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Mathematica Slovaca, Vol. 55 (2005), No. 3, 353--361

Persistent URL: http://dml.cz/dmlcz/131165

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TIME DEPENDENT DIRICHLET SPACE, MIXED DIRICHLET PROBLEM AND PSEUDO-DIFFERENTIAL OPERATORS

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(Communicated by Michal Fečkan)

ABSTRACT. In this paper we study the time dependent Dirichlet space which is generated by pseudo-differential operators. Also we find the set of inequalities defining an optimal control of a system governed by pseudo-differential operators with symbols defined in terms of conditionally exponential convex function.

1. Time dependent Dirichlet form on $\mathbb{R}^n$

Let us introduce certain function spaces on $\mathbb{R} \times \mathbb{R}^n$.

**Definition 1.1.** A real valued function $a: \mathbb{R}^n \to \mathbb{R}$ is said to be conditionally exponential convex if for any $x_1, \ldots, x_n \in \mathbb{R}^n$ and $C_1, \ldots, C_n \in \mathbb{R}$ we have

$$\sum_{j,k=1}^n [a(x_j) + a(x_k) - a(x_j + x_k)] C_j C_k \geq 0. \quad (1.1)$$

**Lemma 1.1.** Let $a^2: \mathbb{R}^n \to \mathbb{R}$ be a continuous conditionally exponential convex function. Then

$$0 \leq a^2(\xi) \leq C_{\alpha} (1 + |\xi|^2), \quad (1.2)$$

$$a^2(\xi) = C - Q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} \left[1 - \exp(x, \xi) + \frac{(x, \xi)}{1 + \|x\|^2} \right] \frac{1 + \|x\|^2}{\|x\|^2} \, d\mu(x) \quad (1.3)$$

where $C \geq 0$ is a constant, $Q: \mathbb{R}^n \to \mathbb{R}$ is a continuous negative quadratic form on $\mathbb{R}^n$ and $\mu$ is a positive bounded measure on $\mathbb{R}^n \setminus \{0\}$, and

$$|a^2(\xi) - a^2(\eta)| \leq 4a(\xi)a(\xi - \eta) + a^2(\xi - \eta), \quad (1.4)$$

$$|a(\xi) - a(\eta)| \leq a(\xi + \eta). \quad (1.5)$$

2000 Mathematics Subject Classification: Primary 49J20, 49K20, 35S15.
Keywords: conditionally exponential convex function, optimal control, pseudodifferential operator.
Remark. The estimate (1.2) can be found in [3], [7], [8] and (1.3) in [4]. We have taken (1.4) and (1.5) from [1], [2].

For any continuous conditionally exponential convex function $a^2 : \mathbb{R}^n \to \mathbb{R}$ and for any $S \geq 0$, we introduce the Hilbert space

$$H^{a^2,S}(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) : \|u\|_{S,a^2} < \infty \}$$

where

$$\|u\|^2_{S,a^2} = \int_{\mathbb{R}^n} (1 + a^2(\xi))^S |\hat{u}(\xi)|^2 \, d\xi,$$

where $\hat{u}$ is Fourier transform of $u$.

Clearly $H^{a^2,0}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, and if we identify $[L^2(\mathbb{R}^n)]^*$ with $L^2(\mathbb{R}^n)$, we have (see [2], [6])

$$[H^{a^2,S}(\mathbb{R}^n)]^* = H^{a^2,-S}(\mathbb{R}^n),$$

where

$$H^{a^2,-S}(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) : \|u\|_{-S,a^2} < \infty \}$$

and the negative norm is given on $L^2(\mathbb{R}^n)$ by

$$\|u\|^2_{-S,a^2} = \int_{\mathbb{R}^n} (1 + a^2(\xi))^{-2S} |\hat{u}(\xi)|^2 \, d\xi = \sup_{0 \neq v \in H^{a^2,S}(\mathbb{R}^n)} \frac{|(u,v)_0|}{\|v\|_{S,a^2}}.$$

Later we will often assume that $a^2$ also satisfies

$$a^2(\xi) \geq C_r |\xi|^r,$$

for some $r > 0$ and all $\xi \in \mathbb{R}^n$, $|\xi| \geq \sigma \geq 0$. In this case $H^{a^2,S}(\mathbb{R}^n)$ is continuously embedded in the usual Sobolev space $H^{Sr}(\mathbb{R}^n)$ and for $Sr > \frac{n}{2}$ we find $H^{a^2,S}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$ with a continuous embedding.

