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COMPRESSIBLE GROUPS WITH GENERAL COMPARABILITY

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ABSTRACT. Compressible groups generalize the order-unit space of self-adjoint operators on Hilbert space, the directed additive group of self-adjoint elements of a unital C*-algebra, and interpolation groups with order units. In a compressible group with general comparability, each element g may be written canonically as a difference $g = g^+ - g^-$ of elements in the positive cone G^+, and the absolute value $|g|$ is defined by $|g| := g^+ + g^-$. In such a group G, we define and study a "pseudo-meet" $g \sqcap h$ and a "pseudo-join" $g \sqcup h$. If G is lattice ordered, $g \sqcap h$ and $g \sqcup h$ coincide with the usual meet and join; in the general case, they retain a number of properties of the latter. We also introduce and study a so-called Rickart projection property suggested by an analogous property in Rickart C*-algebras.

1. Compressible groups

In this article we continue the study of compressible groups with the general comparability property as initiated in [3], focusing on the consequences of the fact that in such a group each element g has a canonical decomposition $g = g^+ - g^-$ with $0 \le g^+, g^-$. Also, we shall prepare the ground for subsequent articles in which, among other things, it will be shown that a sort of "spectral theory", suggested by Example 1.2 below, is available for this class of partially ordered abelian groups. For the reader's convenience, we begin with a brief review of pertinent definitions and nomenclature.

Let G be an additively-written partially ordered abelian group with positive cone $G^+ = \{g \in G :\ 0 \le g\}$. If G^+ generates G, i.e., if $G = G^+ - G^+$, then G is said to be *directed*. We say that G is *unperforated* if and only if it satisfies the condition that if for all $g \in G$ and every positive integer n, $0 \le ng \implies 0 \le g$.

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There are various definitions of "archimedean groups" in the literature. We use the definition in [6; p. 20], so that G is *archimedean* if and only if, whenever $g, h \in G$ and $ng \leq h$ for all positive integers n, then $g \leq 0$.

A *unital group* is a directed abelian group G with a distinguished element $u \in G^+$, called the *unit*, such that the set $E := \{e \in G : \ 0 \leq e \leq u\}$, called the *unit interval*, generates G^+ in the sense that every element in G^+ is a finite linear combination with nonnegative integer coefficients of elements of E. The unit interval E in the unital group G forms a so-called *effect algebra* under the restriction of $+$ to E ([1]). Thus, elements of the unit interval in a unital group are referred to as *effects*.

As usual, we denote the ordered field of real numbers, the ordered subfield of rational numbers, and the ordered ring of integers by $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$, respectively. Regarded as additive abelian groups, and with 1 as the unit, each of $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ is an archimedean unital group with the standard positive cones $\mathbb{R}^+ = \{x^2 : x \in \mathbb{R}\}$, $\mathbb{Q}^+ = \mathbb{Q} \cap \mathbb{R}^+$, and $\mathbb{Z}^+ = \mathbb{Z} \cap \mathbb{Q}^+$.

Let G be a unital group with unit u and unit interval E. A mapping $J \colon G \to G$ is called a *retraction* on G if and only if it is an order-preserving group endomorphism such that $J(u) \leq u$ and, for all $e \in E$, $e \leq J(u) \implies J(e) = e$. If J is a retraction on G, then J is idempotent, i.e., $J \circ J = J$. A retraction J on G is called a *compression* if and only if its kernel $\ker(J) = J^{-1}(0)$ satisfies the condition $\ker(J) \cap E = \{e \in E : \ e + J(u) \in E\}$ ([4]). If J is a retraction on G, then $J(u)$ is called the *focus* of J. Two retractions I and J on G are said to be *quasicomplements* of each other if and only if, for all $g \in G^+$, $I(g) = g \iff J(g) = 0$ and $J(g) = g \iff I(g) = 0$. If I and J are quasicomplements, they are necessarily compressions.

A *compressible group* is a unital group G such that every retraction on G has a quasicomplementary retraction, and every retraction on G is uniquely determined by its focus ([3]). If G is a compressible group with unit u, then an element $p \in G$ is called a *projection* if and only if it is the focus $p = J(u)$ of a retraction (hence a compression) J on G.

Let G be a compressible group with unit u and let P be the set of projections in G. In what follows, we shall denote by J_p the *unique compression on G with the projection $p \in P$ as its focus.* If $p \in P$, then the unique compression on G that is quasicomplementary to J_p is J_{u-p}, whence $p \in P \implies u - p \in P$. Also, $0, u \in P$ and, under the restriction of the partial order on G, P forms an orthomodular poset ([10]) with $p \mapsto u - p$ as the orthocomplementation. As such, P is a sub-effect algebra of the unit interval E in G, hence, if $p, q \in P$, then $p + q \in P \iff p + q \in E$ ([3; Theorem 5.1]). Therefore, by induction on n, if $p_1, p_2, \ldots, p_n \in P$ and $p := \sum_{i=1}^{n} p_i$, then $p \leq u \iff p \in P$.

If $p, q \in P$ and the infimum r (respectively, the supremum s) of p and q

as calculated in P exists, we write $r = p \wedge q$ (respectively, $s = p \vee q$). Existing infima and suprema as calculated in other subsets of G, e.g., E, G^+, or G itself, will be denoted by using appropriate subscripts. For instance, if $a, b \in G$ and c is the infimum of a and b as calculated in G, we write $a \wedge_G b = c$. If $M \subseteq G$, $a, b, c \in M$, and we write $a \wedge_M b = c$, we mean that the infimum $a \wedge_M b$ of a and b, calculated in M, exists and equals c. A similar convention applies to $a \vee_M b$.

The unital groups $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ are compressible groups, and in all three cases the set of projections is $P = \{0, 1\}$, which may be regarded as the two-element Boolean algebra. The following additional examples will provide much of the motivation for the developments in this article.

EXAMPLE 1.1. Let A be a C*-algebra with unit 1 and let G be the additive group of self-adjoint elements in A. Then G forms an archimedean unital group with unit 1 and positive cone $G^+ := \{aa^* : a \in A\}$. The unital group G is a compressible group, the orthomodular poset P consists of all idempotent elements of G, and $p \in P$, $g \in G \implies J_p(g) = pgp$ ([4]).

EXAMPLE 1.2. In Example 1.1, suppose that A is a von Neumann algebra. Then A is a *Rickart* C*-algebra, i.e., there is a uniquely determined mapping $' \colon A \to P$ such that, for all $a, b \in A$, $ab = 0 \iff b = a'b$. Evidently, $a' = (a^*a)'$, so the mapping $a \mapsto a'$ is determined by its restriction $g \mapsto g'$ to elements $g \in G$. In this case, the orthomodular poset P is a complete orthomodular lattice, and if $p, q \in P$, then $p' = 1 - p$ and $p \wedge q = (qp'q)'q = q(qp'q)'$. If $e \in E$, then $e'' := (e')' = 1 - e' \in P$ is the *projection cover* of the effect e in the sense that e'' is the smallest projection that dominates e ([3; Definition 6.1]).

Let $g \in G$. The *absolute value*, the *positive part* and *negative part* of g are defined by $|g| := \sqrt{g^2}$, $g^+ := (|g| + g)/2$, and $g^- := (|g| - g)/2 = (-g)^+$, respectively. Then $g = g^+ - g^-$ with $0 \le g^+, g^-$, and $g'' = (g^+)'' + (g^-)'' = |g|''$. Define $P^{\pm}(g)$ to be the set of all projections $p \in P$ such that p commutes with every projection in P that commutes with g (hence, p and p' commute with g) and $p'g \le 0 \le pg$. The set $P^{\pm}(g)$ has a smallest element $(g^+)''$ and a largest element $(g^-)' = (g^+)'' + g'$. If $p \in P^{\pm}(g)$, then $g^+ = gp = pg$ and $g^- = p'(-g) = -gp'$.

EXAMPLE 1.3. An *interpolation group* is a partially ordered abelian group such that, for all $a, b, c, d \in G$ with $a, b \le c, d$, there exists $t \in G$ such that $a, b \le t \le c, d$ ([6]). Let G be an interpolation group with an order unit u. Then G is a compressible group and the orthomodular poset P of projections consists of all the effects $p \in E = \{e \in G : 0 \le e \le u\}$ such that $p \wedge_G (u - p) = 0$, i.e., the so-called characteristic elements of G ([6; p. 127]). In this case, P forms a Boolean algebra ([3; Theorem 3.5]). If $p \in P$, let $G_p = \{h \in G : (\exists n \in \mathbb{Z}^+)(-np \le h \le np)\}$. Then G_p is a subgroup of G and, under the restriction of the partial order on G, G_p forms an interpolation group with p as an order unit; in fact, G_p is a compressible group in its own right. If $p \in P$, then G is the internal direct sum of G_p and G_{u-p} as partially ordered abelian groups, and $J_p\colon G \to J_p(G) = G_p$ is the corresponding projection mapping ([6; Lemma 8.2]). If $p \in P$ and $e \in E$, then $J_p(e) = p \wedge_E e = p \wedge_G e$ is the infimum of p and e as calculated either in E or in G. If $g \in G$, we can write $g = \sum_{i=1}^{n} k_i e_i$ with $e_i \in E$, $k_i \in \mathbb{Z}$, and we have $J_p(g) = \sum_{i=1}^{n} k_i(p \wedge_E e_i)$. Furthermore, $g = J_p(g) + J_{u-p}(g)$.

