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A few remarks on almost $C$-polynomial functions


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A FEW REMARKS 
ON ALMOST C-POLYNOMIAL FUNCTIONS 

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ABSTRACT. We give some sufficient conditions for a function transforming a 
commutative semigroup to a commutative group to be a polynomial function. 
Some stability results are also given. 

Introduction 

Let \((X,+)\) be a commutative semigroup and let \((Y,+\)) be a commutative 
group. If \(f: X \to Y\) is a function and \(h \in X\), then we define the difference 
operator \(\Delta_h\) in the following way 

\[ \Delta_h f(x) := f(x + h) - f(x), \quad x \in X. \]

The superposition of several operators \(\Delta_{h_1}, \ldots, \Delta_{h_p}\) will be denoted briefly by 

\[ \Delta_{h_1, \ldots, h_p} := \Delta_{h_1} \cdots \Delta_{h_p}, \quad p = 1, 2, \ldots. \]

If \(h_1 = \cdots = h_p = h\), we will write \(\Delta^p_h\) instead of \(\Delta_{h_1, \ldots, h_p}\). It is well known 
([4], for example) that if \(f, g: X \to Y\), \(u, v, x \in X\), then 

\[ \Delta_{u,v} = \Delta_{v,u}, \quad \Delta_{-u} f(x) = -\Delta_{u} f(x - u), \quad \Delta_{u} (f + g) = \Delta_{u} f + \Delta_{u} g. \]

A function \(f: X \to Y\) is called strongly polynomial function of \(p\)th order if and 
only if 

\[ \Delta_{h_1, \ldots, h_{p+1}} f(x) = 0 \quad (1) \]

for all \(x, h_1, \ldots, h_{p+1} \in X\). If we assume that condition (1) holds for all \(x, h \in X\) 
and \(h_1 = h_2 = \cdots = h_{p+1} = h\), i.e. 

\[ \Delta^{p+1}_h f(x) = 0, \quad (2) \]

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then \( f \) is said to be a polynomial function of \( p \)th order. Let \( C \) be a subset of \( X \). A function \( f: X \to Y \) is called strongly \( C \)-polynomial function of \( p \)th order if and only if condition (1) is satisfied for every \( x \in X \) and all \( h_1, \ldots, h_{p+1} \in C \). Analogously, \( f \) is said to be \( C \)-polynomial function of \( p \)th order if and only if condition (2) is satisfied for every \( x \in X \) and each \( h \in C \).

It follows from Djoković’s theorem ([2; Corollar 1], also [5]) that if \( Y \) has the property

\[
\forall y \left( \left[ y \in Y \land ((p + 1)) y = 0 \right] \implies y = 0 \right),
\]

then \( f: X \to Y \) is a polynomial function of \( p \)th order if and only if it is strongly polynomial function of \( p \)th order as well. We say that \( f: X \to Y \) is a polynomial of \( p \)th order if there exist a constant \( a_0 \) and symmetric \( i \)-additive functions \( a_i: X^i \to Y \), \( i = 1, \ldots, p \) (i.e. additive in each variable) such that

\[
f(x) = a_0 + \sum_{i=1}^{p} a_i(x, \ldots, x), \quad x \in X.
\]

1. \( C \)-polynomial functions

In [3] it is proven that if \( X \) and \( Y \) are uniquely divisible by \((p + 1))!\), \( C - C = X \), \( C + C \subset C \) and \( \frac{1}{(p + 1))!} C \subset C \), then every \( C \)-polynomial function of \( p \)th order is a polynomial of \( p \)th order. In this part of the paper, we will obtain some other results of this type. We start with the following lemma.

**Lemma 1.** Let \( X \) be a commutative semigroup and let \( Y \) be a commutative group. If \( f: X \to Y \) is a function, then for arbitrary \( x, h_j^i \in X \), \( j = 1, 2, \ldots, p \), \( i = 0, 1 \), we have

\[
\Delta_{h_1^0 + h_1^1 + \ldots, h_p^0 + h_p^1} f(x) = \sum_{\varepsilon_1, \ldots, \varepsilon_p = 0}^1 \Delta_{h_1^\varepsilon_1, \ldots, h_p^\varepsilon_p} f \left( x + \sum_{k=1}^p (1 - \varepsilon_k) h_k^1 \right). \tag{3}
\]

If, moreover, \( X \) is a group, then

\[
\Delta_{h_1^0 - h_1^1 - \ldots, h_p^0 - h_p^1} f(x) = \sum_{\varepsilon_1, \ldots, \varepsilon_p = 0}^1 (-1)^{\varepsilon_1 + \ldots + \varepsilon_p} \Delta_{h_1^\varepsilon_1, \ldots, h_p^\varepsilon_p} f \left( x - \sum_{k=1}^p h_k^1 \right). \tag{4}
\]