Now we formulate the main new results of this paper. We will introduce certain function spaces on $]0,T[ \times \mathbb{R}^n \subset \mathbb{R} \times \mathbb{R}^n$.

Let $L_2(0,T, H^{a^2,S}(\mathbb{R}^n))$ denote the space of all measurable functions $t \mapsto f(t) : ]0,T[ \to H^{a^2,S}(\mathbb{R}^n)$ where the variable $t$ denotes “time”. We assume that $t \in ]0,T[ , T < \infty$, with Lebesgue measure $dt$ on $]0,T[$ such that

$$\left( \int_0^T \|f(t)\|^2_{H^{a^2,S}(\mathbb{R}^n)} \, dt \right)^{\frac{1}{2}} = \|f\|_{L_2(0,T, H^{a^2,S}(\mathbb{R}^n))}$$

and $L_2(0,T, H^{a^2,S}(\mathbb{R}^n))$ is endowed with the scalar product

$$(f,g)_{L_2(0,T, H^{a^2,S}(\mathbb{R}^n))} = \int_0^T (f(t),g(t))_{H^{a^2,S}(\mathbb{R}^n)},$$
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which is a Hilbert space ([5], [6]).

Analogously, we define the spaces $L_2(0, T, L_2(\mathbb{R}^n))$ and $L_2(0, T, H^{a^2,-S}(\mathbb{R}^n))$ and then we have a chain in the form

$$L_2(0, T, H^{a^2,S}(\mathbb{R}^n)) \subseteq L_2(0, T, L_2(\mathbb{R}^n)) \subseteq L_2(0, T, H^{a^2,-S}(\mathbb{R}^n)). \quad (1.9)$$

Now, let us define the continuous conditionally exponential convex function $a^2 : \mathbb{R}^n \rightarrow \mathbb{R}$ by $a^2(\xi) = \sum_{j=1}^{n} a^2_j(\xi_j) \ (\xi \in \mathbb{R}^n, \xi_j \in \mathbb{R})$ where $a^2_j : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous conditionally exponential convex function, $1 \leq j \leq n$.

Further, let $b_j : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$, $(t, x) \mapsto b_j(t, x)$, $1 \leq j \leq n$, be a function satisfying the following conditions:

(i) $b_j$ is independent of $x_j$;
(ii) $b_j(t, \cdot)$ is bounded and measurable;
(iii) $t \mapsto b_j(t, x)$ is a continuous function;
(iv) $b_j(t, x) \geq d_0 > 0$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $1 \leq j \leq n$.

On $C^\infty_0(\mathbb{R}^n)$ we consider the family of pseudo-differential operators

$$L(t)(x, D)u(x) = \sum_{j=1}^{n} b_j(t, x) a_j^2(D_j) u(x). \quad (1.10)$$

We can associate with $L(t)(x, D)$ the bilinear form

$$E(t)(u, v) = \int_{\mathbb{R}^n} L(t)(x, D)u(x) \cdot v(x) \, dx$$

$$= \sum_{j=1}^{n} (b_j(t, \cdot) a_j(D_j) u, a_j(D_j) v)_0. \quad (1.11)$$

Now using (i) $\implies$ (iv) we get, as in [2]:

