MAXIMUMS OF DARBOUX BAIRE ONE FUNCTIONS

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ABSTRACT. In 1974 A. M. Bruckner, J. G. Ceder, and T. L. Pearson asked whether each Baire one function which can be written as the maximum of Darboux functions, can be written as the maximum of two Darboux Baire one functions. We provide the affirmative answer to this question.

The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. For each $A \subset \mathbb{R}$ the symbol $\text{card} A$ stands for the cardinality of $A$. We write $\mathfrak{c} = \text{card} \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$. We define the oscillation of $f$ at a point $x \in \mathbb{R}$ as

$$\text{osc}(f, x) = \lim_{\delta \to 0+} \sup \{|f(x_1) - f(x_2)| : x_1, x_2 \in (x-\delta, x+\delta)\}.$$ 

For each nondegenerate interval $I \subset \mathbb{R}$ we define

$$\mathfrak{c}\text{-sup}(f, I) = \sup \left\{y \in \mathbb{R} : \text{card}\{x \in I : f(x) > y\} = \mathfrak{c}\right\}.$$ 

For each $x \in \mathbb{R}$ we denote

$$\mathfrak{c}\text{-lim}_{t \to x^-} f(t) = \lim_{\delta \to 0^+} \mathfrak{c}\text{-sup}(f, (x-\delta, x)),$$

and similarly we define the symbol $\mathfrak{c}\text{-lim}_{t \to x^+} f(t)$. We say that $f$ is Darboux if it maps intervals onto connected sets.

In 1974 A. M. Bruckner, J. G. Ceder, and T. L. Pearson proved that a function $f$ is the maximum of Darboux functions $g_0$ and $g_1$ if and only if

$$\min \left\{\mathfrak{c}\text{-lim}_{t \to x^-} f(t), \mathfrak{c}\text{-lim}_{t \to x^+} f(t)\right\} \geq f(x) \quad \text{for each} \quad x \in \mathbb{R},$$

(1)

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and we can conclude that \( g_0 \) and \( g_1 \) are Lebesgue measurable (belong to Baire class \( \alpha \), \( \alpha \geq 2 \)) provided that \( f \) is so ([4; Theorem 3]). They asked also, whether the latter result holds true for \( \alpha = 1 \). We provide the affirmative answer to this question. (See Corollary 4.)

We will need two auxiliary results. Proposition 1 is an immediate consequence of a lemma proved in 1968 by A. M. Bruckner, J. G. Ceder, and R. Keston [3; Lemma 2].

**Proposition 1.** Let \( P \subset (a, b) \) be a first category set and let \( \tau \) be a non-negative extended real number. Then there exists a Darboux Baire one function \( h: \mathbb{R} \to [0, \tau) \) such that \( h(x) = 0 \) except for a first category set disjoint from \( P \) and

\[
\lim_{x \to a^+} h(x) = \lim_{x \to b^-} h(x) = \tau.
\]

Proposition 2 contains a condition equivalent to the Darboux property of a Baire one function. It is due to H. Sen and J. L. Massera. (See [2; Theorem 6.1] or [1; p. 9, Theorem 1.1].)

**Proposition 2.** Assume that \( f: \mathbb{R} \to \mathbb{R} \) is a Baire one function. The following conditions are equivalent:

1. \( f \) is Darboux,
2. for each \( x \in \mathbb{R} \), we have
   \[
   \max\left\{ \lim_{t \to x^-} f(t), \lim_{t \to x^+} f(t) \right\} \leq f(x) \leq \min\left\{ \lim_{t \to x^-} f(t), \lim_{t \to x^+} f(t) \right\}.
   \]

Now we are ready to prove the main result.

**Theorem 3.** Assume that a Baire one function \( f: \mathbb{R} \to \mathbb{R} \) fulfills the following condition:

\[
\min\left\{ \lim_{t \to x^-} f(t), \lim_{t \to x^+} f(t) \right\} \geq f(x) \quad \text{for each} \quad x \in \mathbb{R}.
\]

Then \( f \) is the maximum of two Darboux Baire one functions.

**Proof.** First assume that \( f \) is nonpositive. Let \( \tau_0 = \infty \) and \( A_0 = \emptyset \). For each \( n \in \mathbb{N} \) define \( \tau_n = 2^{-n} \) and

\[
A_n = \{ x \in \mathbb{R} : \text{osc}(f, x) \geq \tau_n \}.
\]

Then

\[
C \overset{df}{=} \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} A_n
\]

is the set of points of continuity of \( f \), which is residual by Baire theorem. It is easy to show that the oscillation function is upper semicontinuous. So, each set
$A_n$ is closed and nowhere dense. For each $n$ arrange all components of $\mathbb{R} \setminus A_n$ in a sequence $\{(a_{nk}, b_{nk}) : k \in N_n\}$, where $N_n \subset \mathbb{N}$.

For all $i < 2$, $n \in \mathbb{N}$, and $k \in N_n$ use Proposition 1 to construct a Darboux Baire one function $h_{ink} : \mathbb{R} \to [0, \tau_{n-1})$ such that $h_{ink}(x) = 0$ except for a first category set

$$P_{ink} \subset (a_{nk}, b_{nk}) \cap C \setminus \bigcup_{j<2m<n} \bigcup_{l \in N_n} P_{jml}$$

and

$$\lim_{x \to a_{nk}^+} h_{ink}(x) = \lim_{x \to b_{nk}^-} h_{ink}(x) = \tau_{n-1};$$

moreover we require that

$$P_{0nk} \cap P_{1nk} = \emptyset.$$ 

Fix an $i < 2$. Put $h_i = \sum_{n \in \mathbb{N}} \sum_{k \in N_n} h_{ink}$. Notice that for each nondegenerate interval $I$, since $P_{ink}$ are pairwise disjoint first category sets (by condition (3)) and $h_{ink}(x) = 0$ outside of $P_{ink}$, the image

$$h_i[I] = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in N_n} h_{ink}[I]$$

is the union of a family of connected sets each of which contains 0. Therefore this set is connected as well, and $h_i$ is a nonnegative Darboux function.

