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ON A THEOREM OF BROWDER

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ABSTRACT. In this paper, there is proved Browder's theorem about the existence of solutions of the Cauchy problem in a Hilbert space for the equation $u'(t) = f(t, u(t))$, where the weak continuity of f is replaced by more general conditions.

Let H be a complex Hilbert space with inner product $(\cdot\,,\cdot)$ and norm $\|\cdot\|$, and let $\mathbb{R}^+$ be the set of nonnegative real numbers. Suppose that $f\colon \mathbb{R}^+ \times H \to H$ is a mapping. It is well known that Peano's methods can be applied to prove that the Cauchy problem:

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = f(t, u(t)), \qquad 0 \le t \le T, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where $u_0 \in H$, has solutions when $H = \mathbb{R}^n$, the n-dimensional Euclidean space, and f is a continuous mapping. This method cannot be generalized to the infinite dimensional case, as was shown by Dieudonné [2, p. 287], even if we assume the continuity of f. Browder [1, Th. 7] has proved the following:

THEOREM 1. *Let H_w be the Hilbert space H endowed with the weak topology and let $f\colon \mathbb{R}^+ \times H \to H$ be a weakly continuous mapping (i.e. f is continuous as a mapping from $\mathbb{R}^+ \times H_w$ into H_w). Then for each $r > 0$, there exists $a(r) > 0$ such that for each $u_0 \in H$ with $\|u_0\| < r$, there exists a C^1 solution u of system (1), (2) for $0 \le t \le a(r)$.*

In this paper we show that for the existence of differentiable solution u of (1), (2) in Theorem 1 it suffices to suppose (instead of the weak continuity of f) the following conditions:

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(3a) for all a, $r > 0$ the image $f\big([0,a] \times \{v \in H : \|v\| \le r\}\big)$ is a bounded subset in H;

(3b) for every $v \in H$ the section $t \mapsto f(t,v)$ is a derivative, i.e. for each $t_0 \in \mathbb{R}^+$

$$\lim_{t \to t_0} \frac{1}{(t-t_0)} \int_{t_0}^{t} f(s,v)\, ds = f(t_0, v);$$

and

(3c) for all a, r, $s > 0$ and $v \in H$ there are $p > 0$ and $v_1, \dots, v_m \in H$ such that if $y_1, y_2 \in H$, $\|y_1\|, \|y_2\| \le r$, $|(y_1 - y_2, v_i)| < p$ for $i = 1, 2, \dots, m$, then $\big|\big(f(t,y_1) - f(t,y_2), v\big)\big| < s$ for all $t \in [0,a]$.

Remark 1. Let C_w and D respectively denote the class of all weakly continuous functions $f\colon \mathbb{R}^+ \times H \to H$ and the class of all functions $f\colon \mathbb{R}^+ \times H \to H$ satisfying the conditions (3a) – (3c).

Define for $f, g \in D$,

$$p(f,g) = \min\Big(1, \sup_{(t,v) \in \mathbb{R}^+ \times H} \|f(t,v) - g(t,v)\|\Big).$$

Remark that (D,p) is a complete metric space and $C_w \subset D$ is a closed subset of D. We shall prove that C_w is a nondense subset of D. Fix $1 > s > 0$ and $v_0 \in H$, $\|v_0\| = 1$. There is a discontinuous derivative $h\colon \mathbb{R}^+ \to [0,1]$ (see [5]). If $f \in C_w$, then the function

$$g(t,v) = f(t,v) + (s/2)h(t)v_0$$

is in $D - C_w$ and $p(f,g) < s$.

Hence C_w is a nondense subset of D.

We shall now apply the basic idea of the proof of Browder's Theorem 1 to the proof of the following:

THEOREM 2. *Let a function $f\colon \mathbb{R}^+ \times H \to H$ satisfy the conditions (3a) – (3c). Then for every $r > 0$ there exists $a(r) > 0$ such that, for each u_0 in H with $\|u_0\| < r$, there exists a solution u of the Cauchy problem*

$$\frac{du}{dt}(t) = f\big(t, u(t)\big), \qquad 0 \le t \le a(t),$$

with

$$u(0) = u_0.$$

Proof. From (3a) it follows that for each $k, r > 0$ there exists $M(k,r) > 0$ such that $\|f(t,u)\| \le M(k,r)$ for each $t \in \mathbb{R}^+$, $u \in H$ with $t \le k$, $\|u\| \le r$. Moreover $M(k,r)$ may be chosen increasing in each variable.

We consider first the case of finite-dimensional space H, and recapitulate the proof of the Peano existence theorem (see also [3]).