EXAMPLE 1.4. A lattice-ordered abelian group is automatically an interpolation group. Let G be a lattice-ordered abelian group with order unit u and unit interval $E = \{e \in G : \ 0 \le e \le u\}$. Then, as in Example 1.3, G is a compressible group and $P = \{p \in E : \ p \wedge_G (u-p) = 0\}$ is a Boolean algebra. In this case, the set of effects $E \subseteq G$ forms a so-called *MV-algebra* ([2]). Conversely, by a theorem of D. Mundici, every MV-algebra can be realized as the set of effects in a lattice-ordered abelian group G with order unit, and G is uniquely determined up to an isomorphism of unital groups ([9]).

In the sequel, *we assume once and for all that $G \ne \{0\}$ is a compressible group, u is the unit in G, E is the unit interval* (*i.e., the set of effects*) *in G, and P is the orthomodular poset of projections in G.*

If H is a subgroup of G, we understand that H is organized into a partially ordered abelian group under the restriction to H of the partial order on G, whence $H^+ = H \cap G^+$. For instance, if $p \in P$, then the image $H := J_p(G)$ of G under J_p forms a compressible group with unit p. The orthomodular poset $P(H)$ of projections in $H = J_p(G)$ is the interval $P(H) = \{q \in P : \ q \le p\}$ in P, and if $q \in P(H)$, then the corresponding compression on H is the restriction $J_q|_H$ to H of the compression J_q on G ([3; Theorem 5.9]). The passage from G to $H = J_p(G)$ is the analogue for the compressible group G of the passage from A to pAp in Example 1.1.

2. Compatibility

The notion of *compatibility* in part (i) of the following definition was originally introduced in [3; Definition 4.1].

DEFINITION 2.1. Let $g, h \in G$ and $p, q \in P$.

(i) $C(p) := \{g \in G : \ g = J_p(g) + J_{u-p}(g)\}$. Elements $g \in C(p)$ are said to be *compatible* with the projection p.

(ii) For projections p and q, we often write the condition $p \in C(q)$ in the alternative form pCq.

(iii) $CPC(g) := \bigcap_{p\in P,\, g\in C(p)} C(p)$.

(iv) By definition, $g \leftrightarrow_P h$ means that $g \in CPC(h)$ and $h \in CPC(g)$.

(v) $C(P) := \bigcap_{p\in P} C(p)$.

(vi) G is a *compatible group* if and only if $G = C(P)$.

Let $g, h, k \in G$. The condition $h \in CPC(g)$ means that h is compatible with every projection p with which g is compatible, and $h \leftrightarrow_P g$ means that h and g are compatible with the same projections in P. If $h \in CPC(g)$ and $g \in CPC(k)$, then $h \in CPC(k)$. Evidently, $\leftrightarrow_P$ is an equivalence relation on G. The condition $g \in C(P)$ holds if and only if g is compatible with every projection $p \in P$. For instance, $u \in C(P) = CPC(u)$. If $p \in P$ and $g \in G$, then $C(p)$, $CPC(g)$, and $C(P)$ are subgroups of G, $C(p) = C(u-p)$, $u \in C(P) \subseteq C(p) \cap CPC(g)$, and $g \in CPC(g)$.

In Example 1.1, $g \in C(p)$ if and only if $gp = pg$, so $C(P)$ is the set of all self-adjoint elements in A that commute with every projection in A. In Example 1.2, A is a von Neumann factor if and only if $C(P) = \{\lambda 1 : \lambda \in \mathbb{R}\}$, and (by the spectral theorem) $g \in CPC(h)$ if and only if g commutes with every self-adjoint element that commutes with h. Thus, G is a compatible group if and only if A is a commutative von Neumann algebra. In Example 1.3, the interpolation group G is a compatible group, so $g \leftrightarrow_P h$ for all $g, h \in G$.

Let $p, q \in P$. By [3; Theorem 5.4], pCq if and only if p and q are (Mackey) compatible elements of the orthomodular poset P, i.e., if and only if there are projections $p_1, q_1, d, r \in P$ such that $p_1 + q_1 + d + r = u$, $p = p_1 + d$, and $q = q_1 + d$. In this case, $d = p \wedge q = p \wedge_E q$ is the infimum of p and q as calculated either in P or in E, and $p_1 + q_1 + d = p \vee q = p \vee_E q$ is the supremum of p and q as calculated either in P or in E ([3; Corollary 5.6]). Also, $pCq \iff qCp \iff J_p \circ J_q = J_q \circ J_p$. In fact, $pCq \implies J_p \circ J_q = J_q \circ J_p = J_{p\wedge q}$.

By [3; Corollary 5.8], P is a Boolean algebra if and only if $P \subseteq C(P)$. Furthermore, by [3; Example 3.7], every Boolean algebra can be realized as the system P of projections in a compatible compressible group G for which $E = P \subseteq C(P)$. Conversely, by [3; Theorem 6.5], if $E = P$, then G is a compatible group and P is a Boolean algebra.

If $p \in P$, then, with the induced partial order, $D := C(p)$ is a compressible group with unit u. The set $P(D)$ of projections in D is given by $P(D) = \{q \in D : qCp\}$, and if $q \in P(D)$, then the corresponding compression on D is the restriction $J_q|_D$ to D of J_q ([3; Theorem 5.10]).

LEMMA 2.2. *Let $g \in G$, $w \in C(P)$ and suppose that G is torsion free. Then, if n is any nonzero integer, $g \leftrightarrow_P (ng + w)$.*

P r o o f. Assume the hypotheses. As $w \in C(p)$, we have $ng \in C(p) \iff ng + w \in C(p)$. If $ng \in C(p)$, then $ng = J_p(ng) + J_{u-p}(ng) = n\big(J_p(g) + J_{u-p}(g)\big)$

and, since G is torsion free, it follows that $g = J_p(g) + J_{u-p}(g)$, i.e., $g \in C(p)$. Conversely, $g \in C(p) \implies ng \in C(p)$. □

THEOREM 2.3. *Let $p, q, r, s \in P$ and let $p_1, p_2, \ldots, p_n \in P$ with $\sum_{i=1}^{n} p_i \le u$. Then:*

(i) *If $p + q + r \le u$, then $J_{p+q} \circ J_{q+r} = J_{q+r} \circ J_{p+q} = J_q$.*

(ii) *If $p = \sum_{i=1}^{n} p_i$ and $g \in \bigcap_{i=1}^{n} C(p_i)$, then $g \in C(p)$ and $J_p(g) = \sum_{i=1}^{n} J_{p_i}(g)$.*

(iii) *If $\sum_{i=1}^{n} p_i = u$ and $g \in G$ with $g = \sum_{i=1}^{n} J_{p_i}(g)$, then $g \in \bigcap_{i=1}^{n} C(p_i)$.*

(iv) *If $p+q+r+s = u$, then $C(p+q) \cap C(q+r) \subseteq C(p) \cap C(q) \cap C(r) \cap C(s)$.*

Proof.

(i) As $p + q + r \le u$, we have $(p + q)C(q + r)$ with $q = (p + q) \wedge (q + r)$, whence $J_{p+q} \circ J_{q+r} = J_{q+r} \circ J_{p+q} = J_{(p+q)\wedge(q+r)} = J_q$.

(ii) The proof of (ii) is by induction on n. Assume the hypotheses. If $n = 1$, there is nothing to prove. Let $n > 1$ and let $q := \sum_{i=1}^{n-1} p_i$, so that $p = q + p_n$. By the induction hypothesis, we may assume that $g \in C(q)$ and that $J_q(g) = \sum_{i=1}^{n-1} J_{p_i}(g)$. Let $r := u - p$, so that $u = p + r = q + p_n + r$. As $g \in C(q)$ and $u - q = p_n + r$, it follows from (i) that

$$J_p(g) = J_{q+p_n}(g) = J_{q+p_n}\big(J_q(g) + J_{p_n+r}(g)\big) = J_q(g) + J_{p_n}(g) = \sum_{i=1}^{n} J_{p_i}(g).$$

Likewise, as $g \in C(p_n)$ and $u - p_n = q + r$, it follows from (i) that

$$\begin{aligned} g &= J_q(g) + J_{p_n+r}(g) = J_q(g) + J_{p_n+r}\big(J_{p_n}(g) + J_{q+r}(g)\big) \\ &= J_q(g) + J_{p_n}(g) + J_r(g) = J_p(g) + J_r(g) = J_p(g) + J_{u-p}(g), \end{aligned}$$

whence $g \in C(p)$.

