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CONVERGENCES PRESERVING THE FIXED POINT PROPERTY

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ABSTRACT. Coincidence theorems for very general (non-Hausdorff) topological spaces X and Y are proved. E.g., if $\{f_\gamma\}_{\gamma\in\Gamma}$, $\{g_\gamma\}_{\gamma\in\Gamma}$ are two nets of functions from X to Y satisfying $(\forall\,\gamma\in\Gamma)(\exists\,x_\gamma\in X)\big(f_\gamma(x_\gamma)=g_\gamma(x_\gamma)\big)$, $\{f_\gamma\}_{\gamma\in\Gamma}$ converges strongly to f and $\{g_\gamma\}_{\gamma\in\Gamma}$ converges strongly to g, then (under certain conditions posed on f and g) the equation $f(x)=g(x)$ has a solution. The paper shows that strong convergence and some other convergences preserve the fixed point property.

1. Introduction

In this paper, three coincidence theorems (Theorems 1, 2 and 4) are proved. We also show that for some convergences the property $f\colon X\to X$ *has a fixed point* is preserved. Four types of convergences are investigated. First we obtain some results concerning strong convergence (Definition 2). Then, we investigate convergences with respect to the fine topology, graph topology, and open-cover topology.

Before giving an exact definition of strong convergence, let us describe it in simple terms. It could be compared with the uniform convergence as follows: Uniform convergence of a sequence of functions with values in a metric space Y is defined using ε-*covers* of Y, i.e. using the open covers of Y containing all open balls with radius $\varepsilon>0$. The use of all covers, not only of the ε-ones, will give us a stronger convergence, which can be defined without using a uniform structure on Y.

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DEFINITION 1. Let X be a topological space, let c be an open cover of X and let $a \in X$. We denote

$$c^*(a) = \{z \in X:\ (\exists V \in c)(\{a, z\} \subseteq V)\} = \bigcup\{V:\ V \in c \ \&\ a \in V\}.$$

The following definition appeared in [KuT].

DEFINITION 2. ([KuT; Definition 3]) Let (X, T), (Y, τ) be topological spaces. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to Y. Let p be an open cover of Y. We say, that a net $\{f_\gamma\}_{\gamma\in\Gamma}$ *converges to a function* $f\colon X \to Y$ p*-uniformly* if

$$(\exists \gamma_0 \in \Gamma)(\forall x \in X)(\forall \gamma \geq \gamma_0)\big(f_\gamma(x) \in p^*(f(x))\big). \tag{u}$$

We say that a net $\{f_\gamma\}_{\gamma\in\Gamma}$ *converges to a function* $f\colon X \to Y$ *strongly* if and only if $\{f_\gamma\}_{\gamma\in\Gamma}$ converges to f p-uniformly for every open cover p of the space Y, and we write $f_\gamma \xrightarrow{s} f$.

The definition of the strong convergence was published for the first time in [KuT]. Nevertheless, as Péraire later communicated us, a very similar kind of convergence was used back in 1940 by Tukey in [Tu] (for more details, see [Pe]). Tukey did not consider all open covers as we did, but only a special family of covers. This enabled him to define a uniformity on the target space. Our goal, unlike Tukey's, is to work with non-uniformizable target spaces.

Some results concerning the strong convergence were published in [GK], [Ku], [KuT], [Pe] and [To].

2. Coincidence theorems and strong convergence

First we present a topological condition which is necessary and sufficient for the existence of a solution of an equation $F(x) = G(x)$. In what follows, if $H\colon X \to Y$ is a multifunction, we denote

$$\operatorname{Gr} H = \{[x, y]:\ y \in H(x)\}.$$

If X is a topological space, by $X \times X$ we denote the topological space $X \times X$ endowed with the product topology.

In this paper we consider mainly (multi)functions with closed graph. Properties of such functions are studied for example in [HM1] and [Sm].

LEMMA 1. *Let X, Y be arbitrary topological spaces. let $F\colon X \to Y$ and $G\colon X \to Y$ be two multifunctions such that $F(X) \subseteq G(X)$ and $\operatorname{Gr}(G^{-1} \circ F$ is a closed subset of the space $X \times X$. Then the following two properties a equivalent:*

(i) *There exists x from X such that $F(x) \cap G(x) \neq \emptyset$.*

(ii) *For every open cover c of X there exists O from c such that $F(O) \cap G(O) \neq \emptyset$.*

Proof. It is easy to see that (i) implies (ii).