We choose

$$a(r) = \min\Big(1, r\big(M(1,2r)\big)^{-1}\Big)$$

and, for each $z > 0$ $(z < z_0)$, we define $u_z(t)$ on the interval $0 \le t \le a(r)$ by

$$u_z(t) = \begin{cases} u_0, & 0 \le t \le z, \\ u_0 + \int_z^t f\big(s, u_z(s-z)\big)\,\mathrm{d}s, & z \le t \le a(r). \end{cases}$$

This formula enables us to compute $u_z(t)$ on the interval $kz \le t \le (k+1)z$ knowing its value on $\big[(k-1)z, kz\big]$.

In the case of finite-dimensional space H the condition (3c) denotes the equicontinuity of all sections $u \mapsto f(t,u)$. So from [3, Theorem 1] it follows that the functions $t \mapsto f\big(t, u_z(t-z)\big)$ are derivatives. Hence u_z is differentiable and satisfies the equations

$$\frac{\mathrm{d}u_z}{\mathrm{d}t}(t) = f\big(t, u_z(t-z)\big),$$
$$u_z(0) = u_0.$$

Moreover, on the interval $\big[kz, (k+1)z\big]$

$$\|u_z(t)\| \le \|u_0\| + \int_z^t \big\|f\big(s, u_z(s-z)\big)\big\|\,\mathrm{d}s,$$

and if we have verified by induction that $\|u_z(t)\| \le 2r$ for $t \le kz$, then

$$\|u_z(t)\| \le \|u_0\| + M(1,2r)(t-z) \le r + M(1,2r)a(r) \le 2r,$$

for $t \le (k+1)z$. So $\|u_z(t)\| \le 2r$ on $0 \le t \le a(r)$. Moreover,

$$\left\|\frac{\mathrm{d}u_z}{\mathrm{d}t}(t)\right\| = \big\|f\big(t, u_z(t-z)\big)\big\| \le M\big(a(r), 2r\big) \le M(1,2r).$$

Hence (u_z) is a bounded equi-continuous set of functions on $0 \le t \le a(r)$. Choosing a uniformly convergent subsequence (for $z \to 0$), we see that its limit $t \mapsto u(t)$ must verify the equation

$$u(t) = u_0 + \int_0^t f\big(s, u(s)\big)\, \mathrm{d}s \qquad \text{for } 0 \le t \le a(r),$$

i.e.,

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = f\big(t, u(t)\big), \qquad 0 \le t \le a(r),$$
$$u(0) = u_0 .$$

For this function u we have moreover

$$\|u(t)\| \le 2r,$$
$$\left\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right\| \le M(1, 2r), \qquad \text{for } 0 \le t \le a(r).$$

We pass now to the case of a general Hilbert space H. Let A be the family of finite-dimensional subspaces of H, ordered by inclusion. For $F \in A$, let P be the orthogonal projection of H on F. We form the approximating equations

$$\frac{\mathrm{d}u_F}{\mathrm{d}t}(t) = Pf\big(t, u_F(t)\big) = f_1\big(t, u_F(t)\big),$$
$$u_F(0) = Pu_0,$$

for a function $u_F \colon I \to F$ ($I \subset \mathbb{R}^+$). If we remark that

$$\|f_1(t,u)\| = \|Pf(t,u)\| \le \|f(t,u)\| \le M(k,r) \qquad \text{for } t \le k, \quad \|u\| \le r,$$

it follows from the preceding discussion that we may find a solution u_F of the approximating equation on $0 \le t \le a(r)$ for $\|u_0\| \le r$ such that

$$\|u_F(t)\| = 2r,$$
$$\left\|\frac{\mathrm{d}u_F}{\mathrm{d}t}(t)\right\| \le M(1, 2r), \qquad \text{for } 0 \le t \le a(r).$$

Considering the functions u_F as mappings of $\big[0, a(r)\big]$ into the closed set $\big\{u : \|u\| \le 2r\big\}$ in H_w, it follows that the family (u_F) is equi-continuous, and that the union of their ranges is contained in a compact set. Hence there exists a continuous function u from $[0, a(r)]$ to H such that, for each F_0 in A, $z > 0$, and each finite set $(w_1, \dots, w_r)$ in H, there exists F in A with $F_0 \subset F$ such that

$$|(u_F(t) - u(t), w_j)| < z \qquad \text{for} \quad 0 \le t \le a(r), \quad 1 \le j \le r$$

(see [1, pp. 519_6–520^1]).