**Proposition 1.1.** For all $u, v \in H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$,

$$|E(t)(u, v)| \leq C\|u\|_{\frac{1}{2}, a^2}\|v\|_{\frac{1}{2}, a^2}, \quad (1.12)$$

$$E(t)(u, u) \geq d_0\|u\|_{\frac{1}{2}, a^2}^2 - d_0\|u\|_0^2. \quad (1.13)$$

**Proof.** Since $C^\infty_0(\mathbb{R}^n)$ is dense in $H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$, it is sufficient to prove (1.12) and (1.13) for $u, v \in C^\infty_0(\mathbb{R}^n)$. It follows that

$$|E(t)(u, v)| = |(L(t)(x, D)u, v)_0|$$

$$= \left| \left( \sum_{j=1}^{n} b_j(t, \cdot) a_j^2(D_j) u, v \right)_0 \right|$$

$$\leq C \sum_{j=1}^{n} \|a_j(D_j) u\|_0 \|a_j(D_j) v\|_0,$$
but \( \|a_j(D_j)u\|_0 \leq C\|u\|_{\frac{1}{2}, a^2} \) for all \( u \in C_0^\infty(\mathbb{R}^n) \).

To prove (1.13), by (1.11) we find
\[
E(t)(u, u) = (L(t)(x, D)u, u)_0
= \sum_{j=1}^{n} (b_j(t, x)a_j(D_j)u, a_j(D_j)u)_0
= \sum_{j=1}^{n} \int_{\mathbb{R}^n} b_j(t, x)a_j(D_j)u \cdot a_j(D_j)u \, dx
\geq d_0 \sum_{j=1}^{n} \int_{\mathbb{R}^n} a_j(D_j)u \cdot a_j(D_j)u \, dx
= d_0 \sum_{j=1}^{n} \|a_j(D_j)u\|_0^2,
\]
i.e. \( E(t)(u, u) \geq 0 \).

But
\[
E(t)(u, u) \geq d_0 \int_{\mathbb{R}^n} a_j^2(\xi_j)|\tilde{u}(\xi)|^2 \, d\xi
= d_0 \left( 1 + \sum_{j=1}^{n} a_j^2(\xi_j) \right) |\tilde{u}(\xi)|^2 \, d\xi - d_0 \int_{\mathbb{R}^n} |\tilde{u}(\xi)|^2 \, d\xi
= d_0\|u\|_{\frac{1}{2}, a^2}^2 - d_0\|u\|_0^2.
\]

From (1.12) and (1.13) it follows that \( E(t) \), with domain \( H^{a^2, \frac{1}{2}}(\mathbb{R}^n) \), is a closed symmetric bilinear form on \( L^2(\mathbb{R}^n) \).

Clearly for all \( u, v \in H^{a^2, \frac{1}{2}}(\mathbb{R}^n) \) the function \( t \mapsto E(t)(u, v) \) is measurable. As in [1; Theorem 2.1], we find, for \( u, v \in H^{a^2, \frac{1}{2}}(\mathbb{R}^n) \),
\[
E(t)(u, v) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}^n} (u(x+y) - u(x))(v(x+y) - v(x)) \cdot \sum_{j=1}^{n} b_j(t, x) \mu_j(dy) \, dx,
\]
where \( \mu_j \) is the image of \( \tilde{\mu}_j \) under the mapping
\[
T_j : \mathbb{R} \to \mathbb{R}^n, \quad \xi_j \mapsto (0, \ldots, 0, \xi_j, 0, \ldots, 0),
\]
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i.e. $\xi_j$ is in the $j$th position. Thus
\[
E(u, v) = \frac{1}{2} \int \int \int_{\mathbb{R}^n} (u(t, x + y) - u(t, x))(v(t, x + y) - v(t, x)) \cdot 
\sum_{j=1}^{n} b_j(t, x) \mu_j(dy) \, dx \, dt - \int \int_{\mathbb{R}^n} \frac{\partial u}{\partial t}(t, x) \cdot v(t, x) \, dx \, dt
\]
defined for
\[
u \in F = \{w \in L^2(0, T, H^{a^2, \frac{1}{2}}(\mathbb{R}^n)) : \frac{\partial w}{\partial t} \in L^2(0, T, H^{a^2, -\frac{1}{2}}(\mathbb{R}^n))\},
\]
and $v \in L^2(0, T, H^{a^2, \frac{1}{2}}(\mathbb{R}^n))$ gives a time dependent Dirichlet form. So we have defined parabolic Dirichlet space $(F, E)$.