Assume that (ii) is satisfied. We show that (i) holds. Put $A = \mathrm{Gr}(G^{-1} \circ F)$, $D = \{[x,x] : x \in X\}$. Suppose that, contrary to what we wish to prove, for every x from X, $F(x) \cap G(x) = \emptyset$. Then $A \cap D = \emptyset$. Since A is closed, the set $W = X \times X - A$ is an open neighborhood of D in the space $X \times X$. Then for every x from X there exists an open set U_x such that $x \in U_x$ and $U_x \times U_x$ is a subset of W, so $(U_x \times U_x) \cap A$ is empty.

Let $p = \{U_x : x \in X\}$ be an open cover of X. Since (ii) is true, there exists $U_x \in p$ such that $F(U_x) \cap G(U_x) \neq \emptyset$ holds. Hence there exist three points $y \in Y$, $a, b \in U_x$ such that $y \in F(a) \cap G(b)$. Therefore $b \in G^{-1} \circ F(a)$ and $[a,b] \in A$. So $\{[a,b]\} \subseteq A \cap (U_x \times U_x) \subseteq A \cap W$ and the set $A \cap W$ is nonempty. This is a contradiction. □

Lemma 1 implies the following assertion, proved by the author in 1993 in [GK].

COROLLARY 1. ([GK; Lemma 1]) *Let X be an arbitrary topological space and let $F\colon X \to X$ be a multifunction with a closed graph. Let p be an open cover of X. Then F has a fixed point if and only if*

$$(\forall p)(\exists U \in p)\big(F(U) \cap U \neq \emptyset\big). \qquad (\star)$$

Before presenting one of our main results we need some more definitions.

DEFINITION 3. Let X, Y be topological spaces, let $n = \{f_\gamma\}_{\gamma \in \Gamma}$, $m = \{g_\gamma\}_{\gamma \in \Gamma}$ be two nets of functions $f_\gamma\colon X \to Y$, $g_\gamma\colon X \to Y$. Let c be an open cover of Y. We say that m and n are *approaching each other* if and only if

$$(\forall c)(\exists \alpha \in \Gamma)(\forall \beta \geq \alpha)(\exists x \in X)(\exists U \in c)\big(\{f_\beta(x), g_\beta(x)\} \subseteq U\big).$$

Remark 1. Of course, the assumption *there exists an index α such that for all indices $\gamma \geq \alpha$ there exists a point x_γ such that $f_\gamma(x_\gamma) = g_\gamma(x_\gamma)$* is a sufficient condition for m and n to be approaching each other.

DEFINITION 4. (See [Ke]) A topological space X is called *fully normal* if and only if for every open cover r of X there exists an open cover u of X such that

$$(\forall x \in X)(\exists V \in r)\big(u^*(x) \subseteq V\big).$$

Such a cover u is said to be a *star refinement* of r.

We are now ready for the following theorem.

THEOREM 1. *Let X be a topological space. Let Y be a fully normal topological space. Let $m = \{g_\gamma\}_{\gamma\in\Gamma}$, $n = \{f_\gamma\}_{\gamma\in\Gamma}$ be two nets of functions $g_\gamma \colon X \to Y$, $f_\gamma \colon X \to Y$ such that m and n are approaching each other. Let $f \colon X \to Y$ be a function with a closed graph, let $g \colon X \to Y$ be an injective, continuous, open function such that $g(X) \supseteq \overline{f(X)}$.*

Let $f_\gamma \xrightarrow{s} f$ and $g_\gamma \xrightarrow{s} g$. Then there exists x from X such that $f(x) = g(x)$.

Proof. We shall prove that f and g satisfy the conditions of Lemma 1.

