorem for compact manifolds. In 1967, A. Granas extended the Lefschetz fixed point theorem to the case of absolute neighbourhood retracts. The proof of the theorem was based on the fact that all compact absolute neighbourhood retracts are homotopically equivalent with polyhedra. Then the case of noncompact absolute neighbourhood retracts was reduced to the compact case by using the generalized trace theory introduced by J. Leray. We recommend [4], [10], [14] for details.

In the present paper we would like to present current results concerning this theorem for metric spaces. We shall prove an abstract version of the Lefschetz fixed point theorem (comp. Theorem (2.12)) from which we shall deduce not only well-known results but also some new results mainly connected with condensing and k-set contraction mappings.

Moreover, relative versions of the Lefschetz fixed point theorem are discussed.

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1. Topological and homological preliminaries

We shall restrict our considerations to metric spaces only. Following K. Borsuk [1] we define:

(1.1) Definition. A space X is called an *absolute neighbourhood retract* (ANR) provided for every space Y and for every homeomorphism $h\colon X \to Y$ such that $h(X)$ is a closed subset of Y there exists an open neighbourhood U of $h(X)$ in Y and a continuous map (called a retraction map) $r\colon U \to h(X)$ such that $r(u) = u$ for every $u \in h(X)$, i.e. $h(X)$ is a retract of U; X is called an *absolute retract* ($X \in$ AR) provided the above holds true for $U = Y$, i.e. $h(X)$ is a retract of Y.

In other words $X \in$ ANR ($X \in$ AR) if and only if X has the neighbourhood extension (extension) property (comp. [2], [7]).

To understand better how large the class of ANR-s (AR-s) is we recall:

(1.2) Proposition. ([2], [7])

(1.2.1) $X \in$ ANR *if and only if there exists a normed space E and an open subset U of E such that X is homeomorphic to a retract of U;*

(1.2.2) $X \in$ AR *if and only if there exists a normed space E and a convex subset W of E such that X is homeomorphic to a retract of W.*

In particular, any open subset in a normed space or any finite polyhedron is an ANR-space; respectively any convex subset of an arbitrary normed space is an AR-space. Note that any AR-space is contractible and any ANR-space is locally contractible.

We shall consider the category of pairs of metric spaces and continuous mappings. By a pair of spaces (X, X_0) we understand a pair consisting of a metric space X and one of its subsets X_0. A pair of the form $(X, \emptyset)$ will be identified with the space X. By a map $f\colon (X, X_0) \to (Y, Y_0)$ we understand a continuous map $f\colon X \to Y$ such that $f(X_0) \subset Y_0$. In what follows having a map of pairs

$$f\colon (X, X_0) \to (Y, Y_0)$$

we shall denote by

$$f_X\colon X \to Y \qquad \text{and} \qquad f_{X_0}\colon X_0 \to Y_0$$

the respective mappings induced by f.

Let H be the Čech homology functor with compact carriers ([8] or [9]) and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of all pairs of spaces and all maps between such pairs, to the category of graded vector spaces over $\mathbb{Q}$ and linear maps of degree zero. Thus

$$H(X, X_0) = \{H_q(X, X_0)\}$$

is a graded vector space, $H_q(X, X_0)$ being the q-dimensional Čech homology with compact carriers of X. For a map $f\colon (X, X_0) \to (Y, Y_0)$, $H(f)$ is the induced linear map $f_* = \{f_{*q}\}$, where $f_{*q}\colon H_q(X, X_0) \to H_q(Y, Y_0)$.

A non-empty space X is called *acyclic* provided:

(i) $H_q(X) = 0$ for all $q \geq 1$,

(ii) $H_o(X) \approx \mathbb{Q}$.

Let $u\colon E \to E$ be an endomorphism of an arbitrary vector space. Let us put $N(u) = \{x \in E : \ u^n(x) = 0 \text{ for some } n\}$, where u^n is nth iterate of u and $\widetilde{E} = E|_{N(u)}$. Since $u(N(u) \subset N(u))$, we have the induced endomorphism $\widetilde{u}\colon \widetilde{E} \to \widetilde{E}$. We call u *admissible* provided $\dim \widetilde{E} < \infty$.

Let $u = \{u_q\}\colon E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call u a *Leray endomorphism* if

(i) all u_q are admissible,

(ii) almost all $\widetilde{E}_q$ are trivial.