(iii) Assume the hypotheses. By symmetry, it will be sufficient to prove that $g \in C(p_1)$. As $J_{u-p_1} \circ J_{p_1} = J_0$ and $J_{u-p_1} \circ J_{p_i} = J_{p_i}$ for $i \ne 1$, we have $J_{u-p_1}(g) = \sum_{i=2}^{n} J_{p_i}(g)$, whence $J_{p_1}(g) + J_{u-p_1}(g) = g$, i.e., $g \in C(p_1)$.

(iv) Suppose $g \in C(p+q) \cap C(q+r)$. Then

$$\begin{aligned} g &= J_{p+q}(g) + J_{r+s}(g) = J_{p+q}\big(J_{q+r}(g) + J_{p+s}(g)\big) + J_{r+s}\big(J_{q+r}(g) + J_{p+s}(g)\big) \\ &= J_q(g) + J_p(g) + J_r(g) + J_s(g), \end{aligned}$$

whence $g \in C(p) \cap C(q) \cap C(r) \cap C(s)$ by (iii). □

COROLLARY 2.4. *Let $p, q \in P$ with pCq. Then*

$$C(p) \cap C(q) \subseteq C(p \wedge q) \cap C(p \vee q).$$

Proof. Since pCq, there are projections $p_1, q_1, d, r \in P$ with $p_1 + q_1 + d + r = u$, $p = p_1 + d$, and $q = q_1 + d$. By Theorem 2.3(iv), $C(p) \cap C(q) \subseteq C(d) \cap C(r) = C(d) \cap C(u-r) = C(p \wedge q) \cap C(p \vee q)$. □

By the following theorem, the orthomodular poset P has the property sometimes referred to in the literature as "regularity" ([8]).

THEOREM 2.5. *Let $p_1, p_2, \ldots, p_n$ be pairwise compatible elements of P. Then the infimum $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ and the supremum $p_1 \vee p_2 \vee \cdots \vee p_n$ exist in P and $p_1, p_2, \ldots, p_n$ are jointly compatible in P, i.e., there is a Boolean subalgebra B of P with $p_1, p_2, \ldots, p_n \in B$. Furthermore, if B is the Boolean subalgebra of P generated by $p_1, p_2, \ldots, p_n$, then $\bigcap_{i=1}^{n} C(p_i) = \bigcap_{b \in B} C(b)$.*

Proof. By Corollary 2.4, if p, q, r are elements of the orthomodular poset P, then $pCq, qCr, rCp \implies (p \wedge q)Cr$, and the conclusions follow from the basic theory of orthomodular posets. □

THEOREM 2.6. *Let $p, q \in P$ with pCq and suppose that $g \in C(p) \cap C(q)$ with $J_{u-p}(g), J_{u-q}(g) \leq 0 \leq J_p(g), J_q(g)$. Then:*

(i) $J_{p\wedge(u-q)}(g) = J_{(u-p)\wedge q}(g) = 0$.

(ii) $J_p(g) = J_q(g) = J_{p\wedge q}(g) = J_{p\vee q}(g)$.

(iii) $J_{u-p}(g) = J_{u-q}(g) = J_{u-(p\wedge q)}(g) = J_{u-(p\vee q)}(g)$.

Proof. As pCq, we have $pC(u-q)$, $(u-p)Cq$, and $(u-p)C(u-q)$. Also, as $g \in C(p) \cap C(q)$, we have $g \in C\big(p \wedge (u-q)\big)$, $g \in C\big((u-p) \wedge q\big)$, $g \in C\big((u-p) \wedge (u-q)\big)$, and $g \in C(p \vee q)$ by Corollary 2.4.

(i) Since $J_{p\wedge(u-q)}(g) = J_p\big(J_{u-q}(g)\big) \leq 0 \leq J_{u-q}\big(J_p(g)\big) = J_{p\wedge(u-q)}(g)$, it follows that $J_{p\wedge(u-q)}(g) = 0$. By symmetry, $J_{(u-p)\wedge q}(g) = 0$.

(ii) We have $u = (p \wedge q) + \big(p \wedge (u-q)\big) + \big((u-p) \wedge q\big) + (u-p) \wedge (u-q)$, whence by (i) and Theorem 2.3(ii),

$$g = J_u(g) = J_{p\wedge q}(g) + J_{(u-p)\wedge(u-q)}(g). \tag{1}$$

As $g \in C(p \vee q)$, it follows that

$$g = J_{p\vee q}(g) + J_{u-(p\vee q)}(g) = J_{p\vee q}(g) + J_{(u-p)\wedge(u-q)}(g). \tag{2}$$

From (1) and (2), it follows that

$$J_{p\wedge q}(g) = J_{p\vee q}(g). \tag{3}$$

As $p \vee q = p + \big((u-p) \wedge q\big)$, we also have

$$J_{p\vee q}(g) = J_p(g) + J_{(u-p)\wedge q}(g) = J_p(g) \tag{4}$$

by (i) and Theorem 2.3(ii). By symmetry,

$$J_q(g) = J_{p\vee q}(g)\,, \tag{5}$$

and (ii) follows from (3), (4), and (5).

(iii) Follows from (ii) upon replacing g by $-g$, p by $u-p$, and q by $u-q$. □

3. General comparability

DEFINITION 3.1. If $g \in G$, then

$$P^{\pm}(g) := \{p \in P \cap CPC(g) :\ g \in C(p) \text{ and } J_{u-p}(g) \le 0 \le J_p(g)\}\,.$$

If $p \in P^{\pm}(g)$, then p splits $g = J_p(g) + J_{u-p}(g)$ into a "positive part" $J_p(g)$ and a "negative part" $J_{u-p}(g)$.

THEOREM 3.2. *Let $g \in G$, $r \in P$, and suppose that $p, q \in P^{\pm}(g)$. Then:*

(i) pCq.

(ii) $r \in P^{\pm}(g) \iff u - r \in P^{\pm}(-g)$.

(iii) $0 \le J_p(g) = J_q(g) = J_{p\wedge q}(g) = J_{p\vee q}(g)$.

(iv) $J_{u-p}(g) = J_{u-q}(g) = J_{u-(p\wedge q)}(g) = J_{u-(p\vee q)}(g) \le 0$.

(v) $p\wedge q, p \vee q \in P^{\pm}(g)$.

(vi) *A minimal (respectively, maximal) element of $P^{\pm}(g)$, if it exists, is necessarily the smallest (respectively, the largest) element of $P^{\pm}(g)$.*

Proof.

(i) Since $p \in CPC(g)$ and $g \in C(q)$, it follows that pCq.

Part (ii) follows easily from Definition 3.1, and parts (iii) and (iv) follow directly from Theorem 2.6(ii) and (iii).

(v) By (i) and Corollary 2.4, $g \in C(p \wedge q)$. Suppose $r \in P$ and $g \in C(r)$. Since $p, q \in CPC(g)$, it follows that pCr and qCr, and again by Corollary 2.4, $rC(p\wedge q)$. By (iii) and (iv), $J_{u-(p\wedge q)}(g) \le 0 \le J_{p\wedge q}(g)$, whence $p \wedge q \in P^{\pm}(g)$. A similar argument shows that $p \vee q \in P^{\pm}(g)$.

(vi) Suppose q is a minimal element of $P^{\pm}(g)$. By (v), $p\wedge q \in P^{\pm}(g)$ and, since $p \wedge q \le q$, we have $q = p\wedge q$, i.e., $q \le p$. Since p is an arbitrary element of $P^{\pm}(g)$, it follows that q is the smallest element of $P^{\pm}(g)$. Similarly, a maximal element of $P^{\pm}(g)$ is necessarily the largest element of $P^{\pm}(g)$. □

LEMMA 3.3. *Suppose G is unperforated, n and m are positive integers, $g, h \in G$, $ng \leq mh$, and $g \leftrightarrow_P h$. Then, if $p \in P^{\pm}(g)$ and $q \in P^{\pm}(h)$, it follows that pCq, $p \wedge q \in P^{\pm}(g)$, and $p \vee q \in P^{\pm}(h)$.*

Proof. We have $h \in C(q)$ and $g \in CPC(h)$, so $g \in C(q)$, whence the fact that $p \in CPC(g)$ implies pCq. As $g \in C(p) \cap C(q)$, Corollary 2.4 implies that $g \in C(p \wedge q)$. Suppose $r \in P$ and $g \in C(r)$. As $p \in CPC(g)$, we have pCr. As $h \in CPC(g)$, we also have $h \in C(r)$, whence the fact that $q \in CPC(h)$ implies qCr. Therefore, $(p \wedge q)Cr$, and it follows that $p \wedge q \in CPC(g)$.

