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SOME AXIOMATIZATIONS OF B-ALGEBRAS

ANDRZEJ WALENDZIAK

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ABSTRACT. Some systems of axioms defining a B-algebra are given with a proof of the independence of the axioms. In addition, we obtain a simplified axiomatization of commutative B-algebras.

1. Introduction

B-algebras have been introduced by J. Neggers and H. S. Kim in [4]. They defined a B-algebra as an algebra \((A, *, 0)\) of type \((2, 0)\) satisfying the following axioms:

A1. \(x * x = 0\),
A2. \(x * 0 = x\),
A3. \((x * y) * z = x * (z * (0 * y))\).

We will denote by \(B\) the class of all B-algebras. In [1], J. R. Cho and H. S. Kim proved that every B-algebra is a quasigroup. M. Kondoo and Y. B. Jun [3] showed that the class \(B\) is equivalent in one sense to the class of groups. In [2], Y. B. Jun, E. H. Roh and H. S. Kim introduced the notion of \(BH\)-algebras, which is a generalization of \(BCH/BCI/BCK\)-algebras. Moreover, \(B\) is a proper subclass of the class of \(BH\)-algebras (cf. [4; Lemma 2.9]). For another useful generalization of B-algebras see [6].

2. Some axiomatizations of B-algebras

THEOREM 2.1. Let \((A, -, +, 0)\) be an algebra of type \((2, 2, 0)\) satisfying the following axioms:

B1. \(x - x = 0\),
B2. \(x - 0 = x\),
B3. \((x - y) - z = x - (z + y)\),
B4. \(x + y = x - (0 - y)\).

Then \((A, -, 0)\) is a B-algebra.

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301
Conversely, if \((A, -, 0) \in \mathcal{B}\), and if we define \(x + y\) by \(x \ast (0 \ast y)\), then \((A, -, +, 0)\) obeys the equations B1-B4.

Proof. Straightforward.

In [6], J. Neggers and H. S. Kim introduced the notion of \(\beta\)-algebras. They defined a \(\beta\)-algebra as an algebra \((A, -, +, 0)\) of type \((2, 2, 0)\) that obeys B2, B3, and the following axiom:

\[(0 - x) + x = 0.\]

It is easy to verify that if an algebra \((A, -, +, 0)\) satisfies B1–B4, then it is a \(\beta\)-algebra.

**Theorem 2.2.** Let \(A = (A, *, 0)\) be an algebra of type \((2, 0)\). Then \(A \in \mathcal{B}\) if and only if \(A\) obeys the laws:

- C1. \(x \ast x = 0\),
- C2. \(0 \ast (0 \ast x) = x\),
- C3. \((x \ast z) \ast (y \ast z) = x \ast y\).

Proof. Suppose that \(A\) is a \(B\)-algebra. For each \(x \in A\) we have \(0 \ast (0 \ast x) = x\) (see [4; Lemma 2.9]). Consequently, C2 is valid in \(A\). By A3 we obtain

\[(x \ast z) \ast (y \ast z) = x \ast [(y \ast z) \ast (0 \ast z)] = x \ast [y \ast ((0 \ast z) \ast (0 \ast z))].\]

Hence applying A1 and A2 we get C3.

Conversely, assume that C1–C3 hold in \(A\). Then we have

\[x = 0 \ast (0 \ast x) = (x \ast x) \ast (0 \ast x) = x \ast 0.\]

From this and from C3 we deduce that

\[(x \ast y) \ast (0 \ast y) = x.\] (1)

Combining (1) with C3 we get

\[x \ast (z \ast (0 \ast y)) = [(x \ast y) \ast (0 \ast y)] \ast [(0 \ast y) \ast x] = (x \ast y) \ast z,\]

i.e., A3 holds. Therefore \(A \in \mathcal{B}\). \(\square\)

**Lemma 2.3.** Let \((A, *, 0)\) be an algebra of type \((2, 0)\) obeying the following laws:

- D1. \(x \ast x = 0\),
- D2. \(x \ast [(0 \ast y) \ast z] = (0 \ast x) \ast z\).

Then:

- (i) \(x \ast 0 = x\),
- (ii) \(0 \ast (0 \ast x) = x\),
- (iii) \(0 \ast x = 0 \ast y \implies x = y\),
- (iv) \((x \ast y) \ast (0 \ast y) = x\),
- (v) \(x \ast y = 0 \ast (y \ast x)\).
SOME AXIOMATIZATIONS OF $B$-ALGEBRAS

Proof.
(i): To obtain (i), substitute $x$ for $y$ in D2 and then use D1.
(ii): Substituting $x = 0$, $y = x$, and $z = 0$, D2 becomes
$$0 \ast \{[(0 \ast x) \ast 0] \ast [(0 \ast 0) \ast 0]\} = x.$$ Applying (i) we obtain (ii).
(iii) follows from (ii).
(iv): Let $a, b \in A$. Using D2 with $x = 0$, $y = 0 \ast a$, $z = b$ we have
$$0 \ast \{[(0 \ast (0 \ast a)) \ast b] \ast [(0 \ast 0) \ast b]\} = 0 \ast a.$$ Hence applying (i) and (ii) we conclude that
$$0 \ast [(a \ast b) \ast (0 \ast b)] = 0 \ast a.$$ That $(a \ast b) \ast (0 \ast b) = a$ follows from (iii).
(v): Let $a, b \in A$. Substituting $x = a$, $y = 0 \ast (b \ast a)$, $z = 0$ in D2 we deduce that
$$a \ast \{[(0 \ast (0 \ast (b \ast a))) \ast 0] \ast [(0 \ast a) \ast 0]\} = 0 \ast (b \ast a).$$ Then $a \ast [(b \ast a) \ast (0 \ast a)] = 0 \ast (b \ast a)$. By (iv), $a \ast b = 0 \ast (b \ast a)$, verifying (v).

