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TIME DEPENDENT DIRICHLET SPACE, MIXED DIRICHLET PROBLEM AND PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. In this paper we study the time dependent Dirichlet space which is generated by pseudo-differential operators. Also we find the set of inequalities defining an optimal control of a system governed by pseudo-differential operators with symbols defined in terms of conditionally exponential convex function.

1. Time dependent Dirichlet form on $\mathbb{R}^n$

Let us introduce certain function spaces on $\mathbb{R} \times \mathbb{R}^n$.

DEFINITION 1.1. A real valued function $a\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *conditionally exponential convex* if for any $x_1, \dots, x_n \in \mathbb{R}^n$ and $C_1, \dots, C_n \in \mathbb{R}$ we have

$$\sum_{j,k=1}^{n} \big[a(x_j) + a(x_k) - a(x_j + x_k)\big] C_j C_k \ge 0 . \tag{1.1}$$

LEMMA 1.1. *Let $a^2\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous conditionally exponential convex function. Then*

$$0 \le a^2(\xi) \le C_\alpha\big(1 + |\xi|^2\big) , \tag{1.2}$$

$$a^2(\xi) = C - Q(\xi) + \int\limits_{\mathbb{R}^n\setminus\{0\}} \left[1 - \exp(x,\xi) + \frac{(x,\xi)}{1+\|x\|^2}\right] \frac{1+\|x\|^2}{\|x\|^2}\, \mathrm{d}\mu(x) \tag{1.3}$$

where $C \ge 0$ is a constant, $Q\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous negative quadratic form on $\mathbb{R}^n$ and μ is a positive bounded measure on $\mathbb{R}^n \setminus \{0\}$, and

$$|a^2(\xi) - a^2(\eta)| \le 4a(\xi)a(\xi-\eta) + a^2(\xi-\eta) , \tag{1.4}$$

$$|a(\xi) - a(\eta)| \le a(\xi+\eta) . \tag{1.5}$$

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Remark. The estimate (1.2) can be found in [3], [7], [8] and (1.3) in [4]. We have taken (1.4) and (1.5) from [1], [2].

For any continuous conditionally exponential convex function $a^2\colon \mathbb{R}^n \to \mathbb{R}$ and for any $S \geq 0$, we introduce the Hilbert space

$$H^{a^2,S}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \ \|u\|_{S,a^2} < \infty\} \tag{1.6}$$

where

$$\|u\|^2_{S,a^2} = \int_{\mathbb{R}^n} \left(1 + a^2(\xi)\right)^{2S} |\tilde{u}(\xi)|^2 \, \mathrm{d}\xi \,, \tag{1.7}$$

where $\tilde{u}$ is Fourier transform of u.

Clearly $H^{a^2,0}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, and if we identify $[L^2(\mathbb{R}^n)]^*$ with $L^2(\mathbb{R}^n)$, we have (see [2], [6])

$$[H^{a^2,S}(\mathbb{R}^n)]^* = H^{a^2,-S}(\mathbb{R}^n)\,,$$

where

$$H^{a^2,-S}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \ \|u\|_{-S,a^2} < \infty\}$$

and the negative norm is given on $L^2(\mathbb{R}^n)$ by

$$\|u\|^2_{-S,a^2} = \int_{\mathbb{R}^n} \left(1 + a^2(\xi)\right)^{-2S} |\tilde{u}(\xi)|^2 \, \mathrm{d}\xi = \sup_{0 \neq v \in H^{a^2,S}(\mathbb{R}^n)} \frac{|(u,v)_0|}{\|v\|_{S,a^2}} \,.$$

Later we will often assume that a^2 also satisfies

$$a^2(\xi) \geq C_r |\xi|^r \,, \tag{1.8}$$

for some $r > 0$ and all $\xi \in \mathbb{R}^n$, $|\xi| \geq \sigma \geq 0$. In this case $H^{a^2,S}(\mathbb{R}^n)$ is continuously embedded in the usual Sobolev space $H^{Sr}(\mathbb{R}^n)$ and for $Sr > \frac{n}{2}$ we find $H^{a^2,S}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$ with a continuous embedding.

Now we formulate the main new results of this paper. We will introduce certain function spaces on $]0,T[\times \mathbb{R}^n \subset \mathbb{R} \times \mathbb{R}^n$.