I. First we show that $H = \operatorname{Gr}(g^{-1} \circ f)$ is closed in $X \times X$. Let $\{[x_\beta, y_\beta]\}_{\beta\in\Lambda}$ be a net of points of H, converging to a point $[x,y]$. We have to show that $[x,y] \in H$. Since $[x_\beta, y_\beta] \in H$ for each $\beta \in \Lambda$, the point y_β is an element of the set $g^{-1}(f(x_\beta))$, so $g(y_\beta) = f(x_\beta)$. Since g is continuous and the net $\{y_\beta\}_{\beta\in\Lambda}$ converges to y, then the net $\{f(x_\beta)\}_{\beta\in\Lambda} = \{g(y_\beta)\}_{\beta\in\Lambda}$ converges to $g(y)$. Hence the net $\{x_\beta, f(x_\beta)\}_{\beta\in\Lambda}$ converges to $[x, g(y)]$. and since $\operatorname{Gr} f$ is closed, $[x, g(y)]$ is an element of $\operatorname{Gr} f$. Therefore $f(x) = g(y)$. But this implies $y \in (g^{-1} \circ f(x))$, so $[x,y] \in H$. Thus H is proved to be closed.

II. It suffices to prove now that for every open cover p of X there exists O from p such that $f(O) \cap g(O) \neq \emptyset$. Let p be an open cover of X. We define

$$r = \{g(V) : \ V \in p\} \cup \{Y - \overline{f(X)}\,\};$$

r is an open cover of Y. Since Y is fully normal, there exists an open cover c of Y which is a star refinement of r. Now let us take an open cover u of Y which is a star refinement of c. Since the nets n and m converge strongly to f and g respectively, there exists an index $\gamma_0 \in \Gamma$ such that

$$(\forall\, x \in X)(\forall\, \gamma \geq \gamma_0)(\exists\, O, W \in u)\big(\{f(x), f_\gamma(x)\} \subseteq O \ \ \& \ \ \{g(x), g_\gamma(x)\} \subseteq W\big).$$

Since m and n are approaching each other, there exists an index $\alpha \in \Gamma$ such that

$$(\forall\, \beta \geq \alpha)(\exists\, x \in X)(\exists\, U \in u)\big(\{f_\beta(x), g_\beta(x)\} \subseteq U\big).$$

Let us take an index δ such that $\delta > \gamma_0$ and $\delta > \alpha$. Then there exists a point z from X and an open set Z from u such that $\{f_\delta(z), g_\delta(z)\} \subseteq Z$. Moreover, since $\delta > \gamma_0$, there exist two open sets V_1, V_2 from u such that $\{f_\delta(z), f(z)\} \subseteq V_1$ and $\{g_\delta(z), g(z)\} \subseteq V_2$. The three inclusions mentioned above imply

$$\{f(z), f_\delta(z), g_\delta(z)\} \subseteq u^*(f_\delta(z)), \qquad \{g(z), f_\delta(z), g_\delta(z)\} \subseteq u^*(g_\delta(z)).$$

So there exist two open sets W_1, W_2 from c such that

$$\{f(z), f_\delta(z), g_\delta(z)\} \subseteq W_1 \qquad \text{and} \qquad \{g(z), f_\delta(z), g_\delta(z)\} \subseteq W_2.$$

Hence $\{f(z), g(z), f_\delta(z), g_\delta(z)\} \subseteq c^*(g_\delta(z))$.

Therefore there exists an open set V from r such that $c^*(g_\delta(z)) \subseteq V$. Since $f(z) \in V$, then $V \neq Y - \overline{f(X)}$, so there exists an open set O from p such that $g(O) = V$. Since $g(z) \in V$ we have $z \in O$. Therefore $f(z) \in f(O) \cap g(O)$. This completes the proof. □

The next example shows that the hypothesis $g(X) \supseteq \overline{f(X)}$ is not superfluous in the statement of Theorem 1.

EXAMPLE 1. Let $X = Y = \mathbb{R}$. Let f, f_n, g, g_n be functions from $\mathbb{R}$ to $\mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} -x & \text{for } x \in (-\infty, 0\rangle, \\ 0 & \text{for } x \in (0, +\infty); \end{cases}$$

for all integers $n \geq 2$,

$$f_n(x) = f(x) \qquad \text{for} \quad x \in \mathbb{R};$$

$$g(x) = \begin{cases} 2 - x & \text{for } x \in (-\infty, 1\rangle, \\ \frac{1}{x} & \text{for } x \in (1, +\infty); \end{cases}$$

for all $n \geq 2$,

$$g_n(x) = \begin{cases} g(x) & \text{for } x \in \left(-\infty, n - \frac{1}{n}\right\rangle \cup \left\langle n + \frac{1}{n}, +\infty\right), \\ \frac{n^2}{n^2-1} \cdot (n - x) & \text{for } x \in \left\langle n - \frac{1}{n}, n\right\rangle, \\ \frac{n^2}{n^2+1} \cdot (x - n) & \text{for } x \in \left\langle n, n + \frac{1}{n}\right\rangle. \end{cases}$$