THEOREM 2.4. An algebra $A = (A, \ast, 0)$ of type $(2,0)$ is a $B$-algebra if and only if the equations D1 and D2 are valid in $A$.

Proof. Let $A$ satisfy D1 and D2. C1 holds in $A$ by D1. From Lemma 2.3(iii) we conclude that $A$ obeys C2. If we let $x = a \ast c$, $y = a \ast b$ and $z = 0 \ast a$ in D2, then we have
$$(a \ast c) \ast \{[(0 \ast (a \ast b)) \ast (0 \ast a)] \ast [(0 \ast (a \ast c)) \ast (0 \ast a)]\} = a \ast b.$$ By Lemma 2.3,
$$(0 \ast (a \ast b)) \ast (0 \ast a) = (b \ast a) \ast (0 \ast a) = b,$$ and similarly, $(0 \ast (a \ast c)) \ast (0 \ast a) = c$. Consequently,
$$(a \ast c) \ast (b \ast c) = a \ast b.$$ This shows that $A$ also satisfies C3. Then $A \in B$ by Theorem 2.2.

For the converse, suppose that $A$ is a $B$-algebra. Obviously D1 is valid in $A$. From Theorem 2.2 we see that C3 holds in $A$, and therefore
$$[(0 \ast y) \ast z] \ast [(0 \ast x) \ast z] = (0 \ast y) \ast (0 \ast x).$$
It follows that
\[ x \ast \left\{ \left[ (0 \ast y) \ast z \right] \ast \left[ (0 \ast x) \ast z \right] \right\} = x \ast \left[ (0 \ast y) \ast (0 \ast x) \right] \quad \text{(by (2))} \]
\[ = (x \ast x) \ast (0 \ast y) \quad \text{(by A3)} \]
\[ = 0 \ast (0 \ast y) \quad \text{(by A1)} \]
\[ = y \quad \text{(by C2)}, \]
proving D2. The proof is finished. \( \square \)

Following J. Neggers and H. S. Kim [4] (see also [1]) we give:

**Definition 2.5.** A B-algebra \( (A, \ast, 0) \) is said to be 0-commutative if \( a \ast (0 \ast b) = b \ast (0 \ast a) \) for all \( a, b \in A \).

In [1], J. R. Cho and H. S. Kim showed that a B-algebra \( A = (A, \ast, 0) \) is 0-commutative if and only if the equation
\[ C2' \quad y \ast (y \ast x) = x \]
holds in \( A \).

From this and from Theorem 2.2 we have:

**Corollary 2.6.** An algebra \( (A, \ast, 0) \) of type \((2,0)\) is a 0-commutative B-algebra if and only if it obeys the laws C1, C2', and C3.

### 3. Proof of the independence of the axioms

The independence of the axioms A1, A2, and A3 was proved by J. Neggers and H. S. Kim in [4].

**Theorem 3.1.** The axioms B1–B4 are independent, i.e., none of them can be deduced from the others.

**Proof.** We are going to give some examples of algebras in which only three of the axioms hold.

Let \( A = \{0, 1\} \). Define binary operations \( \ominus \) and \( \oplus \) on \( A \) as follows:
\[ x \ominus y = x \quad \text{for all } x, y \in A, \]
\[ x \oplus y = 0 \quad \text{for all } x, y \in A. \]

Then \( (A, \ominus, \ominus, 0) \) fulfils the axioms B2–B4, but not B1, since \( 1 \ominus 1 = 1 \neq 0 \). (Independence of B1.)

It is easily seen that \( (A, \oplus, \ominus, 0) \) satisfies B1, B3, and B4, but not B2 (independence of B2).
SOME AXIOMATIZATIONS OF $B$-ALGEBRAS

Now we define the binary operations $-$ and $+$ on $A$ by the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x - y$</th>
<th>$x + y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
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<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The equations $B_1$, $B_2$, and $B_4$ are valid in $(A, -, +, 0)$, but $B_3$ does not hold because $(1 - 1) - 0 = 0$, while $1 - (0 + 1) = 1$. (Independence of $B_3$.)

Finally, let $A = (A, -, +, 0)$ be the algebra, where $-$ is given in the above table and $+$ is defined by

$$x + y = \begin{cases} 0 & \text{if } x = y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

(3)

Obviously, $B_1$–$B_3$ hold in $A$, while $B_4$ does not (independence of $B_4$).

**THEOREM 3.2.** *The system of axioms C1–C3 is independent.*

**Proof.** Let $A = \{0, 1\}$. We use the table below in order to define $\ast$, $\circ$, and $\cdot$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \ast y$</th>
<th>$x \circ y$</th>
<th>$x \cdot y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>1</td>
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<td>0</td>
</tr>
</tbody>
</table>

We can see that the algebra $(A, \ast, 0)$ satisfies C2–C3, but not C1. The axioms C1 and C3 hold in $(A, \circ, 0)$, while C2 does not. It is evident that $(A, \ast, 0)$ obeys C1 and C2. The axiom C3 does not hold because $(0 \ast 1) \ast (1 \ast 1) = 0$, while $0 \ast 1 = 1$. \(\square\)

**Remark 3.3.** It is easy to see that the axiom system D1–D2 of $B$-algebras is independent.
REFERENCES


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