For such u, we define the (*generalized*) *Lefschetz number* $\Lambda(u)$ by putting

$$\Lambda(u) = \sum_q (-1)\,\mathrm{tr}(\widetilde{u}_q),$$

where $\mathrm{tr}(\widetilde{u}_q)$ is the ordinary trace of $\widetilde{u}_q$ (comp. [9]). The following important property of the Leray endomorphism is a consequence of the well-known formula $\mathrm{tr}(u \circ v) = \mathrm{tr}(v \circ u)$ for the ordinary trace.

(1.3) Proposition. *Assume that, in the category of graded vector spaces, the following diagram commutes*

$$\begin{array}{ccc} E' & \xrightarrow{\ u\ } & E'' \\ {\scriptstyle u'}\uparrow & \nwarrow^{v} & \uparrow{\scriptstyle u''} \\ E' & \xrightarrow[\ u\ ]{} & E'' \end{array}$$

Then, if u' or u'' is a Leray endomorphism, so is the other; and, in that case, $\Lambda(u') = \Lambda(u'')$.

An endomorphism $u\colon E \to E$ of a graded vector space E is called *weakly nilpotent* if for every $q \geq 0$ and for every $x \in E_q$, there exists an integer n such that $u_q^n(x) = 0$. Since, for a weakly nilpotent endomorphism $u\colon E \to E$, we have $N(u) = E$, so:

(1.4) Proposition. *If $u\colon E \to E$ is a weakly-nilpotent endomorphism, then* $\Lambda(u) = 0$.

Let $f\colon (X, X_0) \to (X, X_0)$ be a map, $f_*\colon H(X, X_0) \to H(X, X_0)$ is a Leray endomorphism. For such f, we define the Lefschetz number $\Lambda(f)$ of f by putting

$\Lambda(f) = \Lambda(f_*)$. Clearly, if f and g are homotopic, $f \sim g$, then f is a Lefschetz map if and only if g is a Lefschetz map; and, in this case, $\Lambda(f) = \Lambda(g)$.

Let us observe that if X is an acyclic space or, in particular, contractible, then for every $f: X \to X$ the endomorphism $f_*: H(X) \to H(X)$ is a Leray endomorphism and $\Lambda(f_*) = 1$.

Consequently, if $X \in \mathrm{AR}$ or X is a convex subset in a normed space, then for every continuous map $f: X \to X$ the Lefschetz number $\Lambda(f) = \Lambda(f_*) = 1$.

We have the following lemma (see: [3], [6], [10]).

(1.5) Lemma. *Let $f: (X, X_0) \to (X, X_0)$ be a map of pairs. If two of endomorphisms $f_*: H(X, X_0) \to H(X, X_0)$, $(f_X)_*: H(X) \to H(X)$, $(f_{X_0})_*: H(X_0) \to H(X_0)$ are Leray endomorphisms, then so is the third; in that case:*

$$\Lambda(f_*) = \Lambda\big((f_X)_*\big) - \Lambda\big((f_{X_0})_*\big)$$

or equivalently:

$$\Lambda(f) = \Lambda(f_X) - \Lambda\big(f_{X_0}\big).$$

2. Lefschetz mappings

It is convenient to introduce the following notion.

(2.1) Definition. A continuous map $f: X \to X$ is called a *Lefschetz map* provided the generalized Lefschetz number $\Lambda(f)$ of f is well defined and $\Lambda(f) \neq 0$ implies that the set $\mathrm{Fix}(f) = \{x \in X : f(x) = x\}$ is nonempty.

In 1969, A. Granas [10], (see also [11]) proved:

(2.2) Theorem. *Let $X \in \mathrm{ANR}$ and let $f: X \to X$ be a continuous and compact map (i.e., $\overline{f(X)}$ is a compact set), then f is a Lefschetz map.*

To formulate the result proved in 1977 by R. Nussbaum [14] we need some notations.