In other words, if we consider the function g_n restricted on the interval $\left\langle n - \frac{1}{n}, n + \frac{1}{n}\right\rangle$, then its graph consists of two segments, joining the point $(n, 0)$ with the points $\left(n - \frac{1}{n}, g\left(n - \frac{1}{n}\right)\right)$ and $\left(n + \frac{1}{n}, g\left(n + \frac{1}{n}\right)\right)$.

The space $\mathbb{R}$ is fully normal. Since for each $n \geq 2$ $g_n(n) = f_n(n) = f(n)$, the sequences $\{g_n\}_{n=2}^{\infty}$ and $\{f_n\}_{n=2}^{\infty}$ are approaching each other. Only one hypothesis of Theorem 1 is not fulfilled.

The set $\overline{f(X)}$ is not a subset of $g(X)$. The equation $g(x) = f(x)$ has no solution. The example can be regarded also in a different way. Put $g_1 = f$, $f_1 = g$. We have $g_1(X) \supseteq \overline{f_1(X)}$. The sequence $\{g_n\}_{n=2}^{\infty}$ converges strongly to f_1 and the sequence $\{f_n\}_{n=2}^{\infty}$ converges strongly to g_1. Only two hypothesis of Theorem 1 are not satisfied. Namely, the function g_1 is not injective and it is not open.

From Theorem 1, we get a number of previously published results. We recall a definition and a lemma.

DEFINITION 5. ([GK; Definition 1]) Let X be a topological space, let $n = \{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions $f_\gamma\colon X \to X$, and let c be an open cover of X. We say that n is *approaching the diagonal* if and only if

$$(\forall c)(\exists \alpha)(\forall \beta \geq \alpha)(\exists x \in X)(\exists U \in c)\big(\{f_\beta(x), x\} \subseteq U\big).$$

In what follows, $G \in F_p(X)$ means, that G is a multifunction (function) from X to X which has the fixed point property. The proof of the following lemma is trivial.

LEMMA 2. ([GK; Lemma 2]) *Let X be an arbitrary topological space, let $f \in F_p(X)$ and let $m = \{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to X converging to f pointwise ($f_\gamma \to f$). Then m is approaching the diagonal.*

COROLLARY 2. ([GK; Theorem 1]) *Let $f\colon X \to X$ have a closed graph. Let X be a fully normal topological space. Let $f_\gamma \xrightarrow{s} f$ and let the net $\{f_\gamma\}_{\gamma\in\Gamma}$ is approaching the diagonal. Then $f \in F_p(X)$.*

Proof. Put $g(x) = x$ for each x in X and define $g_\gamma = g$ for all $\gamma \in \Gamma$. It is easy to see that the nets $\{f_\gamma\}_{\gamma\in\Gamma}$ and $\{g_\gamma\}_{\gamma\in\Gamma}$ are approaching each other. Since g is the identity function on X, also all other hypotheses of Theorem 1 are satisfied. □

The following theorem and its corollary, concerning the preservation of the fixed point property are variants of Theorem 1.

THEOREM 2. *Let X, Y be arbitrary topological spaces. Let $n = \{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions $f_\gamma\colon X \to Y$. Let $f\colon X \to Y$ be a function with a closed graph. let $g\colon X \to Y$ be an injective, continuous, open function such that $g(X) \supseteq \overline{f(X)}$ holds. Let there exist an index $\alpha \in \Gamma$ such that for each $\gamma \geq \alpha$ there exists x_γ such that $f_\gamma(x_\gamma) = g(x_\gamma)$. Let $f_\gamma \xrightarrow{s} f$. Then there exists x from X such that $f(x) = g(x)$.*