Let $L_2(0,T,H^{a^2,S}(\mathbb{R}^n))$ denote the space of all measurable functions $t \mapsto f(t)\colon \,]0,T[\to H^{a^2,S}(\mathbb{R}^n)$ where the variable t denotes "time". We assume that $t \in \,]0,T[$, $T < \infty$, with Lebesgue measure $\mathrm{d}t$ on $]0,T[$ such that

$$\left(\int_0^T \|f(t)\|^2_{H^{a^2,S}(\mathbb{R}^n)} \, \mathrm{d}t \right)^{\frac{1}{2}} = \|f\|_{L_2(0,T,H^{a^2,S}(\mathbb{R}^n))}$$

and $L_2(0,T,H^{a^2,S}(\mathbb{R}^n))$ is endowed with the scalar product

$$(f,g)_{L_2(0,T,H^{a^2,S}(\mathbb{R}^n))} = \int_0^T \left(f(t), g(t)\right)_{H^{a^2,S}(\mathbb{R}^n)} \,,$$

which is a Hilbert space ([5], [6]).

Analogously, we define the spaces $L_2(0,T,L_2(\mathbb{R}^n))$ and $L_2(0,T,H^{a^2,-S}(\mathbb{R}^n))$ and then we have a chain in the form

$$L_2(0,T,H^{a^2,S}(\mathbb{R}^n)) \subseteq L_2(0,T,L_2(\mathbb{R}^n)) \subseteq L_2(0,T,H^{a^2,-S}(\mathbb{R}^n)) . \tag{1.9}$$

Now, let us define the continuous conditionally exponential convex function $a^2\colon \mathbb{R}^n \to \mathbb{R}$ by $a^2(\xi) = \sum_{j=1}^{n} a_j^2(\xi_j)$ ($\xi \in \mathbb{R}^n$, $\xi_j \in \mathbb{R}$) where $a_j^2\colon \mathbb{R} \to \mathbb{R}$ is a continuous conditionally exponential convex function, $1 \le j \le n$.

Further, let $b_j\colon\,]0,T[\, \times \mathbb{R}^n \to \mathbb{R}$, $(t,x) \mapsto b_j(t,x)$, $1 \le j \le n$, be a function satisfying the following conditions:

(i) b_j is independent of x_j;
(ii) $b_j(t,\cdot)$ is bounded and measurable;
(iii) $t \mapsto b_j(t,x)$ is a continuous function;
(iv) $b_j(t,x) \ge d_0 > 0$ for all $(t,x) \in\,]0,T[\, \times \mathbb{R}^n$ and $1 \le j \le n$.

On $C_0^\infty(\mathbb{R}^n)$ we consider the family of psedo-differential operators

$$L^{(t)}(x,\mathrm{D})u(x) = \sum_{j=1}^{n} b_j(t,x) a_j^2(\mathrm{D}_j)u(x) . \tag{1.10}$$

We can associate with $L^{(t)}(x,\mathrm{D})$ the bilinear form

$$\begin{aligned} E^{(t)}(u,v) &= \int_{\mathbb{R}^n} L^{(t)}(x,\mathrm{D})u(x) \cdot v(x)\ \mathrm{d}x \\ &= \sum_{j=1}^{n} \big(b_j(t,\cdot)a_j(\mathrm{D}_j)u, a_j(\mathrm{D}_j)v\big)_0 . \end{aligned} \tag{1.11}$$

Now using (i) $\implies$ (iv) we get, as in [2]:

PROPOSITION 1.1. *For all $u,v \in H^{a^2,\frac12}(\mathbb{R}^n)$,*

$$|E^{(t)}(u,v)| \le C\|u\|_{\frac12,a^2}\|v\|_{\frac12,a^2} , \tag{1.12}$$

$$E^{(t)}(u,u) \ge d_0\|u\|^2_{\frac12,a^2} - d_0\|u\|_0^2 . \tag{1.13}$$

Proof. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $H^{a^2,\frac12}(\mathbb{R}^n)$, it is sufficient to prove (1.12) and (1.13) for $u,v \in C_0^\infty(\mathbb{R}^n)$. It follows that

$$\begin{aligned} |E^{(t)}(u,v)| &= \big|\big(L^{(t)}(x,\mathrm{D})u,v\big)_0\big| \\ &= \Big|\Big(\sum_{j=1}^{n} b_j(t,\cdot)a_j^2(\mathrm{D}_j)u, v\Big)_0\Big| \\ &\le C\sum_{j=1}^{n} \|a_j(\mathrm{D}_j)u\|_0 \|a_j(\mathrm{D}_j)v\|_0 , \end{aligned}$$

but $\|a_j(\mathrm{D}_j)u\|_0 \le C\|u\|_{\frac{1}{2},a^2}$ for all $u \in C_0^\infty(\mathbb{R}^n)$.