2. Formulation of the control problem

Analogous to (1.9), we have a chain of the form
\[
L_2(0, T, H^{a^2, 1}_0(\mathbb{R}^n)) \subseteq L_2(0, T, L^2(\mathbb{R}^n)) = L_2(0, T, H^{a^2, -\frac{1}{2}}(\mathbb{R}^n)) \quad (2.1)
\]
where $H^{a^2, S}_0(\mathbb{R}^n)$ is the subset of $H^{a^2, S}(\mathbb{R}^n)$ of all functions which vanish on the boundary $\Gamma$ of $\mathbb{R}^n$.

It follows from (1.13) that the continuous bilinear form is coercive, and assume that the function
\[
t \mapsto E(t)(y, \phi) \quad \text{is measurable on } [0, T[ . \quad (2.2)
\]
We can apply the following theorem of Lions [5], [6].

**Theorem 2.1.** Assuming (1.13) and (2.2) hold, then if $f$ is given in $L_2(0, T, H^{a^2, -\frac{1}{2}}(\mathbb{R}^n))$ and $y_0 \in L^2(\mathbb{R}^n)$, there exists a unique $y \in \{\nu \in L_2(0, T, H^{a^2, \frac{1}{2}}(\mathbb{R}^n)) : \frac{\partial \nu}{\partial t} \in L_2(0, T, H^{a^2, -\frac{1}{2}}(\mathbb{R}^n))\}$ satisfying
\[
\frac{\partial y}{\partial t} + L(t)y = f \quad \text{in } Q ,
\]
\[
Q = [0, T[ \times \Omega , \quad \Omega \text{ is an open set of } \mathbb{R}^n ,
\]
\[
y|_{[0, T[ \times \Gamma} = 0 , \quad \Gamma \text{ is boundary of } \Omega ,
\]
\[
y(0, x) = y_0(x) \quad \text{in } \mathbb{R}^n .
\]
The operator
\[
\frac{\partial}{\partial t} + L(t) \in \mathcal{L}(L_2(0, T, H^{a^2, \frac{1}{2}}(\mathbb{R}^n)), L_2(0, T, H^{a^2, -\frac{1}{2}}(\mathbb{R}^n))) . \quad (2.3)
\]
The problem which is defined by the above theorem is known as mixed Dirichlet problem.

Now, we formulate the control problem. Thus the space \( L_2(0, T, L^2(\mathbb{R}^n)) \), being the space of controls is given. A system which is governed by the operator \( L^{(t)} + \frac{\partial}{\partial t} \) is given by (2.3) or by mixed Dirichlet problem. Let \( f \) and \( y_0 \) with \( f \in L_2(0, T, H^{a_2, -\frac{1}{2}}(\mathbb{R}^n)) \) and \( y_0 \in L_2(\mathbb{R}^n) \) be given. We assume that (1.13) and (2.2) hold; then for the control \( u \in L_2(0, T, L_2(\mathbb{R}^n)) \) the state of the system \( y(u) \) which depends on \( x, t \) will be denoted by \( y(t, x; u) \) and is given by the solution of

\[
\frac{\partial y(u)}{\partial t} + L^{(t)} y(u) = f + u \quad \text{in } Q, \\
y(u)\big|_{\Gamma'} = 0, \quad \Gamma' = [0, T] \times \Gamma = \text{Lateral boundary of } Q, \\
y(0, x; u) = y_0(u) \quad \text{in } \mathbb{R}^n, \\
y(u) \in L_2(0, T, H^{a_2, \frac{1}{2}}(\mathbb{R}^n)).
\]

The observation \( Z(u) \) is given by:

\[
Z(u) = y(u),
\]

\( N \) is given as \( N \in \mathcal{L}(L_2(0, T, L_2(\mathbb{R}^n)), L_2(0, T, L_2(\mathbb{R}^n))) \)

\[
(Nu, u)_{L_2(0, T, L_2(\mathbb{R}^n))} \geq \gamma \|u\|^2_{L_2(0, T, L_2(\mathbb{R}^n))}, \quad \gamma > 0. \quad (2.4)
\]

Let \( L_2(0, T, L_2(\mathbb{R}^n)) = L_2(Q) \).