Proof. It suffices to verify that all hypotheses of Lemma 1 are satisfied. Let p be an open cover of X. We define

$$r = \{g(V):\ V \in p\} \cup \{Y - \overline{f(X)}\};$$

r is an open cover of Y. Since the net n is converging to f strongly, there exists an index $\gamma \geq \alpha$ such that for all x in X $f_\gamma(x) \in r^*\big(f(x)\big)$ holds. Choose x_γ such that $f_\gamma(x_\gamma) = g(x_\gamma)$. Since $f_\gamma(x_\gamma) \in r^*(f(x_\gamma))$ holds, there exists an open set $V \in r$ such that $\{f_\gamma(x_\gamma), f(x_\gamma)\} \subseteq V$. But $f_\gamma(x_\gamma) = g(x_\gamma)$, and due to the definition of r there exists an open set $O \in p$ such that $g(O) = V$. So we obtain $\{f_\gamma(x_\gamma), g(x_\gamma)\} \subseteq g(O)$ and $x_\gamma \in O$. Therefore $f(x_\gamma) \in f(O) \cap g(O)$. We have found a set O in p such that $f(O) \cap g(O)$ is nonempty. The rest is obvious, or similar to the proof of Theorem 1. □

COROLLARY 3. *Let X be a topological space, let $f\colon X \to X$ has a closed graph, let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to X which converges to f strongly. Suppose there exists $\gamma \in \Gamma$ such that $(\forall\,\alpha \geq \gamma)\big(f_\alpha \in F_p(X)\big)$. Then $f \in F_p(X)$.*

Two results similar to our Corollary 3 can be found in [FG; Statement 1.2] and [GK; Theorem 3].

The following example shows that uniform convergence does not preserve the fixed point property.

EXAMPLE 2. Let $X = \langle 1, +\infty)$. Let $f\colon X \to X$ be defined by $f(x) = x + \frac{1}{x}$ for all x in X. Let for every integer $n \geq 2$ a function $f_n\colon X \to X$ be defined as follows:

$$f_n(x) = \begin{cases} f(x) = x + \frac{1}{x} & \text{for } x \in \langle 1, n)\,, \\ n + 1 + (n + 1 - x)\big(\frac{1}{n} - 1\big) & \text{for } x \in \langle n, n+1)\,, \\ x & \text{for } x \in \langle n+1, +\infty)\,. \end{cases}$$

Then the sequence f_n converges to f uniformly, $(\forall\,n \geq 2)\big(f_n \in F_p(X)\big)$, but f has no fixed point.

3. Fixed points and some function spaces

In what follows, let X and Y be topological spaces and $C(X,Y)$ be the space of all continuous functions from X to Y. If Y is a uniform space, then strong convergence of a net of functions always implies uniform, and if X is compact then the converse is true ([KuT]). In what follows, we discuss some other function space topologies and their relations with strong convergence. At the end of this section, we show that if $C(X,X)$ is equipped with some appropriate topologies, then the set of the functions from $C(X,X)$ which have a fixed point is closed in $C(X,X)$.

The *open-cover topology* τ^* on $C(X,Y)$ (see [Ir] or [DHHM]) can be defined as follows. Let $G(Y)$ denote the set of all open covers of Y, and for each $c \in G(Y)$ and each $f \in C(X,Y)$ let

$$c(f) = \big\{g \in C(X,Y):\ (\forall\,x \in X)\big(g(x) \in c^*\big(f(x)\big)\big)\big\}.$$

The open-cover topology is generated by the subbase

$$\big\{u(f):\ u \in G(Y)\ \&\ f \in C(X,Y)\big\}.$$

The *graph topology* τ_Γ was introduced by Naimpally in [Na] and has, as its basic open sets, sets of the form $\big\{f \in C(X,Y):\ \operatorname{Gr} f \subseteq G\big\}$, where G is an open subset of $X \times Y$.

Further topologies on graphs of continuous functions can be found in the literature (see [Be], [DHH]).

To define the *fine topology* τ_ω, let (Y,d) be a metric space. Then a base for τ_ω consists of sets of the form

$$B(f,\varepsilon)=\{g\in C(X,Y):\ (\forall x\in X)\big(d(f(x),g(x))<\varepsilon(x)\big)\},$$

where $\varepsilon\colon X\to\mathbb{R}$ is a strictly positive continuous function. (See [DHHM], [Ho]).