To prove (1.13), by (1.11) we find

$$\begin{aligned} E^{(t)}(u,u) &= \big(L^{(t)}(x,\mathrm{D})u, u\big)_0 \\ &= \sum_{j=1}^{n} \big(b_j(t,x)a_j(\mathrm{D}_j)u, a_j(\mathrm{D}_j)u\big)_0 \\ &= \sum_{j=1}^{n} \int_{\mathbb{R}^n} b_j(t,x)a_j(\mathrm{D}_j)u \cdot a_j(\mathrm{D}_j)u \,\mathrm{d}x \\ &\ge d_0 \sum_{j=1}^{n} \int_{\mathbb{R}^n} a_j(\mathrm{D}_j)u \cdot a_j(\mathrm{D}_j)u \,\mathrm{d}x \\ &= d_0 \sum_{j=1}^{n} \|a_j(\mathrm{D}_j)u\|_0^2 , \end{aligned}$$

i.e. $E^{(t)}(u,u) \ge 0$.

But

$$\begin{aligned} E^{(t)}(u,u) &\ge d_0 \int_{\mathbb{R}^n} a_j^2(\xi_j)|\tilde{u}(\xi)|^2 \,\mathrm{d}\xi \\ &= d_0 \int_{\mathbb{R}^n} \Big(1 + \sum_{j=1}^{n} a_j^2(\xi_j)\Big)|\tilde{u}(\xi)|^2 \,\mathrm{d}\xi - d_0 \int_{\mathbb{R}^n} |\tilde{u}(\xi)|^2 \,\mathrm{d}\xi \\ &= d_0\|u\|_{\frac{1}{2},a^2} - d_0\|u\|_0^2 . \end{aligned}$$

From (1.12) and (1.13) it follows that $E^{(t)}$, with domain $H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$, is a closed symmetric bilinear form on $L^2(\mathbb{R}^n)$.

Clearly for all $u,v \in H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$ the function $t \mapsto E^{(t)}(u,v)$ is measurable. As in [1; Theorem 2.1], we find, for $u,v \in H^{a^2,\frac{1}{2}}(\mathbb{R}^n)$,

$$E^{(t)}(u,v) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \big(u(x+y) - u(x)\big)\big(v(x+y) - v(x)\big) \cdot \sum_{j=1}^{n} b_j(t,x)\, \mu_j(\mathrm{d}y)\,\mathrm{d}x , \tag{1.14}$$

where μ_j is the image of $\tilde{\mu}_j$ under the mapping

$$T_j\colon \mathbb{R} \to \mathbb{R}^n , \qquad \xi_j \mapsto (0,\dots,0,\xi_j,0,\dots,0) ,$$

i.e. ξ_j is in the jth position. Thus

$$E(u,v)=\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(u(t,x+y)-u(t,x)\big)\big(v(t,x+y)-v(t,x)\big)\cdot \cdot\sum_{j=1}^{n} b_j(t,x)\,\mu_j(\mathrm{d}y)\,\mathrm{d}x\,\mathrm{d}t-\int_{\mathbb{R}}\int_{\mathbb{R}^n}\frac{\partial u}{\partial t}(t,x)\cdot v(t,x)\,\mathrm{d}x\,\mathrm{d}t \tag{1.15}$$

defined for

$$u\in F=\Big\{w\in L^2\big(0,T,H^{a^2,\frac{1}{2}}(\mathbb{R}^n)\big):\ \tfrac{\partial w}{\partial t}\in L^2\big(0,T,H^{a^2,-\frac{1}{2}}(\mathbb{R}^n)\big)\Big\},$$

and $v\in L^2\big(0,T,H^{a^2,\frac{1}{2}}(\mathbb{R}^n)\big)$ gives a time dependent Dirichlet form. So we have defined parabolic Dirichlet space (F,E). □

2. Formulation of the control problem

Analogous to (1.9), we have a chain of the form

$$L_2\big(0,T,H_0^{a^2,1}(\mathbb{R}^n)\big)\subseteq L_2\big(0,T,L_2(\mathbb{R}^n)\big)=L_2(Q)\subseteq L_2\big(0,T,H_0^{a^2,-1}(\mathbb{R}^n)\big) \tag{2.1}$$

where $H_0^{a^2,S}(\mathbb{R}^n)$ is the subset of $H^{a^2,S}(\mathbb{R}^n)$ of all functions which vanish on the boundary Γ of $\mathbb{R}^n$.