As $0 \leq J_p(g)$, we have $0 \leq J_q(J_p(g)) = J_{p\wedge q}(g)$. Also, as $J_{u-q}(h) \leq 0$, we have $nJ_{u-(p\wedge q)}(g) = J_{u-p}(J_{u-q}(ng)) \leq J_{u-p}(J_{u-q}(mh)) = mJ_{u-p}(J_{u-q}(h)) \leq 0$, whence, since G is unperforated, $J_{u-(p\wedge q)}(g) \leq 0$. Therefore, $p\wedge q \in P^{\pm}(g)$. That $p \vee q \in P^{\pm}(h)$ follows from a similar argument. □

The notions in the following definition were originally introduced in [3; Definition 4.6].

DEFINITION 3.4. The compressible group G has the *general comparability property* (or simply, has *general comparability*) if and only if $g \in G \implies P^{\pm}(g) \neq \emptyset$. It has the *central comparability property* (or simply, has *central comparability*) if and only if, for every $g \in G$, there exists $p \in P^{\pm}(g)$ with $G = C(p)$.

In Example 1.2, the compressible group G of self-adjoint elements in the unital von Neumann algebra A has general comparability. In Example 1.3, the interpolation group G has central comparability if and only if it has general comparability, and general comparability coincides with the property of the same name studied in [6; Chapter 8].

If G has general comparability, it is unperforated and, as an abelian group, it is torsion free ([3; Lemma 4.8]). If G has central comparability, then it is lattice ordered ([3; Theorem 4.9]). On the other hand, if G is a Dedekind σ-complete lattice-ordered abelian group with order unit, then G is a compressible group with central comparability ([6; Theorem 9.9]).

LEMMA 3.5. *If G has general comparability, then G is archimedean if and only if, for all $a, b \in G^+$, $na \leq b$ for all positive integers n only if $a = 0$.*

Proof. If G is archimedean, the given condition obviously holds. Suppose the given condition holds, let $g, h \in G$, and suppose $ng \leq h$ for all positive integers n. Choose $p \in P^{\pm}(g)$. Then $nJ_p(g) \leq J_p(h)$ holds for all positive integers n and, since $J_p(g) \in G^+$, it follows that $J_p(g) = 0$. But then $g = J_p(g) + J_{u-p}(g) = J_{u-p}(g) \leq 0$, so G is archimedean. □

EXAMPLE 3.6. Let X be a compact Hausdorff space that is basically disconnected, i.e., the closure of every open F_σ subset of X is open. Let $C(X,\mathbb{R})$ be the lattice-ordered vector space of all continuous functions $f\colon X \to \mathbb{R}$. Then, with the constant function $u(x) \equiv 1$ as unit, and regarded as a partially ordered additive abelian group, $G := C(X,\mathbb{R})$ is an archimedean compressible group with central comparability. Also, G is a compatible group and P is the σ-complete Boolean algebra of all characteristic set functions of compact open subsets of X.

THEOREM 3.7. *Suppose that G has general comparability, let $g \in G^+$, $w \in C(P)$, and choose any $q_1 \in P^{\pm}(g+w)$. Then there exist $q_2, q_3, \dots \in P$ such that, for all $n = 1, 2, \dots$,*

(i) $q_n \le q_{n+1}$,
(ii) $q_n \in P^{\pm}(ng+w)$,
(iii) $g \in C(q_n)$,
(iv) $q_n \in CPC(g)$.

Proof. As G has general comparability, it is unperforated and torsion free, hence Lemma 2.2 implies that, if $p \in P$ and n is a nonzero integer, then $ng+w \in C(p) \iff g \in C(p)$, whence $(ng+w) \leftrightarrow_P g$. As $\leftrightarrow_P$ is an equivalence relation on G, it follows that $(ng+w) \leftrightarrow_P (mg+w)$ for all nonzero integers n and m.

We construct the sequence $(q_n)_{1\le n}$ inductively, starting with $q_1 \in P^{\pm}(g+w)$. Suppose $q_1 \le q_2 \le \cdots \le q_m$ have already been obtained such that (ii)–(iv) hold for $n = 1, 2, \dots, m$. As $g \in G^+$, we have $mg+w \le (m+1)g+w$. Choose $q \in P^{\pm}\big((m+1)g+w\big)$. As $(mg+w) \leftrightarrow_P \big((m+1)g+w\big)$, Lemma 3.3 implies that $p_m C q$ and $p_m \vee q \in P^{\pm}\big((m+1)g+w\big)$. Define $q_{m+1} := q_m \vee q$, so that $q_m \le q_{m+1} \in P^{\pm}\big((m+1)g+w\big)$. Then $(m+1)g+w \in C(q_{m+1})$, so $g \in C(q_{m+1})$. Also, $q_{m+1} \in CPC\big((m+1)g+w\big) = CPC(g)$. □

As the mapping $p \mapsto u-p$ is an order-reversing bijection on P, it follows that P satisfies the ascending chain condition (i.e., P contains no infinite strictly increasing sequence) if and only if it satisfies the descending chain condition (i.e., P contains no infinite strictly decreasing sequence). If the unital C*-algebra A in Example 1.1 is finite dimensional, then it is a von Neumann algebra as in Example 1.2, the orthomodular lattice P satisfies the chain conditions, and P is a modular lattice. A Boolean algebra, e.g., the system P of projections in a compatible group, satisfies the chain conditions if and only if it is finite.

COROLLARY 3.8. *Suppose that G is archimedean, G has general comparability, and P satisfies the ascending chain condition. If $g \in G^+$, there is a smallest element $q \in P^{\pm}(g)$, there is a positive integer N such that $q \le Ng$, and for every projection $p \in P$, $J_p(g) = g \iff q \le p$.*

Proof. In Theorem 3.7, let $w := -u$ and let $(q_n)_{1 \leq n}$ be the resulting sequence of projections. Since P satisfies the ascending chain condition, there is a positive integer N such that $n \geq N \implies q_n = q_N$. Let $q := q_N$. Then, $q \in CPC(g)$ and $g \in C(q)$ by Theorem 3.7(iii) and (iv). Also, $n \geq N \implies J_{u-q}(ng - u) \leq 0 \leq J_q(ng - u)$. Consequently, $n \geq N \implies nJ_{u-q}(g) \leq u - q$, and since G is archimedean, it follows that $J_{u-q}(g) \leq 0$. But $0 \leq g$ implies that $0 \leq J_{u-q}(g)$, whence $J_{u-q}(g) = 0$. Therefore, $0 \leq g = J_q(g)$, so $q \in P^{\pm}(g)$. Also, $0 \leq J_q(Ng - u) = Ng - q$, i.e., $q \leq Ng$. Let $p \in P$. If $J_p(g) = g$, then $J_{u-p}(g) = 0$, whence $0 \leq J_{u-p}(q) \leq NJ_{u-p}(g) = 0$, so $J_{u-p}(q) = 0$, whereupon $q \leq p$. Conversely, if $q \leq p$, then, since $J_q(g) = g$, we have $J_p(g) = J_p(J_q(g)) = J_q(g) = g$. Finally, if $p \in P^{\pm}(g)$, then $g = g^+ = J_p(g)$, so $q \leq p$, whence q is the smallest element in $P^{\pm}(g)$. □

4. Positive and negative parts

Example 1.2 provides motivation for the following definition.

DEFINITION 4.1. Suppose G has general comparability, let $g \in G$, and choose $p \in P^{\pm}(g)$. By parts (iii) and (iv) of Theorem 3.2, $J_p(g)$ and $J_{u-p}(g)$ are independent of the choice of $p \in P^{\pm}(g)$. Therefore, we can and do define

$$g^+ := J_p(g), \qquad g^- := -J_{u-p}(g) = J_{u-p}(-g), \qquad |g| := g^+ + g^- .$$

LEMMA 4.2. *Suppose G has general comparability, let $p \in P$ and $g \in C(p)$ with $J_{u-p}(g) \leq 0 \leq J_p(g)$. Then $g^+ = J_p(g)$ and $g^- = J_{u-p}(-g)$.*

Proof. Assume the hypotheses and select $q \in P^{\pm}(g)$. As $q \in CPC(g)$ and $g \in C(p)$, it follows that pCq, hence $g^+ = J_q(g) = J_p(g)$ and $g^- = J_{u-q}(g) = J_{u-p}(g)$ by parts (ii) and (iii) of Theorem 2.6. □

LEMMA 4.3. *Suppose G has general comparability and let $g \in G$, $p \in P$. Then:*

(i) $0 \leq g^+, g^-, |g|$.
(ii) $g = g^+ - g^-$.
(iii) $g^- = (-g)^+$.
(iv) $\pm g \leq |g| = |-g|$.
(v) $|g| + g = 2g^+$ *and* $|g| - g = 2g^-$.
(vi) $0 \leq g \iff u \in P^{\pm}(g) \iff g = g^+ \iff g = |g|$.
(vii) $g^+, g^-, |g| \in CPC(g)$.
(viii) $|g| \in \ker(J_p) \iff g \in C(p) \cap \ker(J_p)$.
(ix) $g^+ \wedge_{G^+} g^- = 0$.
(x) $n \in \mathbb{Z}^+ \implies (ng)^+ = ng^+$ *and* $(ng)^- = ng^-$.
(xi) $n \in \mathbb{Z} \implies |ng| = |n||g|$.