We shall need the following Kuratowski or Hausdorff (see [7] or [9]) measure of noncompactness. Let X be a complete metric space and A be a bounded subset of X. We let:

$$\gamma(A) = \inf\{r > 0 : \text{ there exists a finite covering of } A \text{ by subsets of diameter at most } r\}$$

or

$$\gamma(A) = \inf\{r > 0 : \text{ there exists a finite covering of } A \text{ by open balls with radius } r\}.$$

We have the following properties (see [7] or [9]):

(2.3) $0 \leq \gamma(A) \leq \delta(A)$, where $\delta(A)$ is the diameter of A;
(2.4) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$;
(2.5) $\gamma(N_\varepsilon(A)) \leq \gamma(A) + 2\varepsilon$, where $N_\varepsilon(A) = \{x \in E : \; d(x, A) < \varepsilon\}$;
(2.6) $\gamma(A) = 0$ if and only if A is relatively compact;
(2.7) if $K_1 \supset K_2 \supset \cdots \supset K_n \supset \ldots$, where K_n is closed nonempty for any n and $\lim_{n\to\infty} \gamma(K_n) = 0$, then $K_\infty = \bigcap_{n=1}^{\infty} K_n$ is compact and nonempty.

For a map $f\colon X \to X$, a compact subset $A \subset X$ is called an *attractor* provided for any open neighbourhood U of A in X and for every $x \in X$ there exists $n = n_x$ such that $f^n(x) \in U$. In what follows we shall denote family of mappings with compact attractor by CA.

Note that, if $f\colon X \to X$ has a compact attractor A, then $\mathrm{Fix}(f) \subset A$.

A continuous mapping $f\colon X \to X$ is called *condensing* (*k-set contraction*) map provided:

(2.8) if $\gamma(A) \neq 0$, then $\gamma(f(A)) < \gamma(A)$, ($\gamma(f(A)) \leq k \cdot \gamma(A)$ for some $k \in [0,1)$), where we have assumed that X is a complete metric space and $A \subset X$.

Of course, any compact map is a k-set contraction map and any k-set contraction map is a condensing map.

(2.9) THEOREM. ([14], [5], [7]) *Let U be an open subset of a Banach space E. Assume further that $f\colon U \to U$ is a condensing map which has a compact attractor, then f is a Lefschetz map.*

(2.10) DEFINITION. Let $f\colon X \to X$ be a continuous map and X_0 a subset of X. We shall say that X_0 *absorbs* compact sets provided for any compact set $K \subset X$ there exists a natural number $n = n_K$ such that $f^n(K) \subset X_0$.

It is easy to prove the following:

(2.11) PROPOSITION. *Assume that $f\colon X \to X$ is a continuous map and X_0 is an open subset of X which absorbs points. Then X_0 absorbs compact sets.*

For the proof see: [5], [7], [9].

Now we are able to prove the following important result:

(2.12) THEOREM. (Abstract version of the Lefschetz fixed point theorem) *Let $f\colon (X, X_0) \to (X, X_0)$ be a continuous map of pairs. Assume that $f_{X_0}\colon X_0 \to X_0$ is a Lefschetz map and X_0 absorbs compact sets. Then $f_X\colon X \to X$ is a Lefschetz map.*

P r o o f . First, we shall observe that $f_*\colon H(X, X_0) \to H(X, X_0)$ is weakly nilpotent and hence $\Lambda(f) = \Lambda(f_*) = 0$.

We let

$$\begin{aligned} & i\colon X_0 \to X, \qquad i(x) = x \quad \text{for every } x \in X_0, \\ & \widetilde{H}(X) = H(X)/N\big((f_X)_*\big), \\ & \widetilde{H}(X_0) = H(X_0)/N\big((f_{X_0})_*\big), \\ & \widetilde{i}_*\colon \widetilde{H}(X_0) \to \widetilde{H}(X), \qquad \widetilde{i}_*([a]) = [i_*(a)] \quad \text{for every } [a] \in \widetilde{H}(X_0). \end{aligned}$$

Since the considered functor H has compact carriers and f absorbs compact sets we deduce that $\widetilde{i}_*$ is an isomorphism. Consequently from the exactness of the homology sequence for the pair (X, X_0) we infer that $\widetilde{H}(X, X_0) = 0$. Thus $\Lambda(f) = \Lambda(f_*) = 0$ and from (1.5) we obtain:

$$\Lambda(f) = \Lambda(f_*) = 0. \tag{1}$$

By assumption, $f_{X_0}\colon X_0 \to X_0$ is a Lefschetz map. Therefore, in view of Lemma (1.5), we deduce that the Lefschetz number $\Lambda(f_X)$ of f_X is well defined and

$$\Lambda(f) = 0 = \Lambda(f_X) - \Lambda(f_{X_0}). \tag{2}$$

Now, if we assume that $\Lambda(f_X) \neq 0$, then $\Lambda(f_{X_0}) \neq 0$ and hence $\mathrm{Fix}(f_{X_0}) \neq \emptyset$. The proof is completed since $\mathrm{Fix}(f_{X_0}) \subset \mathrm{Fix}(f_X)$. □

In the next section we shall show several applications of Theorem (2.12).