It follows from (1.13) that the continuous bilinear form is coercive, and assume that the function

$$t\mapsto E^{(t)}(y,\phi)\qquad \text{is measurable on}\quad]0,T[\,. \tag{2.2}$$

We can apply the following theorem of Lions [5], [6].

THEOREM 2.1. *Assuming* (1.13) *and* (2.2) *hold, then if* f *is given in* $L_2\big(0,T,H^{a^2,-\frac{1}{2}}(\mathbb{R}^n)\big)$ *and* $y_0\in L_2(\mathbb{R}^n)$*, there exists a unique* $y\in\Big\{\nu\in L_2\big(0,T,H^{a^2,\frac{1}{2}}(\mathbb{R}^n)\big):\ \frac{\partial\nu}{\partial t}\in L_2\big(0,T,H^{a^2,-\frac{1}{2}}(\mathbb{R}^n)\big)\Big\}$ *satisfying*

$$\begin{aligned}
&\frac{\partial y}{\partial t}+L^{(t)}y=f && \textit{in } Q\,,\\
&Q=\,]0,T[\,\times\Omega\,, && \Omega \textit{ is an open set of } \mathbb{R}^n\,,\\
&y\big|_{]0,T[\times\Gamma}=0\,, && \Gamma \textit{ is boundary of } \Omega\,,\\
&y(0,x)=y_0(x) && \textit{in } \mathbb{R}^n\,.
\end{aligned}$$

The operator

$$\frac{\partial}{\partial t}+L^{(t)}\in\mathcal{L}\Big(L_2\big(0,T,H^{a^2,\frac{1}{2}}(\mathbb{R}^n)\big),L_2\big(0,T,H^{a^2,-\frac{1}{2}}(\mathbb{R}^n)\big)\Big)\,. \tag{2.3}$$

The problem which is defined by the above theorem is known as mixed Dirichlet problem.

Now, we formulate the control problem. Thus the space $L_2(0,T,L^2(\mathbb{R}^n))$, being the space of controls is given. A system which is governed by the operator $L^{(t)} + \frac{\partial}{\partial t}$ is given by (2.3) or by mixed Dirichlet problem. Let f and y_0 with $f \in L_2(0,T,H^{a^2,-\frac{1}{2}}(\mathbb{R}^n))$, $y_0 \in L_2(\mathbb{R}^n)$ be given. We assume that (1.13) and (2.2) hold; then for the control $u \in L_2(0,T,L_2(\mathbb{R}^n))$ the state of the system $y(u)$ which depends on x, t will be denoted by $y(t,x;u)$ and is given by the solution of

$$\begin{aligned} \frac{\partial y(u)}{\partial t} + L^{(t)}y(u) = f + u \qquad & \text{in } Q\,, \\ y(u)\big|_{\Gamma'} = 0\,, \qquad & \Gamma' =]0,T[\times\Gamma = \text{Lateral boundary of } Q\,, \\ y(0,x;u) = y_0(u) \qquad & \text{in } \mathbb{R}^n\,, \\ y(u) \in L_2(0,T,H^{a^2,\frac{1}{2}}(\mathbb{R}^n))\,. & \end{aligned}$$

The observation $Z(u)$ is given by:

$$Z(u) = y(u)\,,$$

N is given as $N \in \mathcal{L}\big(L_2(0,T,L_2(\mathbb{R}^n)), L_2(0,T,L_2(\mathbb{R}^n))\big)$

$$(Nu,u)_{L_2(0,T,L_2(\mathbb{R}^n))} \geq \gamma\|u\|^2_{L_2(0,T,L_2(\mathbb{R}^n))}\,, \qquad \gamma > 0\,. \tag{2.4}$$

Let $L_2(0,T,L_2(\mathbb{R}^n)) = L_2(Q)$.

The cost function $J(u)$ is given by

$$\begin{aligned} J(u) &= \|y(u) - Z_d\|^2_{L_2(Q)} + (Nu,u)_{L_2(Q)} \\ &= \int_Q (y(u) - Z_d)^2 \, \mathrm{d}p(x)\,\mathrm{d}t + (Nu,u)_{L_2(Q)} \end{aligned} \tag{2.5}$$

where Z_d is a given element in $L_2(Q)$.