Proof. (i), (ii), (iii), (iv), (v), and (vi) are obvious.

(vii) Suppose $g \in C(p)$ and choose $q \in P^{\pm}(g)$. Then qCp and we have $g^+ = J_q(g) = J_q(J_p(g) + J_{u-p}(g)) = J_p(J_q(g)) + J_{u-p}(J_q(g)) = J_p(g^+) + J_{u-p}(g^+)$, so $g^+ \in C(p)$. Likewise, $g^- \in C(p)$, so $|g| = g^+ + g^- \in C(p)$.

(viii) Suppose $g \in C(p)$ with $J_p(g) = 0$ and choose $q \in P^{\pm}(g)$. Then qCp so $J_p(g^+) = J_p(J_q(g)) = J_q(J_p(g)) = 0$. Likewise, $J_p(g^-) = -J_p(J_{u-q}(g)) = -J_{u-q}(J_p(g)) = 0$, and it follows that $J_p(|g|) = J_p(g^+ + g^-) = 0$. Conversely, suppose $J_p(|g|) = 0$. Then $J_p(g^+) + J_p(g^-) = 0$ and, since $0 \le J_p(g^+), J_p(g^-)$, we have $J_p(g^+) = J_p(g^-) = 0$, whence $g^+, g^- \in C(p)$, and it follows that $g = g^+ - g^- \in C(p)$. Also, $J_p(g) = J_p(g^+) - J_p(g^-) = 0$.

(ix) By (i), 0 is a lower bound in G^+ for g^+ and g^-. Suppose $h \in G^+$ with $h \le g^+, g^-$ and choose $p \in P^{\pm}(g)$. Then $0 \le J_p(h) \le J_p(g^-) = J_p(J_{u-p}(g)) = 0$, and it follows that $J_{u-p}(h) = h$. Thus, $0 \le h = J_{u-p}(h) \le J_{u-p}(g^+) = J_{u-p}(J_p(g)) = 0$, so $h = 0$.

(x) and (xi) Select $p \in P^{\pm}(g)$ and suppose $n \in \mathbb{Z}^+$. Then $J_{u-p}(ng) = nJ_{u-p}(g) \le 0 \le nJ_p(g) = J_p(ng)$, so $(ng)^+ = ng^+$ and $(ng)^- = ng^-$ by Lemma 4.2. Part (xi) follows from (x). □

LEMMA 4.4. *Suppose G has general comparability and let $g, h \in G$ with $h \in CPC(g)$. Then:*

(i) $g \le h \implies g^+ \le h^+$.
(ii) $0, g \le h \iff g^+ \le h$.
(iii) $\pm g \le h \iff |g| \le h$.

Proof.

(i) Assume the hypotheses and select $p \in P^{\pm}(g)$. Then $g \in C(p)$, hence $h \in CPC(g)$ implies that $h \in C(p)$, so $h^+ \in C(p)$ by Lemma 4.3(vii). Therefore, since $0 \le h^+$, we have $J_p(h^+) \le J_p(h^+) + J_{u-p}(h^+) = h^+$. Consequently, $g^+ = J_p(g) \le J_p(h) = J_p(h^+) - J_p(h^-) \le J_p(h^+) \le h^+$.

(ii) If $0, g \le h$, then by (i), $g^+ \le h^+ = h$. Conversely, if $g^+ \le h$, then $0, g \le g^+ \le h$.

(iii) Suppose $\pm g \le h$ and choose $p \in P^{\pm}(g)$. As $g \in C(p)$, it follows that $h \in C(p)$. Also, since $g \le h$ we have $g^+ = J_p(g) \le J_p(h)$. Likewise, since $-g \le h$, we have $g^- = J_{u-p}(-g) \le J_{u-p}(h)$, and therefore $|g| = g^+ + g^- \le J_p(h) + J_{u-p}(h) = h$. The converse implication follows from the fact that $\pm g \le |g|$. □

5. The pseudo-meet and pseudo-join

In the study of operator algebras, the expressions $\frac{1}{2}(A+B-|A-B|)$ and $\frac{1}{2}(A+B+|A-B|)$ have been called the *lower envelope* and the *upper envelope*, respectively, of the self-adjoint operators A and B ([11; p. 279]). If the compressible group G has general comparability, then with the aid of the following lemma, we can form analogous expressions, which we shall call the *pseudo-meet* and the *pseudo join*.

LEMMA 5.1. *If G has general comparability and $g,h\in G$, the equation $2x = g+h-|g-h|$ has a unique solution $x = g-(g-h)^+ = h-(h-g)^+$.*

P r o o f. Let $x := g-(g-h)^+$. As $g-h = (g-h)^+ - (g-h)^- = (g-h)^+ - (h-g)^+$, we have $x = g-(g-h)^+ = h-(h-g)^+$, whence $2x = g-(g-h)^+ + h-(h-g)^+ = g+h-(g-h)^+-(g-h)^- = g+h-|g-h|$. That x is the unique solution of $2x = g+h-|g-h|$ follows from the fact that the group G is torsion free. □

DEFINITION 5.2. If G has general comparability and $g,h\in G$, we define the *pseudo-meet* $g\sqcap h := g-(g-h)^+ = h-(h-g)^+$ and the *pseudo-join* $g\sqcup h := -(-g\sqcap -h) = g+(h-g)^+ = h+(g-h)^+$.

In view of Lemma 5.1, $g\sqcap h$ is the unique solution x of the equation $2x = g+h-|g-h|$ and $g\sqcup h$ is the unique solution y of the equation $2y = g+h+|g-h|$. By the following lemma, even if G is not lattice ordered, the pseudo-meet and pseudo-join enjoy many of the properties of the meet and join in a lattice-ordered abelian group.

LEMMA 5.3. *Suppose G has general comparability and let $g,h,k\in G$. Then:*

(i) $g\sqcap h = h\sqcap g$ *and* $g\sqcup h = h\sqcup g$.
(ii) $g\sqcap h \le g,h \le g\sqcup h$.
(iii) $(g\sqcap h)+k = (g+k)\sqcap(h+k)$ *and* $(g\sqcup h)+k = (g+k)\sqcup(h+k)$.
(iv) $g\le h \iff g = g\sqcap h \iff h = g\sqcup h$.
(v) $g\sqcap h + g\sqcup h = g+h$.
(vi) $g^+ = g\sqcup 0$ *and* $g^- = -(g\sqcap 0)$.
(vii) $g^+\sqcap g^- = 0$.
(viii) $|g| = g\sqcup(-g)$.

P r o o f. Part (i) follows from Lemma 5.1 and Definition 5.1.

(ii) $g\sqcap h = g-(g-h)^+ \le g$ and, by (i), $g\sqcap h\le h$. Similarly, $g,h\le g\sqcup h$.

(iii) $(g+k)\sqcap(h+k) = g+k-\big((g+k)-(h+k)\big)^+ = k+g-(g-h)^+ = k+g\sqcap h$. Similarly, $(g\sqcup h)+k = (g+k)\sqcup(h+k)$.

(iv) If $g\le h$, then $g\sqcap h = h-(h-g)^+ = h-(h-g) = g$. Conversely, if $g = g\sqcap h$, then $g\le h$ by (ii).

(v) $g \sqcap h + g \sqcup h = g - (g-h)^+ + h + (g-h)^+ = g + h$.

(vi) $g \sqcup 0 = g + (0-g)^+ = g + g^- = g^+$, whence $g^- = (-g)^+ = (-g) \sqcup 0 = -(g \sqcap 0)$.

(vii) $g^+ \sqcap g^- = g^+ - (g^+ - g^-)^+ = g^+ - g^+ = 0$.

(viii) $2\big(a \sqcup (-a)\big) = a + (-a) + |a - (-a)| = |2a| = 2|a|$ by Lemma 4.3(xi), whence, as G is torsion free, $a \sqcup (-a) = |a|$. □

THEOREM 5.4. *Suppose G has general comparability and let $g, h \in G$. Then:*

(i) *If $0 \le g, h$, then $g \sqcap h = 0 \iff (\exists p \in P)\big(g = J_p(g)\ \ \&\ \ h = J_{u-p}(h)\big)$.*

(ii) *$g \sqcap h \le 0 \le g, h \implies g \sqcap h = 0$.*

(iii) *$g \sqcap h$ (respectively, $g \sqcup h$) is a maximal lower bound in G (respectively, a minimal upper bound in G) for g and h.*

(iv) *If the infimum $g \wedge_G h$ (respectively, the supremum $g \vee_G h$) of g and h exists in G, then $g \wedge_G h = g \sqcap h$ (respectively, $g \vee_G h = g \sqcup h$).*

(v) *G is lattice ordered if and only if $\sqcap$ (or, equivalently, $\sqcup$) is associative.*

Proof.