3. Consequences of Theorem (2.12)

In what follows all mappings are assumed to be continuous.

Following [6] we recall the notion of compact absorbing contractions.

(3.1) DEFINITION. A mapping $f\colon X \to X$ is called *compact absorbing contractions* (CAC) provided the following conditions are satisfied:

(3.1.1) there exists an open subset U of X such that $\overline{f(U)} \subset U$ and $\overline{f(U)}$ is compact,

(3.1.2) the set U given in (3.1.1) absorbs points.

First, we are going to explain how large the class of CAC-mappings is. Evidently, any compact map $f\colon X \to X$ is a CAC-mapping. In fact, the compact set $\overline{f(X)}$ is an attractor of f and we can take X as an open neighbourhood U of $\overline{f(X)}$. More generally, any *eventually compact* map, i.e., the map $f\colon X \to X$ such that there exists n for which $\overline{f^n(X)}$ is compact, has a compact attractor A to be equal $\overline{f^n(X)}$. It is also easy to see that any CAC-map has a compact attractor A, namely $\overline{f(U)}$ (see (3.1.1)).

We shall say that a map $f\colon X \to X$ is *asymptotically compact* provided for each $x \in X$ the orbit $\{x, f(x), \ldots, f^n(x), \ldots\}$ is relatively compact and the core:

$$C_f = \bigcap_{n=1}^{\infty} \overline{f^n(X)}$$

is nonempty compact.

As is observed in [6; Proposition (6.4)] any asymptotically compact map $f\colon X \to X$ has a compact attractor A to be equal C_f.

If follows from the above that:

(3.2) Proposition.

(3.2.1) *Any compact map has a compact attractor,*
(3.2.2) *any eventually compact map has a compact attractor,*
(3.2.3) *any asymptotically compact map has a compact attractor.*

So the class of mappings with compact attractors is quite large.

To explain the connection between mappings with compact attractors and CAC-mappings we need one more notion.

A map $f\colon X \to X$ is called *locally compact* (LC-map) provided for every $x \in X$ there exists an open neighbourhood U_x of x in X such that $\overline{f(U_x)}$ is compact.

We have:

(3.3) Proposition. ([5], [6], [7]) *Any locally compact map with compact attractor is a* CAC*-mapping.*

All obtained above information we can illustrate in the following:

$$\boxed{\mathrm{CA}+\mathrm{LC}} \subset \boxed{\mathrm{CAC}} \subset \boxed{\mathrm{CA}}.$$

We recommend [15; Theorems 4.7, 4.8] for further information about considered classes of mappings.

Let us mention the first application of (2.12):

(3.4) Theorem. *Let $X \in$ ANR and $f\colon X \to X$ be a* CAC*-map. Then f is a Lefschetz map.*

Proof. Let $f\colon X \to X$ be a CAC-map, where $X \in$ ANR. We choose an open subset $U \subset X$ according to the Definition (3.1). Then $f(U) \subset U$

and $\overline{f(U)} \subset U$ is compact. Therefore, in view of (2.2), the map $\widetilde{f}\colon U \to U$, $\widetilde{f}(x) = f(x)$ is a Lefschetz map. Now our claim follows from (3.1), (3.1.2) and (2.12). □

(3.5) COROLLARY. *If* $X \in \mathrm{AR}$ *and* $f\colon X \to X$ *is a* CAC*-map, then* $\mathrm{Fix}(f) \neq \emptyset$.

(3.6) OPEN PROBLEM. *Is* (3.3) *true for every* CA*-mapping* f*?*

Now, we are going to discuss the Lefschetz fixed point theorem for condensing mappings.

