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THE HEIGHT OF
THE FIRST STIEFEL-WHITNEY CLASS
OF ANY NONORIENTABLE
REAL FLAG MANIFOLD

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ABSTRACT. Using suitable fiberings, we calculate the height of the first Stiefel-Whitney class of any nonorientable real flag manifold $O(n_1 + \cdots + n_q)/O(n_1) \times \cdots \times O(n_q)$.

1. Introduction

Let $n_1, \ldots, n_q$ ($q \geq 2$) be fixed positive integers, and let $F(n_1, \ldots, n_q)$ be the real flag manifold consisting of all $q$-tuples $(S_1, \ldots, S_q)$ of mutually orthogonal vector subspaces in $\mathbb{R}^n$, where $n = n_1 + \cdots + n_q$ and $\dim(S_i) = n_i$. As a homogeneous space, we have

$$F(n_1, \ldots, n_q) \cong O(n)/O(n_1) \times \cdots \times O(n_q).$$

In particular, $F(n_1, n_2)$ is the Grassmann manifold of all $n_1$-dimensional vector subspaces in $\mathbb{R}^n$.

Over the manifold $F(n_1, \ldots, n_q)$, there are $q$ canonical vector bundles $\gamma_1, \ldots, \gamma_q$ with $\dim(\gamma_i) = n_i$. They are characterized by the fact that the fiber of $\gamma_i$ over $(S_1, \ldots, S_q) \in F(n_1, \ldots, n_q)$ is the vector space $S_i$. The direct sum $\bigoplus_{i=1}^q \gamma_i$ is the trivial $n$-dimensional vector bundle.

By Korbas [3], the manifold $F(n_1, \ldots, n_q)$ is nonorientable, hence has its first Stiefel-Whitney class $w_1(F(n_1, \ldots, n_q)) \in H^1(F(n_1, \ldots, n_q);\mathbb{Z}_2)$ non-zero, precisely when not all of the numbers $n_1, \ldots, n_q$ have the same parity.

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In 2000, Ilori and Ajayi [2] calculated the height of \( w_1(F(n_1, \ldots, n_q)) \) (denoted \( \text{height}(F(n_1, \ldots, n_q)) \)) for some of those flag manifolds \( F(n_1, \ldots, n_q) \) which are nonorientable. (Recall that \( \text{height}(F(n_1, \ldots, n_q)) \) is the largest \( c \) such that \( w_1^c(F(n_1, \ldots, n_q)) \in H^*(F(n_1, \ldots, n_q); \mathbb{Z}_2) \) does not vanish.) Their result is the following.

**Proposition 1.1.** (Ilori, Ajayi [2]) Suppose that \( \prod_{i=1}^{q-1} n_i \) is odd, \( n-k \) is even, where \( k = \sum_{i=1}^{q-1} n_i \), and \( 4 \leq 2k \leq n \) with \( 2^s < n \leq 2^{s+1} \). Then

\[
\text{height}(w_1(F(n_1, \ldots, n_{q-1}, n-k))) = \begin{cases} 
2^{s+1} - 2 & \text{if } k = 2 \text{ or } k = 3 \text{ and } n = 2^s + 1, \\
2^{s+1} - 1 & \text{otherwise}.
\end{cases}
\]

Our aim here is to show that a slight modification of the approach used by Ilori and Ajayi leads in fact to the following complete result covering the height of the first Stiefel-Whitney class of any nonorientable real flag manifold.

**Theorem 1.2.** Let \( F(n_1, \ldots, n_q) \), for \( q \geq 2 \), be any nonorientable real flag manifold; hence not all of \( n_1, \ldots, n_q \) have the same parity. Letting \( p \) be the sum of all even numbers among \( n_1, \ldots, n_q \), put \( k = \min\{p, n-p\} \). If \( s \) is the uniquely determined integer such that \( 2^s < n \leq 2^{s+1} \), then we have

\[
\text{height}(w_1(F(n_1, \ldots, n_q))) = \begin{cases} 
n - 1 & \text{if } k = 1, \\
2^{s+1} - 2 & \text{if } k = 2 \text{ or } k = 3 \text{ and } n = 2^s + 1, \\
2^{s+1} - 1 & \text{otherwise}.
\end{cases}
\]

The knowledge of \( \text{height}(F(n_1, \ldots, n_q)) \) is useful for several reasons. For instance, Ilori and Ajayi [2] show how it can be used for deriving a result on immersions of real flag manifolds in Riemannian manifolds. Of course, \( \text{height}(F(n_1, \ldots, n_q)) \) also gives a lower bound for the cup-length. Results of our study of the cup-length for real flag manifolds will be postponed to a forthcoming paper [4].