Let $U \subset \mathbb{R}$ be an open set. If $0 \notin U$, then the set

$$h_i^{-1}(U) = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in N_n} h_{ink}^{-1}(U)$$

is a countable union of $F_\sigma$-sets, whence it is an $F_\sigma$-set as well. In the opposite case choose an $n_0 \in \mathbb{N}$ such that $[0, \tau_{n_0}) \subset U$. Then

$$h_i^{-1}(U) = (\mathbb{R} \setminus C) \cup h_i^{-1}(U \setminus \{0\}) \cup \bigcap_{n \leq n_0} \bigcup_{k \in N_n} ((a_{nk}, b_{nk}) \cap h_{ink}^{-1}(U)),$$

so $h_i^{-1}(U)$ is an $F_\sigma$-set. Therefore $h_i$ is a Baire one function.

Define $g_i = f - h_i$. To prove that $g_i$ is Darboux, we will use Proposition 2. Fix an $x \in \mathbb{R}$. By (2), there is a sequence $x_n \nearrow x$ such that

$$\lim_{n \to \infty} f(x_n) \geq f(x).$$

For each $n \in \mathbb{N}$, if $x_n \in C$, then choose a point

$$t_n \in (x_n - 1/n, x_n) \setminus \bigcup_{m \in \mathbb{N}} \bigcup_{k \in N_n} P_{imk}.$$
such that

$$|f(t_n) - f(x_n)| < 1/n;$$

otherwise set $t_n = x_n$. Then

$$\lim_{t \to x^-} g_i(t) \geq \lim_{n \to \infty} g_i(t_n) = \lim_{n \to \infty} f(t_n) \geq f(x) \geq g_i(x).$$

To prove that $\lim_{t \to x^-} g_i(t) \leq g_i(x)$, we consider two cases.

If $x \in C$, then let $x_n \nearrow x$ be such that $\lim_{n \to \infty} h_i(x_n) \geq h_i(x)$. (Recall that $h_i$ is Darboux.) Then

$$\lim_{t \to x^-} g_i(t) \leq \lim_{n \to \infty} g_i(x_n) = \lim_{n \to \infty} f(x_n) - \lim_{n \to \infty} h_i(x_n)$$

$$\leq f(x) - h_i(x) = g_i(x).$$

Now let $x \notin C$. Then $x \in A_m \setminus A_{m-1}$ for some $m \in \mathbb{N}$. There is a sequence $(b_{mk_n})$ (maybe constant) such that $b_{mk_n} \to x$ and $b_{mk_n} \leq x$ for each $n$. By (4), for each $n \in \mathbb{N}$ there is an $x_n \in (b_{mk_n} - 1/n, b_{mk_n})$ such that

$$h_i(x_n) \geq \min\{\tau_{m-1} - 1/n, n\}.$$ Then

$$\lim_{t \to x^-} g_i(t) \leq \lim_{n \to \infty} g_i(x_n) \leq \lim_{n \to \infty} f(x_n) - \lim_{n \to \infty} h_i(x_n)$$

$$\leq \lim_{t \to x^-} f(t) - \tau_{m-1} \leq f(x) = g_i(x).$$

(We used the fact that the function $f$ is nonpositive and $\text{osc}(f, x) \leq \tau_{m-1}$.)

Similarly we can show that $\lim_{t \to x^+} g_i(t) \leq g_i(x) \leq \lim_{t \to x^+} g_i(t)$. So, $g_i$ is Darboux.

Finally observe that by (3) and (5),

$$\{x \in \mathbb{R} : g_0(x) \neq f(x)\} \subset \{x \in \mathbb{R} : g_1(x) = f(x)\}.$$ So, since $h_0$ and $h_1$ are nonnegative, $f = \max\{g_0, g_1\}$ on $\mathbb{R}$. This completes the proof in case $f$ is nonpositive.

Finally let $f$ be an arbitrary Baire one function fulfilling condition (2). Let $\varphi: (-\infty, 0) \to \mathbb{R}$ be an increasing homeomorphism. By the first part of the proof, there are Darboux Baire one functions $\tilde{g}_0$ and $\tilde{g}_1$ such that $\varphi^{-1} \circ f = \max\{\tilde{g}_0, \tilde{g}_1\}$ on $\mathbb{R}$. Define $g_i = \varphi \circ \tilde{g}_i$ ($i < 2$). One can easily see that $g_0$ and $g_1$ fulfill the requirements of the theorem. □
**Corollary 4.** Let $f: \mathbb{R} \to \mathbb{R}$. The following are equivalent:

(i) $f$ is the maximum of Darboux Baire one functions,
(ii) $f$ is a Baire one function which fulfills condition (1),
(iii) $f$ is a Baire one function which fulfills condition (2).

**Proof.** The implication (ii) $\implies$ (iii) is evident, the implication (iii) $\implies$ (i) follows from Theorem 3, and the implication (i) $\implies$ (ii) follows from [4; Theorem 3]. \qed 

**References**


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