(i) Assume that $0 = g \sqcap h$, i.e., $g = (g-h)^+$, and select $p \in P^{\pm}(g-h)$. Then $g = (g-h)^+ = J_p(g-h) = J_p(g) - J_p(h) \le J_p(g)$. Also, $g - h = J_p(g-h) + J_{u-p}(g-h) = g + J_{u-p}(g-h)$, and it follows that $h = J_{u-p}(h-g) = J_{u-p}(h) - J_{u-p}(g) \le J_{u-p}(h)$. As $0 \le g \le J_p(g)$, we have $J_{u-p}(g) = 0$, whence $g = J_p(g)$. Likewise, $h = J_{u-p}(h)$.

Conversely, suppose $p \in P$, $J_p(g) = g$, and $J_{u-p}(h) = h$. As $0 \le g, h$, it follows that $J_{u-p}(g) = 0$ and $J_p(h) = 0$, whence $g - h = J_p(g-h) + J_{u-p}(g-h)$ with $0 \le g = J_p(g-h)$ and $J_{u-p}(g-h) = -h \le 0$. Consequently, $(g-h)^+ = J_p(g-h) = g$ by Lemma 4.2.

(ii) Suppose that $g \sqcap h \le 0 \le g, h$ and let $p \in P^{\pm}(g-h)$. Then $(g-h)^+ = J_p(g-h)$, so $g - J_p(g-h) = g \sqcap h \le 0$. Applying J_{u-p} to the latter inequality, we obtain $J_{u-p}(g) \le 0$. But, since $0 \le g$, we also have $0 \le J_{u-p}(g)$, and it follows that $J_{u-p}(g) = 0$, whence $J_p(g) = g$. As $p \in P^{\pm}(g-h)$, we have $u - p \in P^{\pm}(h-g)$, so by symmetry, $J_{u-p}(h) = h$, and it follows from (i) that $g \sqcap h = 0$.

(iii) Suppose $k \in G$ and $g \sqcap h \le k \le g, h$. We have to prove that $g \sqcap h = k$. By Lemma 5.3(iii), $g \sqcap h - k = (g-k) \sqcap (h-k) \le 0 \le g-k, h-k$, whence $g \sqcap h = k$ by (i). By duality, $g \sqcup h$ is a minimal upper bound in G for g and h.

(iv) We have $g \sqcap h \le g, h$, so if $g \wedge_G h$ exists, $g \sqcap h \le g \wedge_G h \le g, h$, whence $g \sqcap h = g \wedge_G$ by (iii). By duality, if $g \vee_G h$ exists, then $g \sqcup h = g \vee_G h$.

(v) If G is lattice ordered, then $\sqcap = \wedge_G$ and $\sqcup = \vee_G$ by (iv), whence $\sqcap$ and $\sqcup$ are associative. For $g, h \in G$, $g \sqcup h = -\big((-g) \sqcap (-h)\big)$, so $\sqcap$ is associative if and only if $\sqcup$ is associative. Suppose $\sqcap$ is associative and let $g, h \in G$. By Lemma 5.3(ii), $g \sqcap h$ is a lower bound in G for g and h. Suppose

$k \in G$ with $k \leq g, h$. By Lemma 5.3(iv), $k = k \sqcap g$ and $k = k \sqcap h$, whence $k \sqcap (g \sqcap h) = (k \sqcap g) \sqcap h = k \sqcap h = k$, and it follows that $k \leq g \sqcap h$. Therefore, $g \sqcap h = g \wedge_G h$, and G is lattice ordered. □

THEOREM 5.5. *If G has general comparability, then the following conditions are mutually equivalent:*

(i) *For all $g, h \in G$, $-h \leq g \leq h \iff |g| \leq h$.*
(ii) *For all $g, h \in G$, $|g + h| \leq |g| + |h|$.*
(iii) *For all $g, h \in G$, if $0, g \leq h$, then $g^+ \leq h$.*
(iv) *$g, h \in G^+ \implies g \sqcap h \in G^+$.*
(v) *G is lattice ordered.*
(vi) *G is an interpolation group.*
(vii) *G is a compatible group.*

Proof.

(i) $\implies$ (ii). Assume (i) and let $g, h \in G$. Then, as $\pm g \leq |g|$ and $\pm h \leq |h|$, we have $\pm(g+h) \leq |g| + |h|$, i.e., $-\big(|g| + |h|\big) \leq g + h \leq |g| + |h|$, and it follows from (i) that $|g + h| \leq |g| + |h|$.

(ii) $\implies$ (iii). Assume (ii) and suppose $g, h \in G$ with $0, g \leq h$. Then $|g| = |h + (g - h)| \leq |h| + |g - h| = h + |-(h - g)| = h + |h - g| = h + h - g$, whence $2g^+ = g + |g| \leq 2h$. Since G is unperforated, it follows that $g^+ \leq h$.

(iii) $\implies$ (iv). Assume (iii) and let $g, h \in G^+$. Then, $h - g, 0 \leq h$, so $(h - g)^+ \leq h$ by (iii), and it follows that $0 \leq h - (h - g)^+ = g \sqcap h$.

(iv) $\implies$ (v). Assume (iv) and let $g, h, k \in G$ with $k \leq g, h$. Then $g - k, h - k \in G^+$, so $(g \sqcap h) - k = (g - k) \sqcap (h - k) \in G^+$ by Lemma 5.3(iii), and it follows that $k \leq g \sqcap h$. By Lemma 5.3(ii), $g \sqcap h$ is a lower bound for g and h, so $g \sqcap h$ is the greatest lower bound in G for g and h. Therefore, G is lattice ordered.

(v) $\implies$ (vi) $\implies$ (vii) $\implies$ (v). Clearly, (v) $\implies$ (vi) $\implies$ (vii). As a compatible group with general comparability has central comparability, it is lattice ordered, so (vii) $\implies$ (v).

(v) $\implies$ (i). Assume (v) and let $g, h \in G$ with $-h \leq g \leq h$, i.e., $\pm g \leq h$. Thus, $g \vee_G (-g) \leq h$, and it follows from Theorem 5.4(iv) and Lemma 5.3(viii) that $|g| = g \sqcup (-g) = g \vee_G (-g) \leq h$. Conversely, if $|g| \leq h$, then $\pm g \leq h$ by Lemma 4.3(iv), whence $h \leq g \leq h$. □

If the compressible group G is lattice ordered, then the unit interval $E := \{e \in G : 0 \leq e \leq u\}$ is a *pseudo-Boolean* effect algebra in the sense that disjoint elements of E are orthogonal, i.e., if $e, f \in E$, $e \wedge_E f = 0 \implies e + f \leq u$. With $\wedge_E$ replaced by $\sqcap$, a compressible group with general comparability has an analogous property as per part (i) of the following lemma.

Lemma 5.6. *Suppose G has general comparability, $w \in C(P)$ and $g, h \in G$ with $0 \le g, h \le w$. Then:*

(i) $g \sqcap h = 0 \implies g + h = g \sqcup h \le w$.

(ii) *Every element $k \in G$ with $0 \le k \le 2w$ can be written uniquely in the form $k = g + h$ with $0 \le g \le h \le w$ and $g \sqcap (w - h) = 0$. In fact, the unique elements g and h satisfying these conditions are $g = (k - w)^+$ and $h = k \sqcap w$.*

(iii) $0 \le g \sqcap (w - g) \le g, w - g$.

Proof.

(i) Assume that $w \in C(P)$, $0 \le g, h \le w$, and $g \sqcap h = 0$. By Lemma 5.3(v), $g + h = g \sqcap h + g \sqcup h = g \sqcup h$. Also, by Theorem 5.4(i), there exists $p \in P$ such that $g = J_p(g) \le J_p(w)$ and $h = J_{u-p}(h) \le J_{u-p}(w)$. Therefore, since $w \in C(P) \subseteq C(p)$, it follows that $g + h \le J_p(w) + J_{u-p}(w) = w$.