Let U_{ad} (set of admissible controls) be a closed convex subset of $L_2(Q)$. We seek $\inf J(v)$, $v \in U_{\mathrm{ad}}$.

THEOREM 2.2. *We assume that* (1.13) *and* (2.1) *as well as* (2.4) *hold. The cost function is given by* (2.5). *The optimal control* u *is characterized by the following system of equations and inequalities*

$$
\begin{aligned}
\frac{\partial y(u)}{\partial t} + L^{(t)} y(u) &= f + u && \text{in } Q\,,\\
y(u)\big|_{\Gamma'} &= 0\,, && \Gamma' =]0,T[\times\Gamma\\
& && \quad = \text{Lateral boundary of } Q\,,\\
y(0,x;u) &= y_0(x) && \text{in } \mathbb{R}^n\,,\\
-\frac{\partial P(u)}{\partial t} + L^{(t)} P(u) &= y(u) - Z_d && \text{in } Q\,,\\
P(u)\big|_{\Gamma'} &= 0\,, && \Gamma' =]0,T[\times\Gamma\,,\\
P(T,x;u) &= 0 && \text{in } \mathbb{R}^n\,,\\
u &\in U_{\mathrm{ad}}\,,\\
\big(P(u) + Nu, v - u\big)_{L_2(Q)} &\ge 0 && \text{for all } v \in U_{\mathrm{ad}}\,,
\end{aligned}
$$

i.e.,

$$
\int_Q \big(P(u) + Nu\big)(v-u)\,\mathrm{d}p(x)\,\mathrm{d}t \ge 0 \qquad \text{for all} \quad v \in U_{\mathrm{ad}}\,,
$$

$$
y(u), P(u) \in L_2\big(0,T,H^{a^2,\frac{1}{2}}(\mathbb{R}^n)\big)\,.
$$

Proof. The control $u \in U_{\mathrm{ad}}$ is optimal if and only if

$$
J'(u)(v-u) \ge 0 \qquad \text{for all} \quad v \in U_{\mathrm{ad}}\,,
$$

that is

$$
\big(y(u) - Z_d, y(v) - y(u)\big)_{L_2(Q)} + (Nu, v-u)_{L_2(Q)} \ge 0\,. \tag{2.6}
$$

(2.6) may be written as:

$$
\int_0^T \big(y(u) - Z_d, y(v) - y(u)\big)_{L_2(\mathbb{R}^n)}\,\mathrm{d}t + (Nu, v-u)_{L_2(Q)} \ge 0\,.
$$

We introduce the adjoint state $P(u)$ by

$$
\begin{aligned}
-\frac{\partial}{\partial t} P(u) + L^{(t)} P(u) &= y(u) - Z_d\,,\\
P(T,u) &= 0\,,\\
P(u) &\in L_2\big(0,T,H^{a^2,\frac{1}{2}}(\mathbb{R}^n)\big)\,.
\end{aligned}
$$

Then

$$\int_0^T \big(y(u) - Z_d, y(v) - y(u)\big)\ \mathrm{d}t$$

$$= \int_0^T \Big(-\frac{\partial}{\partial t} P(u), y(v) - y(u)\Big)\ \mathrm{d}t + \int_0^T \big(L^{(t)} P(u), y(v) - y(u)\big)\ \mathrm{d}t$$

$$= \int_0^T \Big(P(u), \frac{\partial}{\partial t}\big(y(v) - y(u)\big)\Big)\ \mathrm{d}t + \int_0^T \Big(P(u), L^{(t)}\big(y(v) - y(u)\big)\Big)\ \mathrm{d}t$$

$$= \int_0^T \Big(P(u), \Big(\frac{\partial}{\partial t} + L^{(t)}\Big)\big(y(v) - y(u)\big)\Big)\ \mathrm{d}t$$

$$= \int_0^T \big(P(u), v - u\big)\ \mathrm{d}t = \big(P(u), v - u\big)_{L_2(Q)}.$$

Hence, (2.6) may be written as:

$$\big(P(u) + Nu, v - u\big)_{L_2(Q)} \geq 0 \qquad \text{for all} \quad v \in U_{\mathrm{ad}}\,,$$

which completes the proof. □

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