In what follows, when we write $C_\Gamma(X,Y)$ or $C_*(X,Y)$, we suppose X and Y to be topological spaces, and we denote by $C_\Gamma(X,Y)$ ($C_*(X,Y)$) the space of all continuous functions from X to Y equipped with the graph (open-cover) topology. The symbol $C_\omega(X,Y)$ will denote the space of all continuous functions from a topological space X to a metric space Y, equipped with the fine topology.

It is easy to see that if a net of functions $\{f_\gamma\}_{\gamma\in\Gamma}$ converges to a function f in a space $C_*(X,Y)$, then it converges to f strongly. The following two assertions say when convergences in $C_\Gamma(X,Y)$ and $C_\omega(X,Y)$ are stronger than the strong one.

THEOREM 3. *Let X, Y be topological spaces. Let $f\in C(X,Y)$ and let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions; $f_\gamma\in C(X,Y)$ for all $\gamma\in\Gamma$. If $\{f_\gamma\}_{\gamma\in\Gamma}$ converges to f in $C_\Gamma(X,Y)$, then $f_\gamma\xrightarrow{s}f$.*

Proof. Let u be an open cover of Y. Define $G\subseteq X\times Y$ by $G=\bigcup\limits_{V\in u}\{f^{-1}(V)\times V\}$. Then G is an open neighborhood of f in the topology τ_Γ. So there exists $\beta\in\Gamma$ such that $(\forall\gamma\geq\beta)(\operatorname{Gr}f_\gamma\subseteq G)$. So for such an index γ the following holds:

$$(\forall x\in X)(\exists V\in u)\big((x,f_\gamma(x))\subseteq(f^{-1}(V),V)\big).$$

But this implies $(\forall x\in X)(\exists V\in u)\big(\{f(x),f_\gamma(x)\}\subseteq V\big)$, so $\{f_\gamma\}_{\gamma\in\Gamma}$ converges to f strongly. □

COROLLARY 4. *Let X be a countably paracompact normal space and (Y,d) be a metric space. Let $f\in C(X,Y)$ and let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions $f_\gamma\in C(X,Y)$ for all $\gamma\in\Gamma$. If $\{f_\gamma\}_{\gamma\in\Gamma}$ converges to f in $C_\omega(X,Y)$, then $f_\gamma\xrightarrow{s}f$.*

Proof. Under these conditions the fine topology τ_ω and the graph topology τ_Γ on $C(X,Y)$ coincide ([DHHM]). □

The next example shows that even if the spaces X and Y are very nice (equal to $\mathbb{R}$), the strong convergence of a net of functions need not imply the convergence of this net with respect to the topology τ^* or τ_Γ or τ_ω.

EXAMPLE 3. Let functions g, f, f_n from $\mathbb{R}$ to $\mathbb{R}$ are defined as follows.

$$f \equiv 0;$$
$$f_n \equiv \frac{1}{n} \quad \text{for all positive integer } n;$$
$$g(x) = \begin{cases} 0 & \text{for all } x \le 0, \\ -\frac{x}{4} & \text{for all } x,\ 0 \le x \le 2, \\ \frac{1}{x} - 1 & \text{for all } x \ge 2. \end{cases}$$

First we show that the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge to f in the open-cover topology. Let us consider the following open cover p of $\mathbb{R}$:

$$p = \{(-1+\varepsilon, \varepsilon) :\ \varepsilon \in (0,1)\} \cup \{(-\infty, -\tfrac{1}{2}), (\tfrac{1}{2}, +\infty)\}.$$

The function f is an element of the open set

$$p(g) = \{h \in C(\mathbb{R}, \mathbb{R}) :\ (\forall x \in \mathbb{R})(h(x) \in p^*(g(x)))\}.$$

To verify this, it suffices to realise that

$$(\forall x \le 0)(f(x) = g(x)),$$
$$(\forall x \in \langle 0,2\rangle)(\{f(x), g(x)\} \subseteq (-\tfrac{3}{4}, \tfrac{1}{4})),$$
$$(\forall x \ge 2)(\exists \varepsilon \in (0,1))(\varepsilon < \tfrac{1}{x}\ \&\ \{0, -1+\tfrac{1}{x}\} \subseteq (-1+\varepsilon, \varepsilon)).$$

Now let m be an integer, $m \ge 2$. We shall show that $f_m \notin p(g)$. Suppose, contrary to what we wish to prove, that $f_m \in p(g)$. Then there exists $V \in p$ such that $\{f_m(m), g(m)\} \subseteq V$ so there exists $\varepsilon \in (0,1)$ such that $\{\frac{1}{m}, \frac{1}{m} - 1\} \subseteq (-1+\varepsilon, \varepsilon)$. The last inclusion implies: $\frac{1}{m} < \varepsilon$ and $\frac{1}{m} - 1 > \varepsilon - 1$ at the same time. This is a contradiction.