(ii) Let $0 \le k \le 2w$, let $g := (k - w)^+$, and let $h := k \sqcap w = k - (k - w)^+ = k - g$, so that $k = g + h$. Choose $p \in P^{\pm}(k - w)$. As $0 \le w \in C(P) \subseteq C(p)$, we have $J_p(w) \le J_p(w) + J_{u-p}(w) = w$. Evidently $0 \le (k - w)^+ = g = J_p(k - w) = J_p(k) - J_p(w)$. Since $k \le 2w$, it follows that $J_p(k) \le 2J_p(w)$, so $g = J_p(k) - J_p(w) \le J_p(w) \le w$, and we have $0 \le g \le w$. By Lemma 5.3(ii), $h = g \sqcap w \le w$. As $k - w \in C(p)$ and $w \in C(p)$, it follows that $k \in C(p)$, so $J_p(k) \le J_p(k) + J_{u-p}(k) = k \le k + J_p(w)$. Therefore, $g = (k - w)^+ = J_p(k) - J_p(w) \le k$, whence $0 \le k - g = h$, and we have $0 \le h \le w$. By Lemma 5.3(iii), $h = k \sqcap w = (g + h) \sqcap w = g \sqcap (w - h) + h$, whence $g \sqcap (w - h) = 0$, and it follows from (i) that $g + (w - h) \le w$, i.e., $g \le h$.

To prove uniqueness, suppose $k = g + h$ with $g \sqcap (w - h) = 0$. Then by Lemma 5.3(iii), $k \sqcap w = (g + h) \sqcap w = g \sqcap (w - h) + h = h$ and $g = k - h = k - k \sqcap w = (k - w)^+$.

(iii) As $0 \le g \le w$, we have $0 \le 2g \le 2w$. Therefore, by (ii) with k replaced by $2g$, $2g = (2g - w)^+ + \big((2g) \sqcap w\big)$ with $0 \le (2g - w)^+ \le (2g) \sqcap w \le w$, whence $2(2g - w)^+ \le (2g - w)^+ + \big((2g) \sqcap w\big) = 2g$. As G is unperforated, it follows that $(2g - w)^+ \le g$, so $0 \le g - (2g - w)^+ = g - \big(g - (w - g)\big)^+ = g \sqcap (w - g)$. Also, by Lemma 5.3(ii), $g \sqcap (w - g) \le g, w - g$. □

An effect $q \in E = \{e \in G : \ 0 \le e \le u\}$ is said to be *sharp* if and only if the infimum $q \wedge_E (u - q)$, as calculated in E, exists and $q \wedge_E (u - q) = 0$, i.e., if and only if 0 is the only effect $e \in E$ with $e \le q, u - q$ ([7]). An effect $q \in E$ is said to be *principal* if and only if, for all $e, f \in E$, the conditions $e, f \le q$ with $e + f \le u$ imply that $e + f \le q$ ([5]). Thus, the next theorem generalizes [5; Theorem 6.8].

THEOREM 5.7. *Suppose the compressible group G has general comparability and let $q \in E$. Then the following conditions are mutually equivalent:*

(i) *q is principal.*
(ii) *q is sharp.*
(iii) $q \sqcap (u - q) = 0$.
(iv) $q \in P$.

Proof.

(i) $\Longrightarrow$ (ii). Assume (i) $e \in E$ with $e \le q, u - q$. Then $e, q \le q$ with $e + q \le u$, and it follows that $e + q \le q$, so $e = 0$. As 0 is a lower bound in E for q and $u - q$, it follows that $q \wedge_E (u - q) = 0$.

(ii) $\Longrightarrow$ (iii) follows from Lemma 5.6(iii) with $w = u$.

(iii) $\Longrightarrow$ (iv). Suppose $q \sqcap (u - q) = 0$. Then by Theorem 5.3(i) there exists $p \in P$ such that $J_p(q) = q$ and $u - p - J_{u-p}(q) = J_{u-p}(u - q) = u - q$. But, $J_{u-p}(q) = J_{u-p}(J_p(q)) = 0$, so $q = p \in P$.

(iv) $\Longrightarrow$ (i). Suppose $q \in P$ and let $0 \le e, f \le q$ with $e + f \le u$. As $0 \le e, f \le q$, we have $J_q(e) = e$ and $J_q(f) = f$, whence $e + f = J_q(e + f) \le J_q(u) = q$. □

6. The Rickart projection property

With Example 1.2 and the more general notion of a Rickart C*-algebra in mind, we make the following definition.

DEFINITION 6.1. The compressible group G has the *Rickart projection property* if and only if there is a mapping $' \colon G \to P$, called the *Rickart mapping*, such that, for all $g \in G$ and all $p \in P$, $p \le g' \iff g \in C(p)$ with $J_p(g) = 0$.

If X is a compact Hausdorff basically-disconnected space, then the compressible group $G = C(X, \mathbb{R})$ in Example 3.6 has the Rickart projection property.

LEMMA 6.2. *Suppose that G has the Rickart projection property. Then, for all $g, h \in G$, all $p \in P$, and all $e \in E$:*

(i) $g \in C(g')$ *and* $J_{g'}(g) = 0$.
(ii) *If* $0 \le g$, *then* $J_p(g) = 0 \iff p \le g'$.
(iii) $p' = u - p$ *and* $g'' := (g')' = u - g'$.
(iv) $g'' \le p \iff g \in C(p)$ *with* $J_p(g) = g$.
(v) $g'' = 0 \iff g = 0$.
(vi) $0 \le g \le h \Longrightarrow h' \le g'$.
(vii) $e \le e''$ *with equality if and only if* $e \in P$.
(viii) $e \le p \iff e'' \le p$.

Proof.

(i) As $g' \in P$ and $g' \leq g'$, we have $g \in C(g')$ and $J_{g'}(g) = 0$.

(ii) If $0 \leq g$, then, $J_p(g) = 0 \implies g \in C(p)$, and (ii) follows.

(iii) If $q \in P$, then by (ii), $q \leq p' \iff J_q(p) = 0 \iff p \leq u - q \iff q \leq u - p$. Therefore, $q \leq p' \iff q \leq u - p$, from which it follows that $p' = u - p$. In particular, since $g' \in P$, we have $g'' = u - g'$.

(iv) If $g \in C(p)$, then $J_p(g) = g \iff J_{u-p}(g) = 0$. Therefore, $g \in C(p)$ with $J_p(g) = g \iff u - p \leq g' \iff g'' = u - g' \leq p$.

(v) Evidently, $0' = u$, so $0'' = u - u = 0$. Conversely, if $g'' = 0$, then by (iv), $0 = J_0(g) = g$.

(vi) If $0 \leq g \leq h$, then $0 \leq J_{h'}(g) \leq J_{h'}(h) = 0$, whence $h' \leq g'$.

(vii) By (i), $J_{e'}(e) = 0$, whence, since $e \in E$, $e = J_{u-e'}(e) \leq u - e' = e''$ by (iii). If $e \in P$, then $e' = u - e$ and $e'' = u - (u - e) = e$ by (iii) again. Conversely, $e'' \in P$, so if $e = e''$, then $e \in P$.

(viii) If $e \leq p$, then by (vi), $p' \leq e'$, so $e'' \leq p'' = p$ by (vii). Conversely, by (vii) again, if $e'' \leq p$, then $e \leq p$. □

The notions in the following definition were originally introduced in [3; Definition 6.1].

DEFINITION 6.3. If $e \in E$ and $c \in P$, then c is a *projection cover* for (or of) e if and only if c is the smallest element in $\{p \in P : e \leq p\}$. The compressible group G has the *projection cover property* if and only if every effect $e \in E$ has a projection cover.

THEOREM 6.4. *Suppose that G has the Rickart projection property. Then:*

(i) *G has the projection cover property and the projection cover of each effect $e \in E$ is $e'' \in P$.*

(ii) *P is an orthomodular lattice and, for all $p, q \in P$, $p \wedge q = J_p\big((J_p(q'))'\big)$.*

(iii) $p, q \in P \implies \big(J_p(q)\big)'' = p \wedge (p' \vee q)$.

(iv) *If $g_1, g_2, \ldots, g_n \in G^+$, then* $\left(\sum_{i=1}^{n} g_i\right)'' = \bigvee_{i=1}^{n} (g_i)''$.

(v) $g \in G^+ \implies g', g'' \in CPC(g)$.

(vi) *If $e \in E$ and $p \in P$, then* $\big(J_p(e)\big)'' = \big(J_p(e'')\big)''$.

(vii) *If $g \in G^+$ and $p \in P$, then* $\big(J_p(g)\big)'' = p \wedge (p' \vee g'')$.

Proof.

(i) Follows directly from Lemma 6.2(viii).

(ii) By [3; Theorem 6.3], P is an orthomodular lattice, and by (i), Lemma 6.2(iii), and [3; Lemma 6.2(vii)], $p \wedge q = J_p\big(u - (J_p(u - q))''\big) = J_p\big((J_p(q'))'\big)$.