**Proof.** Induction. As an example, we give a proof of equality (4). For \( p = 1 \) we have

\[
\Delta_{h_1^0 - h_1^1} f(x) = f(x + h_1^0 - h_1^1) - f(x - h_1^1) + f(x - h_1^1) - f(x) = \Delta_{h_1^0} f(x - h_1^1) + \Delta_{-h_1^1} f(x) = \Delta_{h_1^0} f(x - h_1^1) - \Delta_{h_1^1} f(x - h_1^1)
\]

\[
= \sum_{\varepsilon_1 = 0}^1 (-1)^{\varepsilon_1} \Delta_{h_1^{\varepsilon_1}} f(x - h_1^1).
\]
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Assume (4) and take arbitrary \( x, h_j^i \in X, \ j = 1, \ldots, p+1, \ i = 0,1. \) Then,

\[
\Delta_{h_1^0 - h_1^1, \ldots, h_{p+1}^0 - h_{p+1}^1} f(x) \\
= \Delta_{h_1^0 - h_1^1, \ldots, h_{p+1}^0 - h_{p+1}^1} \left( \sum_{\varepsilon_{p+1} = 0}^{1} (-1)^{\varepsilon_{p+1}} \Delta_{h_{p+1}^\varepsilon} f(x - h_{p+1}^1) \right) \\
= \sum_{\varepsilon_{p+1} = 0}^{1} (-1)^{\varepsilon_{p+1}} \Delta_{h_{p+1}^\varepsilon} \left( \sum_{\varepsilon_1, \ldots, \varepsilon_p = 0}^{1} (-1)^{\varepsilon_1 + \cdots + \varepsilon_p} \Delta_{h_{p+1}^{\varepsilon_1}, \ldots, h_p^{\varepsilon_p}} f(x - h_{p+1}^1 - \sum_{k=1}^{p} h_k^1) \right) \\
= \sum_{\varepsilon_1, \ldots, \varepsilon_{p+1} = 0}^{1} \Delta_{h_{p+1}^{\varepsilon_1}, \ldots, h_p^{\varepsilon_p}} f(x - \sum_{k=1}^{p+1} h_k^1),
\]

which ends the proof. \( \square \)

The next two lemmas are consequences of Lemma 1.

**Lemma 2.** Let \( X \) be a commutative semigroup and let \( Y \) be a commutative group. If \( C \subseteq X \) satisfies the condition

\[
C + C = X, \quad (5)
\]

then every strongly \( C \)-polynomial function of \( p \)th order \( f: X \to Y \) is strongly polynomial of \( p \)th order.

**Proof.** Fix \( x, h_1, \ldots, h_{p+1} \in X. \) According to (5), there exist \( h_j^i \in C, \ j = 1, \ldots, p+1, \ i = 0,1, \) such that \( h_j^i = h_j^0 + h_j^1, \ j = 1, \ldots, p+1. \) By virtue of (3) of Lemma 1 and our assumption we obtain

\[
\Delta_{h_1, \ldots, h_{p+1}} f(x) = \Delta_{h_1^0, \ldots, h_{p+1}^0, h_1^1, \ldots, h_{p+1}^1} f(x) \\
= \sum_{\varepsilon_1, \ldots, \varepsilon_{p+1} = 0}^{1} \Delta_{h_1^{\varepsilon_1}, \ldots, h_{p+1}^{\varepsilon_{p+1}}} f(x + \sum_{k=1}^{p+1} (1 - \varepsilon_k) h_k^1) = 0,
\]

which finishes the proof. \( \square \)

In a similar way one can prove the following lemma.

**Lemma 3.** Let \( X \) and \( Y \) be commutative groups. If \( C \subseteq X \) satisfies the condition

\[
C - C = X, \quad (6)
\]

then every strongly \( C \)-polynomial function of \( p \)th order \( f: X \to Y \) is strongly polynomial of \( p \)th order.

Let \( m \) be a fixed positive integer. We say that a group \( X \) has a \((m-C)\)-property if and only if each element \( h \in X \) has a representation \( h = \sum_{i=1}^{m} h_i \), where
Note that if \( f: X \to Y \) is a strongly \( C \)-polynomial function of \( p \)th order, then it is also strongly \( (C \cup (-C)) \)-polynomial function of \( p \)th order.

**Theorem 1.** Let \( X \) and \( Y \) be commutative groups. If \( X \) has the \((m-C)\)-property with some positive integer \( m \), then every strongly \( C \)-polynomial function of \( p \)th order \( f: X \to Y \) is strongly polynomial of \( p \)th order.