**2. Proof of Theorem 1.2**

We intend to make the proof of Theorem 1.2 as selfcontained as possible.

Let \( w_i(\gamma_j) \) be the \( i \)th Stiefel-Whitney class of the canonical vector bundle \( \gamma_j \) over \( F(n_1, \ldots, n_q) \). Then according to Borel [1; Theorem 11.1], we have

\[
H^*(F(n_1, \ldots, n_q); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\gamma_1), \ldots, w_{n_1}(\gamma_1), \ldots, w_1(\gamma_q), \ldots, w_{n_q}(\gamma_q)]/I,
\]

where \( I = (...) \) is the ideal generated by the relations...
where the ideal $I$ is given by the identity

$$
\prod_{j=1}^{q} (1 + w_1(\gamma_j) + \cdots + w_{n_j}(\gamma_j)) = 1.
$$

Let $\sigma$ be any permutation of the set $\{1, \ldots, q\}$. The map $\tilde{\sigma}: F(n_1, \ldots, n_q) \rightarrow F(n_{\sigma(1)}, \ldots, n_{\sigma(q)})$ given by $\tilde{\sigma}(S_1, \ldots, S_q) = (S_{\sigma(1)}, \ldots, S_{\sigma(q)})$ is a diffeomorphism. Thus we may and shall suppose that there is $t \in \{1, \ldots, q\}$ such that $n_1, \ldots, n_t$ are odd, and $n_{t+1}, \ldots, n_q$ are even. Then the map

$$
\pi: F(n_1, \ldots, n_q) \rightarrow F(n_1, \ldots, n_t, n_{t+1} + \cdots + n_q),
$$

$$
\pi(S_1, \ldots, S_q) = (S_1, \ldots, S_t, S_{t+1} \oplus \cdots \oplus S_q),
$$

defines a smooth fiber bundle (cf. [5; 7.4]) with fiber $F(n_{t+1}, \ldots, n_q)$. We obviously have $\gamma_i = \pi^*(\gamma_i)$ for $i = 1, \ldots, t$.

For the inclusion of the fiber, $i: F(n_{t+1}, \ldots, n_q) \hookrightarrow F(n_1, \ldots, n_q)$, one has $\gamma_j = i^*(\gamma_j)$, $j = t+1, \ldots, q$, and the classes

$$
w_m(\gamma_j) = i^*(w_m(\gamma_j)) \quad \text{for} \quad j = t+1, \ldots, q, \ m = 1, \ldots, n_j
$$
generate $H^*(F(n_{t+1}, \ldots, n_q))$ as a vector space over $\mathbb{Z}_2$. Choosing an appropriate basis we see that the assumptions of the Leray-Hirsch theorem (see, e.g. [7]) are satisfied. This implies that $\pi^*$ is a monomorphism.

From the K o r b a ś formula ([3; Theorem 1.1]) for the first Stiefel-Whitney class of $F(n_1, \ldots, n_q)$, we obtain

$$
\pi^*(w_1(F(n_1, \ldots, n_t, n_{t+1} + \cdots + n_q)))
\begin{align*}
&= \pi^*(w_1(\gamma_1) + \cdots + w_1(\gamma_t)) \\
&= \pi^*(w_1(\gamma_1)) + \cdots + \pi^*(w_1(\gamma_t)) \\
&= w_1(\gamma_1) + \cdots + w_1(\gamma_t) \\
&= w_1(F(n_1, \ldots, n_q)).
\end{align*}
$$

It is needed to analyse two cases.