(iii) By [3; Lemma 6.2(vi)], the mapping $\phi_p\colon P \to P$ defined by $\phi_p(q) := \big(J_p(q)\big)''$ for $q \in P$ is residuated, hence it preserves suprema. Also, if pCq, then $\phi_p(q) = \big(J_p(q)\big)'' = (p \wedge q)'' = p \wedge q$. As $J_p(p') = 0$, we have $\phi_p(p') = 0'' = 0$, and it follows that $\phi_p(q \vee p') = \phi_p(q) \vee \phi_p(p') = \phi_p(q)$. Therefore, since $pC(q \vee p')$ in the orthomodular lattice P, we have $\phi_p(q) = \phi_p(q \vee p') = p \wedge (q \vee p')$.

(iv) Let $g := \sum_{i=1}^{n} g_i$ and let $p \in P$. Then, since $0 \le g, g_1, g_2, \ldots, g_n$, we have $p \le g' \iff J_p(g) = 0 \iff \sum_{i=1}^{n} J_p(g_i) = 0 \iff J_p(g_i) = 0$ for $i = 1, 2, \ldots, n \iff p \le (g_i)'$ for $1 = 1, 2, \ldots, n \iff p \le \bigwedge_{i=1}^{n} (g_i)'$, and it follows that $g' = \bigwedge_{i=1}^{n} (g_i)'$. Therefore, by the deMorgan law in P, $g'' = \bigvee_{i=1}^{n} (g_i)''$.

(v) Suppose $g \in G^+$, $p \in P$, and $g \in C(p)$. Let $a := J_p(g)$ and $b := J_{p'}(g)$. Then $a, b \in G^+ \cap C(p)$, $g = a + b$, $J_{p'}(a) = 0$, and $J_p(b) = 0$. Consequently, $p' \le a'$, $p \le b'$, and by (iv), $g' = a' \wedge b'$. As $p' \le a'$ and $p \le b'$, we have pCa' and pCb', whence $pC(a' \wedge b')$, i.e., pCg'. Therefore, $g' \in CPC(g)$, and also $g'' = u - g' \in CPC(g)$.

(vi) As $e \le e''$, we have $J_p(e) \le J_p(e'')$, whence $\big(J_p(e)\big)'' \le \big(J_p(e'')\big)'' = p \wedge (p' \vee e'')$. Let $q := \big(J_p(e)\big)''$ and let $r := p \wedge (p \wedge q)'$. As $p \wedge q \le p$, it follows that $(p \wedge q)Cp'$ with $r' = p' \vee (p \wedge q) = p' + (p \wedge q)$. Now, $J_p(e) \le p, q$, so $J_p(e) \le p \wedge q$. As $r \le p, (p \wedge q)'$, we have $J_r(e) = J_r\big(J_p(e)\big) \le J_r(p \wedge q) = 0$, whence $r \le e'$, i.e., $e'' \le r'$, therefore $J_p(e'') \le J_p(r') = J_p\big(p' + (p \wedge q)\big) = p \wedge q \le q$, and it follows that $\big(J_p(e'')\big)'' \le q = \big(J_p(e)\big)''$.

(vii) As $g \in G^+$, we can write $g = \sum_{i=1}^{n} e_i$ with $e_i \in E$ for $i = 1, 2, \ldots, n$. Therefore, by (iv), $g'' = \bigvee_{i=1}^{n} (e_i)''$. Also, $J_p(g) = \sum_{i=1}^{n} J_p(e_i)$, so by (iv), (vi), and (iii),

$$\big(J_p(g)\big)'' = \bigvee_{i=1}^{n} \big(J_p(e_i)\big)'' = \bigvee_{i=1}^{n} \big(J_p((e_i)'')\big)'' = \bigvee_{i=1}^{n} \big(p \wedge (p' \vee (e_i)''\big).$$

As the mapping $q \mapsto p \wedge (p' \vee q)$, $q \in P$, preserves suprema in P, it follows that

$$\big(J_p(g)\big)'' = p \wedge \left(p' \vee \bigvee_{i=1}^{n} (e_i)''\right) = p \wedge (p' \vee g'').$$

□

THEOREM 6.5. *Suppose that G has general comparability. Then G has the projection cover property if and only if G has the Rickart projection property. Furthermore, if G has the Rickart projection property and if $g \in G$, $p \in P$, then:*

(i) $g' = |g|' \in CPC(g)$ *and* $g'' = |g|'' \in CPC(g)$.
(ii) $(g^+)'' + (g^-)'' = (g^+)'' \vee (g^-)'' = g''$.
(iii) $(g^+)'' \leq (g^-)'$ *and, if* $q \in P$ *with* $(g^+)'' \leq q \leq (g^-)'$, *then* $J_q(g) = g^+$.
(iv) $(g^-)'' \leq (g^+)'$ *and, if* $r \in P$ *with* $(g^-)'' \leq r \leq (g^+)'$, *then* $J_r(-g) = g^-$.
(v) *If* $(g^+)' = 0$, *then* $0 \leq g$. *If* $(g^+)' = u$, *then* $g \leq 0$.

Proof. By Theorem 6.4(i), if G has the Rickart projection property, then it has the projection cover property. Conversely, suppose that G has the projection cover property and denote the projection cover of each $e \in E$ by $\gamma(e)$. By [3; Theorem 6.3], P is an orthomodular lattice. Let $g \in G$ and let $p \in P$. There are effects $e_1, e_2, \ldots, e_n \in E$ such that $|g| = \sum_{i=1}^{n} e_i$. Define $g' := \bigwedge_{i=1}^{n} (u - \gamma(e_i)) \in P$. Then, for $i = 1, 2, \ldots, n$, $J_p(e_i) = 0 \iff e_i \leq u - p \iff \gamma(e_i) \leq u - p \iff p \leq u - \gamma(e_i)$. Since $0 \leq J_p(e_i)$ for all $i = 1, 2, \ldots, n$, it follows that $J_p(|g|) = 0 \iff \sum_{i=1}^{n} J_p(e_i) = 0 \iff J_p(e_i) = 0$ for $i = 1, 2, \ldots, n$. Therefore, by Lemma 4.3(viii),

$$g \in C(p) \text{ with } J_p(g) = 0 \iff J_p(|g|) = 0 \iff p \leq \bigwedge_{i=1}^{n} (u - p_i) = g',$$

so G has the Rickart projection property.

(i) That $g' = |g|'$ is a direct consequence of Lemma 4.3(viii), and $g' = |g|' \implies g'' = |g|''$. Thus, by Theorem 6.4(v), we have $g', g'' \in CPC(|g|)$, and by Lemma 4.3(vii), $|g| \in CPC(g)$, whence $g', g'' \in CPC(g)$.

(ii) Choose $p \in P^{\pm}(g)$, so that $g^+ = J_p(g)$ and $g^- = J_{p'}(-g)$. Thus, $J_{p'}(g^+) = 0$, so $p' \leq (g^+)'$, and $J_p(g^-) = 0$, so $p \leq (g^-)'$. Consequently, $(g^+)'' \leq p$ and $(g^-)'' \leq p'$, whence $(g^+)'' + (g^-)'' \leq p + p' = u$, and it follows that $(g^+)'' + (g^-)'' = (g^+)'' \vee (g^-)''$. Hence, by (i) and Theorem 6.4(iv), $g'' = |g|'' = (g^+ + g^-)'' = (g^+)'' \vee (g^-)'' = (g^+)'' + (g^-)''$.

(iii) By (ii), $(g^+)'' \leq u - (g^-)'' = (g^-)'$. Let $q \in P$ with $(g^+)'' \leq q \leq (g^-)'$. As $(g^+)'' \leq q$, we have $J_q(g^+) = g^+$, and as $q \leq (g^-)'$, we also have $J_q(g^-) = 0$, whence $J_q(g) = J_q(g^+) - J_q(g^-) = g^+$.

(iv) Analogous to the proof of (iii).

(v) If $(g^+)' = 0$, then $(g^-)'' = 0$ by (iv), whence $g^- = 0$, and it follows that $g = g^+ \geq 0$. If $(g^+)' = u$, then $(g^+)'' = 0$, whence $g^+ = 0$, and it follows that $g = g^- \leq 0$. □

THEOREM 6.6. *If G has general comparability, G is archimedean, and P satisfies the ascending chain condition, then G has the Rickart projection property and, if $g \in G$, there exists a positive integer N such that $g'' \leq N|g|$.*

P r o o f. By Corollary 3.8 with g replaced by $|g|$, there exists $q \in P$ and a positive integer N such that $q \leq N|g|$ and, for all $p \in P$, $q \leq u - p \iff J_{u-p}(|g|) = |g|$. Then, if $p \in P$, $p \leq u - q \iff q \leq u - p \iff J_{u-p}(|g|) = |g| \iff J_p(|g|) = 0$. But, by Lemma 4.3(viii), $J_p(|g|) = 0 \iff g \in C(p)$ with $J_p(g) = 0$. Therefore, G has the Rickart projection property, $g' = u - q$, and $g'' = q \leq N|g|$. □

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