**Proof.** Fix \( x, h_1, \ldots, h_{p+1} \in X \). There exist a positive integer \( m \) and \( h_{j,k} \in C \cup (-C), j = 1, \ldots, p+1, k = 1, \ldots, m \), such that \( h_j = \sum_{k=1}^{m} h_{j,k} \). Thus

\[
\Delta_{h_1, \ldots, h_{p+1}} f(x) = \Delta \left( \frac{\sum_{k=1}^{m} h_{1,k}}{m}, \ldots, \frac{\sum_{k=1}^{m} h_{p+1,k}}{m} \right) f(x)
\]

\[
= \Delta \left( \frac{\sum_{k=1}^{m} h_{1,k}}{m}, \ldots, \frac{\sum_{k=1}^{m} h_{p+1,k}}{m} \right) \sum_{j=1}^{p+1} \left[ f \left( x + \sum_{k=1}^{m} h_{p+1,k} \right) - f \left( x + \sum_{k=1}^{m} h_{p+1,k} \right) \right]
\]

\[
= \sum_{j=1}^{p+1} \Delta_{h_{p+1,j}} f \left( x + \sum_{k=1}^{m} h_{p+1,k} \right) = 0,
\]

because \( f \) is strongly \( (C \cup (-C)) \)-polynomial function of \( p \)th order. This ends the proof. \( \square \)

2. Stability in the sense of Ulam and Hyers

Assume \( X \) is a commutative semigroup and \( Y \) is a real Banach space. Let us fix \( \varepsilon \geq 0 \) and let \( f: X \to Y \) be a function. We are interested in solutions to the inequalities

\[
\|\Delta_{h_1, \ldots, h_{p+1}} f(x)\| \leq \varepsilon, \quad x \in X, \quad h_1, \ldots, h_{p+1} \in C, \quad (7)
\]

and

\[
\|\Delta_{h_{p+1}} f(x)\| \leq \varepsilon, \quad x \in X, \quad h \in C, \quad (8)
\]

where \( C \) is a subset of \( X \). In the case of \( C = X \), the problem was considered by many authors. In particular, M. Albert and J. A. Baker [1] have proved the following theorem.
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**Theorem A-B.** Let $X$ be a commutative semigroup with zero and let $Y$ be a real Banach space. If $f: X \to Y$ satisfies condition (7) with $C = X$, then there exists a unique (up to an additive constant) polynomial $g: X \to Y$ of $p$th order such that

$$\|f(x) - g(x)\| \leq \varepsilon, \quad x \in X.$$ 

The first theorem in this section reads as follows.

**Theorem 2.** Let $X$ be a commutative semigroup with zero and let $Y$ be a real Banach space. If $f: X \to Y$ satisfies condition (7) where $C \subset X$ satisfies one of conditions (5) or (6), then there exists a unique (up to an additive constant) polynomial $g: X \to Y$ of $p$th order such that

$$\|f(x) - g(x)\| \leq 2^{p+1}\varepsilon, \quad x \in X.$$ 

**Proof.** Assume (5) (if (6) is satisfied, then the proof is similar). Let $x, h_1, \ldots, h_{p+1} \in X$ be arbitrary fixed. According to (5), there exist $h_j^i \in C$, $j = 1, \ldots, p+1$, $i = 0, 1$, such that $h_j = h_j^0 + h_j^1$, $j = 1, \ldots, p+1$. By Lemma 1 and (7) we get

$$\|\Delta_{h_1, \ldots, h_{p+1}} f(x)\| \leq \sum_{\varepsilon_1, \ldots, \varepsilon_{p+1} = 0}^{1} \left\|\Delta_{h_{1}^{\varepsilon_1}, \ldots, h_{p+1}^{\varepsilon_{p+1}}} f\left(x + \sum_{k=1}^{p+1} (1 - \varepsilon_k) h_k^1\right)\right\| \leq 2^{p+1}\varepsilon.$$ 