Now let us consider a function $\varepsilon \colon \mathbb{R} \to \mathbb{R}$ defined as follows:

$$\varepsilon(x) = \begin{cases} 1 & \text{for } x \in \langle -1, 1\rangle, \\ \frac{1}{|x|} & \text{for } x \in \mathbb{R} - \langle -1, 1\rangle. \end{cases}$$

Denote by G the set $\{[a,b] :\ a \in \mathbb{R} \text{ and } |b| < \varepsilon(a)\}$. Then the set $W = \{h \in C(R,R) :\ \mathrm{Gr}\, h \subseteq G\}$, is an open neighbourhood of f in the graph topology. For no index n is $f_n \in W$.

Similarly, the set $P = B(f, \varepsilon) = \{h \in C(R,R) :\ d(f(x), h(x)) < \varepsilon(x)$ for all $x \in X\}$ is an open neighbourhood of f in the fine topology and for all positive integers n, $f_n \in C(R,R) - P$. So the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f neither in the graph topology nor in the fine one. Of course, it converges to f strongly (see also Remark 2 below).

The following assertion says that the convergence with respect to the fine topology is *coincidence preserving*. It will be also used in the proof of our final theorem.

THEOREM 4. *Let X be an arbitrary topological space. Let (Y, d) be a metric space. Let $m = \{g_\gamma\}_{\gamma\in\Gamma}$, $n = \{f_\gamma\}_{\gamma\in\Gamma}$ be two nets of continuous functions $g_\gamma\colon X \to Y$, $f_\gamma\colon X \to Y$. Let $f, g \in C(X, Y)$. Let the net m converge to f and let n converge to g, both with respect to the fine topology τ_ω. Let ε be a strictly positive function and*

$$(\forall \varepsilon \in C(X, R))\,(\exists \gamma_0 \in \Gamma)\,(\forall \gamma \geq \gamma_0)\,(\exists x_\gamma)\,\big(d(f_\gamma(x_\gamma), g_\gamma(x_\gamma)) < \varepsilon(x_\gamma)\big)\,. \quad (*)$$

Then there exists x from X such that $f(x) = g(x)$.

Proof. Let us suppose, to the contrary, that for all x in X, $d(f(x), g(x)) > 0$. Let us define a function $\varepsilon\colon X \to \mathbb{R}$ by $\varepsilon(x) = \frac{1}{4}\cdot d\big(f(x), g(x)\big)$ for all x in X. Since ε is a strictly positive continuous function, the set $O_1 = \{h \in C(X,Y) : (\forall x \in X)\big(d(h(x), f(x)) < \varepsilon(x)\big)\}$ ($O_2 = \{h \in C(X,Y) : (\forall x \in X)\big(d(h(x), g(x)) < \varepsilon(x)\big)\}$) is an open neighborhood of f (g) in $C_\omega(X,Y)$. So there exists an index $\alpha \in \Gamma$ such that for all $\beta \geq \alpha$. $f_\beta \in O_1$ and $g_\beta \in O_2$. Moreover, by $(*)$, there exists $\gamma_0 \in \Gamma$ such that

$$(\forall \gamma \geq \gamma_0)(\exists x_\gamma)\big(d(f_\gamma(x_\gamma), g_\gamma(x_\gamma)) < \varepsilon(x_\gamma)\big)\,.$$

Take $\delta \geq \max\{\alpha, \gamma_0\}$. Then there exists x_δ such that $d\big(f_\delta(x_\delta), g_\delta(x_\delta)\big) < \varepsilon(x_\delta)$. Since $f_\delta \in O_1$ and $g_\delta \in O_2$, we obtain:

$$\begin{aligned} d\big(f(x_\delta), g(x_\delta)\big) &\leq d\big(f(x_\delta), f_\delta(x_\delta)\big) + d\big(f_\delta(x_\delta), g_\delta(x_\delta)\big) + d\big(g_\delta(x_\delta), g(x_\delta)\big) \\ &< 3\cdot\varepsilon(x_\delta) < d\big(f(x_\delta), g(x_\delta)\big)\,. \end{aligned}$$

This is a contradiction. The proof is complete. □

The problem to determine conditions under which the strong convergence is topologizable has been recently solved in [Pe].