We prove the following:

(3.7) PROPOSITION. *Let* (X, d) *be a complete bounded space and let* $f\colon X \to X$ *be a condensing map. Then* f *is an asymptotically compact map, in particular* f *has a compact attractor.*

P r o o f. According to [17; Proposition 2] we have:

$$\lim_{n\to\infty} \gamma\big(\overline{f^n(X)}\big) = 0 .$$

It implies, in view of (2.7), that the core

$$C_f = \bigcap_{n=1}^{\infty} \overline{f^n(X)}$$

is compact and nonempty.

Moreover, let $O(x) = \{x, f(x), f^2(x), \ldots\}$ be an orbit of $x \in X$ with respect to f. Then we have: $O(x) = \{x\} \cup f\big(O(x)\big)$ and consequently, if we assume that $\gamma\big(O(x)\big) > 0$, then we get:

$$\gamma\big(O(x)\big) = \gamma\big(f\big(O(x)\big)\big) < \gamma\big(O(x)\big),$$

a contradiction. So f is asymptotically compact and therefore it has a compact attractor. □

(3.8) COROLLARY. *Let* U *be an open subset of a Banach space* E *and let* $f\colon U \to U$ *be a condensing map. If there exists a closed bounded subset* B *of* E *such that* $f(U) \subset B \subset U$*, then* f *has a compact attractor.*

In fact, by applying (3.7) to $\widetilde{f}\colon B \to B$, $\widetilde{f}(x) = f(x)$ for every $x \in B$, we get (3.8).

Now, from (2.9) and (3.8) we get:

(3.9) COROLLARY. *Let* U *and* $f\colon U \to U$ *be the same as in* (3.8)*. Then* f *is a Lefschetz map.*

We need the following definition:

(3.10) DEFINITION. A complete, bounded metric space (X,d) is called a *special* ANR (written $X \in \mathrm{ANR}_s$) provided there exists an open U of a Banach space E and two continuous mappings $r\colon U \to X$ and $s\colon X \to U$ such that:

(3.10.1) $r \circ s = \mathrm{id}_X$,
(3.10.2) r and s are nonexpansive, i.e., $\gamma(r(B)) \le \gamma(B)$ and $\gamma(s(A)) \le \gamma(A)$ for every bounded sets A and B.

We are able to prove the following version of the Lefschetz fixed point theorem:

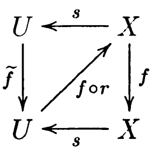
(3.11) THEOREM. *Let $X \in \mathrm{ANR}_s$ and let $f\colon X \to X$ be a condensing map. Then f is a Lefschetz map.*

Proof. From (3.7) we deduce that f has a compact attractor. Let U, $r\colon U \to X$ and $s\colon X \to U$ are according to Definition (3.10).

We define the map $\tilde{f}\colon U \to U$ by putting:

$$\tilde{f} = s \circ f \circ r .$$

In, view of (3.10.2), we deduce that $\tilde{f}$ is a condensing map. Observe that if A is a compact attractor of f, then $s(A)$ is a compact attractor $\tilde{f}$ (see: (3.10.1)). Consequently $\tilde{f}\colon U \to U$ is a condensing with compact attractor map. From the other hand we have the following commutative diagram:



Thus $\Lambda(f) = \Lambda(\tilde{f})$ and our theorem follows from (2.9). □

(3.12) LEMMA. *Let $f\colon X \to X$ be a map. Assume further that A is a compact attractor for f and V is an open neighbourhood A in X. Then there exists an open neighbourhood U of A in X such that*

(3.12.1) $f(U) \subset U$,
(3.12.2) $A \subset U \subset V$.

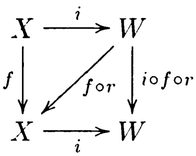
Proof. Let $U = \bigcap_{n=0}^{\infty} f^{-n}(V)$. Then $f(U) \subset U$ and $A \subset U$. We only need to show that U is an open subset of X. On the contrary, suppose that there exists a sequence $\{x_n\} \subset X \setminus U$ such that $\lim_{n\to\infty} x_n = x$ and $x \in U$. Let $K = \{x_n\} \cup \{x\}$. Then K is a compact set and consequently there exists m such that $f^i(K) \subset V$ for all $i \ge m$. Hence $x_n \in \bigcap_{i=m}^{\infty} f^{-i}(V)$. But $x_n \notin U$

so $x_n \notin \bigcap_{i=m}^{\infty} f^{-i}(V)$ and from the continuity of f follows that $x \notin \bigcap_{i=0}^{m} f^{-i}(V)$ which contradicts the fact that $x \in U$. □