Fix $v \in H$ and $z > 0$. There is F_0 in A such that, for all F in A and the corresponding projections P,

$$\|Pv - v\| < z.$$

We know that, for $0 \le t \le a(r)$,

$$(u_F(t), v) = (u_0, v) + \int_0^t (f(s, u_F(s)), Pv) \, ds.$$

It follows from (3c) that there are $p > 0$ ($p < z$) and $u_1, \dots, u_m \in H$ such that for each $t \in [0, a(r)]$ and all y_1, $y_2 \in H$ with $\|y_1\|$, $\|y_2\| < 2r$ if

$$|(y_1 - y_2, u_i)| < p \qquad \text{for} \quad i = 1, \dots, m,$$

then

$$|(f(t, y_1) - f(t, y_2), v)| < z.$$

We may choose $F \supset F_0$ so that

$$\begin{aligned} |(u_F(t) - u(t), u_i)| &< p < z, \qquad i = 1, \dots, m, \\ |(u_F(t) - u(t), v)| &< p < z, \qquad 0 \le t \le a(r). \end{aligned}$$

Evidently

$$|(f(t, u_F(t)) - f(t, u(t)), v)| < z \qquad \text{for} \quad 0 \le t \le a(r).$$

Since

$$|(f(t, u_F(t)), Pv) - (f(t, u_F(t)), v)| \le M(1, 2r)\|v - Pv\| \le zM(1, 2r),$$

we have

$$\begin{aligned}
&\left|\big(u(t),v\big)-(u_0,v)-\int_0^t\big(f\big(s,u(s)\big),v\big)\,\mathrm{d}s\right| \\
&=\left|\big(u(t),v\big)-\big(u_F(t),v\big)+\big(u_F(t),v\big)-(u_0,v)-\int_0^t\big(f\big(s,u(s)\big),v\big)\,\mathrm{d}s\right| \\
&=\left|\big(u(t)-u_F(t),v\big)+\int_0^t\big(f\big(s,u_F(s)\big),Pv\big)\,\mathrm{d}s-\int_0^t\big(f\big(s,u(s)\big),v\big)\,\mathrm{d}s\right| \\
&\le z\big(1+a(r)M(1,2r)\big)\,.
\end{aligned}$$

Since $z>0$ is arbitrary,

$$\big(u(t),v\big)=(u_0,v)+\int_0^t\big(f\big(s,u(s)\big),v\big)\,\mathrm{d}s\,.$$

Since $v\in H$ is arbitrary,

$$u(t)=u_0+\int_0^t f\big(s,u(s)\big)\,\mathrm{d}s \qquad \text{for} \quad 0\le t\le a(r)\,,$$

and the proof is finished.

Remark 2. It is known (see [3]) that if in Theorem 2 we assume that f satisfies a local Lipschitz condition in u, then the local solution $u\colon\big[0,a(r)\big]\to H$ with $u(0)=u_0$ is unique.

Moreover, we have

THEOREM 3. *If in Theorem 2, besides the conditions* (3a) – (3c), *we suppose that*

$$\mathrm{Re}\big(f(t,u)-f(t,v),u-v\big)\le\|u-v\|^2/2t$$

for all u,v in H and $0\le t\le a(r)$, then the solution u is unique.

The proof is a repetition of that of Medeiros's Theorem 3 in [4].

DEFINITION. ([4, Df. 1]). *Let w be a positive real function defined on $[0,T]$. We say that w is a permissible function if it is strictly increasing on $[0,T]$, if $w(0)=0$, and if*

$$\frac{1}{w(z)}\int_s^a \mathrm{d}z \to \infty \qquad as \quad s\to 0\,, \quad s>0\,, \quad 0<a<T\,.$$

THEOREM 4. *If in Theorem 2, besides* (3a)–(3c), *we suppose that*

$$2\,\mathrm{Re}\big(f(t,u)-f(t,v),u-v\big)\le w\big(\|u-v\|^2\big)\,, \qquad 0\le t\le a(r)\,,$$

for some permissible function w, then solution u on $\big[0,a(r)\big]$ is unique.

The proof is a repetition of that of Medeiros's Theorem 3 in [4].

REFERENCES

[1] BROWDER, F. E.: *Non-linear equations of evolution*, Ann. of Math. (2) **80** (1964), 485–523.

[2] DIEUDONNÉ, J.: *Foundations of Modern Analysis*, Academic Press, New York-London, 1960.

[3] GRANDE, Z.: *A theorem about Carathéodory's superposition*, Math. Slovaca **42** (1992), 443–449.

[4] MEDEIROS, L. A.: *On nonlinear differential equations in Hilbert space*, Amer. Math. Monthly **76** (1969), 1024–1027.

[5] ZAHORSKI, Z.: *Sur la première dérivée*, Trans. Amer. Math. Soc. **69** (1950), 1–54.

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