The cost function \( J(u) \) is given by

\[
J(u) = \|y(u) - Z_d\|^2_{L_2(Q)} + (Nu, u)_{L_2(Q)} \\
= \int_Q (y(u) - Z_d)^2 \, dp(x) \, dt + (Nu, u)_{L_2(Q)} \quad (2.5)
\]

where \( Z_d \) is a given element in \( L_2(Q) \).

Let \( U_{ad} \) (set of admissible controls) be a closed convex subset of \( L_2(Q) \). We seek \( \inf J(v), \quad v \in U_{ad} \).
THEOREM 2.2. We assume that (1.13) and (2.1) as well as (2.4) hold. The cost function is given by (2.5). The optimal control $u$ is characterized by the following system of equations and inequalities

\[
\frac{\partial y(u)}{\partial t} + L^{(t)}y(u) = f + u \quad \text{in } Q,
\]
\[
y(u)|_{\Gamma'} = 0, \quad \Gamma' = ]0, T[ \times \Gamma = \text{Lateral boundary of } Q,
\]
\[
y(0, x; u) = y_0(x) \quad \text{in } \mathbb{R}^n,
\]
\[
\frac{\partial P(u)}{\partial t} + L^{(t)}P(u) = y(u) - Z_d \quad \text{in } Q,
\]
\[
P(u)|_{\Gamma'} = 0, \quad \Gamma' = ]0, T[ \times \Gamma, \quad \text{in } \mathbb{R}^n,
\]
\[
(P(u) + Nu, v - u)_{L^2(Q)} \geq 0 \quad \text{for all } v \in U_{ad},
\]

i.e.,
\[
\int_Q (P(u) + Nu)(v - u) \, dp(x) \, dt \geq 0 \quad \text{for all } v \in U_{ad},
\]
\[
y(u), P(u) \in L^2(0, T, H^{a^2, \frac{1}{2}}(\mathbb{R}^n)).
\]

Proof. The control $u \in U_{ad}$ is optimal if and only if
\[
J'(u)(v - u) \geq 0 \quad \text{for all } v \in U_{ad},
\]
that is
\[
(y(u) - Z_d, y(v) - y(u))_{L^2(Q)} + (Nu, v - u)_{L^2(Q)} \geq 0.
\]
(2.6) may be written as:
\[
\int_0^T (y(u) - Z_d, y(v) - y(u))_{L^2(\mathbb{R}^n)} \, dt + (Nu, v - u)_{L^2(Q)} \geq 0.
\]
We introduce the adjoint state $P(u)$ by
\[
-\frac{\partial}{\partial t} P(u) + L^{(t)}P(u) = y(u) - Z_d,
\]
\[
P(T, u) = 0,
\]
\[
P(u) \in L^2(0, T, H^{a^2, \frac{1}{2}}(\mathbb{R}^n)).
\]
Then
\[
\int_{0}^{T} (y(u) - Z dt, y(v) - y(u)) \, dt
\]
\[
= \int_{0}^{T} \left( -\frac{\partial}{\partial t} P(u), y(v) - y(u) \right) \, dt + \int_{0}^{T} \left( L^{(t)} P(u), y(v) - y(u) \right) \, dt
\]
\[
= \int_{0}^{T} \left( P(u), \frac{\partial}{\partial t} (y(v) - y(u)) \right) \, dt + \int_{0}^{T} \left( P(u), L^{(t)} (y(v) - y(u)) \right) \, dt
\]
\[
= \int_{0}^{T} \left( P(u), \left( \frac{\partial}{\partial t} + L^{(t)} \right) (y(v) - y(u)) \right) \, dt
\]
\[
= \int_{0}^{T} (P(u), v - u) \, dt = (P(u), v - u)_{L^2(Q)}.
\]

Hence, (2.6) may be written as:
\[
(P(u) + Nu, v - u)_{L^2(Q)} \geq 0 \quad \text{for all } v \in U_{ad},
\]

which completes the proof.

REFERENCES


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Received May 5, 2003
Revised February 8, 2004

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