We prove:

(3.13) THEOREM. *Assume that X is nonexpansive retract of some open subset W in a Banach space E. Assume further that $f: X \to X$ is CA-mapping with a compact attractor A. If there exists an open neighbourhood V of A in X such that the restriction $f|_V: V \to X$ of f to V is a condensing map, then f is a Lefschetz map.*

Proof. For the proof consider the following diagram:



in which $r: W \to X$ is the nonexpansive retraction and $i: X \to W$ is the inclusion map. Let us put $g = i \circ f \circ r$.

From the commutativity of the above diagram it follows that f is a Lefschetz map if and only if g is a Lefschetz map. Observe also that A is an attractor for g and moreover, $g|_{r^{-1}(V)}: r^{-1}(V) \to W$ is a condensing map. By applying Lemma (3.12) we get an open subset U of W such that $\widetilde{g}: U \to U$, $\widetilde{g}(u) = g(u)$ is a condensing map with compact attractor A. Consequently it follows from (2.9) that $\widetilde{g}$ is a Lefschetz map.

Now, in view of (2.12), we deduce that g is a Lefschetz map and the proof is completed. □

(3.14) Remark. Observe that any k-set contraction map is condensing, so Theorems (3.11) and (3.13) remain true for k-set contraction mappings.

4. The relative version

From the point of view of applications in dynamical systems the relative version of the Lefschetz fixed point theorem is important (see: [1], [3], [10], [16]). In the relative version we get not only the existence of fixed points but also some information of their localization. For the proof of the relative version instead of the Lefschetz number we need the fixed point index for the appropriate class of mappings.

We shall follow the ideas contained in [1]. First we would like to remark the following two facts:

(4.1) the fixed point index is well defined for CAC-mappings on ANR-s (see [1]),
(4.2) the fixed point index is well defined for condensing CA-mappings on open subset of Banach spaces (see [14] or [7]).

We have the following three versions of the relative Lefschetz fixed theorem:

(4.3) **THEOREM.** ([1]) *Let* $X_0 \subset X$ *and* $X, X_0 \in$ ANR. *Assume that* $f\colon (X, X_0) \to (X, X_0)$ *is a map such that* f_X *and* f_{X_0} *are* CAC*-mappings. Then the Lefschetz number* $\Lambda(f)$ *of* f *is well defined and* $\Lambda(f) \neq 0$ *implies that*

$$\mathrm{Fix}(f) \cap \left(\overline{X \setminus X_0}\right) \neq \emptyset .$$

(4.4) **THEOREM.** *Let* W *be an open subset of a Banach space* E *and* W_0 *be an open subset of* W *and let* $f\colon (W, W_0) \to (W, W_0)$ *be a mapping such that:*

(4.4.1) f_W *and* f_{W_0} *are condensing mappings with compact attractors.*

Then the Lefschetz number $\Lambda(f)$ *of* f *is well defined and* $\Lambda(f) \neq 0$ *implies that*

$$\mathrm{Fix}(f) \cap \left(\overline{W \setminus W_0}\right) \neq \emptyset .$$

Similarly, for k-set contraction mappings we get:

(4.5) **THEOREM.** *Let* W *and* W_0 *be the same as in* (4.4) *and* $f\colon (W, W_0) \to (W, W_0)$ *be a mapping such that:*

(4.5.1) f_W *and* f_{W_0} *are* k*-set contractions with relatively compact orbits.*

Then the Lefschetz number $\Lambda(f)$ *of* f *is well defined and* $\Lambda(f) \neq 0$ *implies that*

$$\mathrm{Fix}(f) \cap \left(\overline{W \setminus W_0}\right) \neq \emptyset .$$

Note that the proof of (4.4) and (4.5) is strictly analogous to the proof of (4.3) which is presented in full generality in [1].

Finally, let us add some concluding remarks. We would like to point out that the following topics concerning the Lefschetz fixed point theorem are still possible:

(i) non metric case, i.e., for retracts of open sets in admissible spaces in the sense of Klee (comp. [6] and also [18]),
(ii) periodic fixed point theory (comp. [1], [3]),
(iii) the multivalued case (comp. [8], [9], [7], [6]).

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