Our assertion follows now from Theorem A-B. □

J. H. B. Kemperman ([4; p. 369]) noticed that if $X$ is a commutative group admitting division by $(p + 1)!$, then we can express values of the operator $\Delta_{h_1, \ldots, h_{p+1}}$ as linear combinations of iterates of the $(p + 1)$th order of difference operators depending only on one span. More precisely, if $x, h_1, \ldots, h_{p+1} \in X$ and $f: X \to Y$ is a function, then

$$\Delta_{h_1, \ldots, h_{p+1}} f(x) = \sum_{\varepsilon_1, \ldots, \varepsilon_{p+1} = 0}^{1} (-1)^{\varepsilon_1 + \cdots + \varepsilon_{p+1}} \Delta_{h_{1}^{\varepsilon_1}, \ldots, h_{p+1}^{\varepsilon_{p+1}}}^{p+1} f\left(x + h_{\varepsilon_1, \ldots, \varepsilon_{p+1}}''\right),$$ 

where

$$h_{\varepsilon_1, \ldots, \varepsilon_{p+1}}' = -\sum_{j=1}^{p+1} \frac{\varepsilon_j}{j} h_j,$$

and

$$h_{\varepsilon_1, \ldots, \varepsilon_{p+1}}'' = \sum_{j=1}^{p+1} \varepsilon_j h_j.$$  

The next theorem refers to inequality (8).
THEOREM 3. Let $X$ be a commutative group admitting division by $(p + 1)!$, let $Y$ be a real Banach space. Assume $\frac{1}{(p+1)!} C \subseteq C$, $C + C \subseteq C$ and (6). If $f : X \to Y$ satisfies condition (8), then there exists a unique (up to an additive constant) polynomial $g : X \to Y$ of $p$th order such that

$$\|f(x) - g(x)\| \leq 4^{p+1} \varepsilon, \quad x \in X.$$ 

Proof. Fix arbitrary $x, h_1, \ldots, h_{p+1} \in X$. There exist $h_j^i \in C$, $j = 1, \ldots, p+1$, $i = 0, 1$, such that $h_j = h_j^0 - h_j^1$, $j = 1, \ldots, p+1$. For arbitrary $\varepsilon_j, \delta_j \in \{0, 1\}$, $j = 1, \ldots, p+1$, let us define

$$h_{\varepsilon_1, \ldots, \varepsilon_{p+1}} := \sum_{j=1}^{p+1} \delta_j h_j^\varepsilon_j,$$

$$z_{\delta_1, \ldots, \delta_{p+1}} := x + \sum_{j=1}^{p+1} (1 - \varepsilon_j) h_j^1 + \sum_{j=1}^{p+1} \delta_j h_j^\varepsilon_j + h_{\varepsilon_1, \ldots, \varepsilon_{p+1}}.$$ 

According to Lemma 1 we obtain

$$\Delta_{h_1, \ldots, h_{p+1}} f(x)$$

$$= \Delta_{h_0, \ldots, h_{p+1}} f(x)$$

$$= \sum_{\varepsilon_1, \ldots, \varepsilon_{p+1} = 0}^{1} (-1)^{\varepsilon_1 + \cdots + \varepsilon_{p+1}} h_{\varepsilon_1, \ldots, \varepsilon_{p+1}} f\left(x - \sum_{j=1}^{p+1} (1 - \varepsilon_j) h_j^1\right)$$

$$= -\sum_{\varepsilon_1, \ldots, \varepsilon_{p+1} = 0}^{1} (-1)^{\varepsilon_1 + \cdots + \varepsilon_{p+1}} \delta_1 + \cdots + \delta_{p+1} h_{\delta_1, \ldots, \delta_{p+1}}^p f\left(z_{\delta_1, \ldots, \delta_{p+1}}^{\varepsilon_1, \ldots, \varepsilon_{p+1}}\right).$$

Hence

$$\|\Delta_{h_1, \ldots, h_{p+1}} f(x)\| \leq 4^{p+1} \|\Delta_{h_1, \ldots, h_{p+1}} f\left(z_{\delta_1, \ldots, \delta_{p+1}}^{\varepsilon_1, \ldots, \varepsilon_{p+1}}\right)\|,$$

which together with (8) implies that

$$\|\Delta_{h_1, \ldots, h_{p+1}} f(x)\| \leq 4^{p+1} \varepsilon.$$

Now our assertion follows from Theorem A-B. \qed

As a final remark note that we are able to repeat the argumentation used in the proof of Theorem 3 to obtain the following theorem.
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**THEOREM 4.** Let $X$ be a commutative group admitting division by $(p + 1)!$ and let $Y$ be a commutative group. If $C$ is a subset of $X$ such that

$$\frac{1}{(p+1)!}C \subseteq C, \quad C + C \subseteq C \quad \text{and} \quad C - C = X,$$

then every $C$-polynomial function of $p$th order is a strongly polynomial function of $p$th order.

**Remark.** Recall that ([2; Theorem 3]) if, moreover, $Y$ is a commutative group such that for every $y \in Y$

$$\text{equation } (p!)x = y \text{ has a unique solution } x = \frac{y}{p!},$$

then every polynomial function $f : X \to Y$ of $p$th order is a polynomial of $p$th order, too.

**REFERENCES**


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