PROPOSITION 1. ([Pe; Theorem 3]) *Let X be a set, let (Y, τ) be a fully normal topological space. Then there exists an uniformity U_s on Y such that*

(a) *τ is finer than the topology τ_s of U_s;*
(b) *If Y is T_1 or regular, then τ_s is identical with τ;*
(c) *A net $\{f_\gamma\}_{\gamma\in\Gamma}$ of functions $f_\gamma\colon X \to Y$ converges strongly to a function $f\colon X \to Y$ if and only if it converges to f relatively to U_s.*

In what follows, the above mentioned uniformity U_s will be called Tukey's uniformity. By $C_T(X,Y)$ we shall denote the space of all continuous functions from a topological space X to a fully normal topological space Y, equipped with Tukey's uniformity.

Our last theorem speaks about fixed point property preserving convergences on the space $C(X,Y)$.

THEOREM 5. *Let X be a T_2 topological space. Then $C^F(X)$, the set of all continuous functions from X to X which have the fixed point property, is a closed subset of the spaces $C_\Gamma(X,X)$ and $C_*(X,X)$. If X is fully normal, then the set $C^F(X)$ is closed in $C_T(X,X)$, and if X is a metric space, then $C^F(X)$ is a closed subset of $C_\omega(X,X)$.*

Proof.

(I) Let $n = \{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions. Let for all $\gamma \in \Gamma$, $f_\gamma \in C^F(X)$. Let n converges to a continuous function f in $C_\Gamma(X,X)$. Put $g = \mathrm{id}_X$, i.e. $(\forall x \in X)\big(g(x) = x\big)$. Then the net n and the functions f and g satisfy the hypothesis of Theorem 2. So there exists an x in X such that $f(x) = g(x) = x$.

(II) Our assertions concerning $C_*(X,X)$ and $C_T(X,X)$ can be proved by the same arguments.

(III) Let $n = \{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions. Let for all $\gamma \in \Gamma$, $f_\gamma \in C^F(X)$. Let n converges to a continuous function f in $C_\omega(X,X)$ with respect to the topology τ_ω. Put $g = \mathrm{id}_X$ and $n = \{g_\gamma \equiv g\}_{\gamma\in\Gamma}$, and apply Theorem 4. □

Theorem 5 deals with closed subsets of some function spaces. In [HM2] some results concerning compact subsets of some of these spaces can be found.

OPEN QUESTION. As we have seen, if a net of functions from $C(X,Y)$ converges to a function f with respect to the open cover topology τ^* (or the graph topology τ_Γ), then it converges to f strongly. It is an open question whether, in general, for every topological space X and every metric space Y the τ_ω-convergence in $C(X,Y)$ imply strong convergence.

Remark 2. Strong convergence has another good property, which the convergences mentioned above do not have. It is *target respecting.* We mean by this the following property:

> Let X be a set, let Y be a topological space. Let $\{y_\gamma\}_{\gamma\in\Gamma}$ be a net converging to a point $y \in Y$. Then the net of constant functions $\{f_\gamma\}_{\gamma\in\Gamma}$ defined by $(\forall \gamma \in \Gamma)(\forall x \in X)\big(f_\gamma(x) = y_\gamma\big)$ converges strongly to the constant function $f \equiv y$.

To see this, let us consider an open cover u of Y. There exists $V \in u$ such that $y \in V$. Then

$$(\exists\, \alpha \in \Gamma)(\forall\, \gamma \geq \alpha)(\forall\, x \in X)\big(\{f_\gamma(x) = y_\gamma,\ f(x)\} \subseteq V\big)\,.$$

So $\{f_\gamma\}_{\gamma\in\Gamma}$ is proved to converge strongly to f.

Example 3 shows that the convergences induced by the topologies τ_ω, τ_Γ, τ_* are not *target respecting*, even if X and Y are very nice.

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