**Case of $k = 1$:**

Now certainly $n_1 = 1$ and $n_2, \ldots, n_q$ are even, hence $t = 1$. The relevant fiber bundle (see (1)) is now

$$
\pi: F(1, n_2, \ldots, n_q) \rightarrow F(1, n_2 + \cdots + n_q);
$$

its base is the (nonorientable) $(n-1)$-dimensional real projective space $F(1, n_2 + \cdots + n_q) = \mathbb{RP}^{n-1}$. As it is well known,

$$
\text{height}(w_1(\mathbb{RP}^{n-1})) = n - 1,
$$

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and by (2), \( \pi^*(w_1(\mathbb{R}P^{n-1})) = w_1(F(1,n_2,\ldots,n_q)) \). Since \( \pi^* \) is a ring monomorphism, we have

\[
0 \neq \pi^*(w_1^{n-1}(\mathbb{R}P^{n-1})) = w_1^{n-1}(F(1,n_2,\ldots,n_q)),
\]

while

\[
0 = \pi^*(w_1^n(\mathbb{R}P^{n-1})) = w_1^n(F(1,n_2,\ldots,n_q)).
\]

This proves the theorem in case of \( k = 1 \).

Case of \( k \geq 2 \):
From (2) and the fact that \( \pi^* \) is a monomorphism, we know that \( w_1^c(F(n_1,\ldots,n_q)) = 0 \) if and only if \( w_1^c(F(n_1,\ldots,n_t,n_{t+1} + \cdots + n_q)) = 0 \). Therefore the height of \( w_1(F(n_1,\ldots,n_t,n_{t+1} + \cdots + n_q)) \) is the same as the height of \( w_1(F(n_1,\ldots,n_q)) \).

We know that now \( w_1(F(n_1,\ldots,n_t,n_{t+1} + \cdots + n_q)) \neq 0 \). Further consider the fiber bundle

\[
p: F(n_1,\ldots,n_t,n_{t+1} + \cdots + n_q) \to F(n_1 + \cdots + n_t,n_{t+1} + \cdots + n_q),
\]

\[
p(S_1,\ldots,S_{t+1}) = (S_1 \oplus \cdots \oplus S_t,S_{t+1}),
\]

with fiber \( F(n_1,\ldots,n_t) \). Of course, its base space is nothing but the Grassmann manifold \( F(n-p,p) \). In addition to this, the K o r b a s formula (cf. [3]) yields

\[
\begin{align*}
\pi^*(w_1(F(n_1,\ldots,n_t,n_{t+1} + \cdots + n_q))) &= w_1^1 = w_1(F(n_1,\ldots,n_t,n_{t+1} + \cdots + n_q)), \\
\pi^*(w_1(F(n_1,\ldots,n_t,n_{t+1} + \cdots + n_q))) &= w_1^1 = w_1(F(n_1,\ldots,n_t,n_{t+1} + \cdots + n_q)),
\end{align*}
\]

Note that for the Grassmann manifolds the Whitney sum of their two canonical vector bundles is trivial, hence their first Stiefel-Whitney classes coincide. The Leray-Hirsch theorem now again applies, and it implies that the height of \( w_1(F(n_1,\ldots,n_t,n_{t+1} + \cdots + n_q)) \) coincides with the height of \( w_1(\gamma_1) = w_1(\gamma_2) \in H^*(F(n-p,p)) \) (\( \gamma_1 \) is the \( (n-p) \)-plane bundle over \( F(n-p,p) \)). But the height of \( w_1(\gamma_1) = w_1(\gamma_2) \in H^*(F(n-p,p)) \) is known (St o n g [6]):

\[
\text{height}(w_1(\gamma_1)) = \begin{cases} 
  n - 1 & \text{if } k = 1, \\
  2^{s+1} - 2 & \text{if } k = 2 \text{ or } \\
  2^{s+1} - 1 & \text{if } k = 3 \text{ and } n = 2^s + 1, \\
  2^{s+1} - 1 & \text{otherwise.}
\end{cases}
\]

This completes the proof of Theorem 1.2.
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