Karla Čipková; Danica Jakubíková-Studenovská
Graph homomorphisms of partial monounary algebras

Mathematica Slovaca, Vol. 55 (2005), No. 2, 155--167

Persistent URL: http://dml.cz/dmlcz/131431

Math. Slovaca, **55** (2005), No. 2, 155–167

Mathematica Slovaca
© 2005
Mathematical Institute
Slovak Academy of Sciences

GRAPH HOMOMORPHISMS OF PARTIAL MONOUNARY ALGEBRAS

Karla Čipková* — Danica Jakubíková-Studenovská**

(*Communicated by Pavol Zlatoš*)

ABSTRACT. The aim of the paper is to study mappings of partial monounary algebras that respect their graph representation. The idea is based on the notion of strong homomorphisms of graphs. A mapping of a partial monounary algebra (V_1, f_1) into (V_2, f_2) is said to be a graph homomorphism if it is a strong homomorphism of the graphs corresponding to the partial algebras. We describe graph homomorphisms of partial monounary algebras.

0. Introduction

The advantage of partial monounary algebras is their relatively simple visualization. They can be represented by a graph, both oriented and unoriented; furthermore, the graph of a partial monounary algebra is always planar, hence easy to draw. The objective of this paper is to study mappings on partial monounary algebras that respect their graph representations. Namely, the idea is based on the notion of strong homomorphism of graphs. To each partial monounary algebra (V, f) there corresponds a graph $\mathcal{G}(V, f)$. Then a mapping $\varphi\colon V_1 \to V_2$ is called a *graph homomorphism* ($\mathcal{G}$*-homomorphism*) of a partial monounary algebra (V_1, f_1) into (V_2, f_2) if φ is a strong homomorphism of the graph $\mathcal{G}(V_1, f_1)$ into $\mathcal{G}(V_2, f_2)$. We will describe $\mathcal{G}$-homomorphisms of partial monounary algebras.

Let us remark that we consider two possibilities of defining a graph corresponding to a given monounary algebra either without loops or admitting loops, which yields to the notions of a $\mathcal{G}$-homomorphism and a $\mathcal{G}'$-homomorphism.

2000 Mathematics Subject Classification: Primary 08A60.
Keywords: (partial) monounary algebra, graph, graph homomorphism, graph algebra.

Supported by grant VEGA 1/0423/03.

1. Preliminaries

By a *partial monounary algebra* we will understand a pair (V, f) where V is a nonempty set and f is a partial unary operation defined on $A \subseteq V$. i.e. to each $a \in \operatorname{dom} f \subseteq V$ an element $f(a) \in V$ is assigned (cf. e.g. [7]). If $\operatorname{dom} f = V$, then (V, f) is called a (*total*) *monounary algebra.* Monounary algebras and partial monounary algebras were studied by several authors, e.g. [3], [6].

Let (V, f) be a partial monounary algebra. We denote by $\mathcal{G}(V, f) = (V, E)$ the graph such that

$$\big(\forall x \in V\big)\big(\forall y \in V\big)\big((x, y) \in E \iff f(x) = y\big).$$

Let (V, f) be a partial monounary algebra and let $x, y \in V$. We use the notation $f(x) = y$ if $f(x)$ exists (is defined), i.e. if $x \in \operatorname{dom} f$, and is equal to y. The notation $f(x) \neq y$ means that either $f(x)$ does not exist, or $f(x)$ exists but is not equal to y. In any other formula where the symbol $f(x)$ appears, we suppose that $f(x)$ is defined, if we do not explicitly state otherwise. For instance: suppose that φ is a mapping of V into V and $x, y \in V$. Then the formula $f\big(\varphi(y)\big) = \varphi\Big(f\big(f\big(\varphi(x)\big)\big)\Big)$ means that $f\big(\varphi(y)\big)$ is defined, both $f\big(\varphi(x)\big)$ and $f\big(f\big(\varphi(x)\big)\big)$ are defined as well and the mapping φ assigns the element $f\big(\varphi(y)\big)$ to the element $f\big(f\big(\varphi(x)\big)\big)$.

For any $x \in V$ we set $f^0(x) = x$. If $n \in \mathbb{N} \cup \{0\}$, $f^n(x) = y$ and if $f(y)$ exists, then we set $f^{n+1}(x) = f(y)$. If $M \subseteq V$, we denote by $f^n(M)$ the set of $f^n(x)$ for all $x \in M$ where $f^n(x)$ exists. Further, for arbitrary $k \in \mathbb{N}$ we denote by $f^{-k}(x)$ the set of all elements $y \in V$ such that $f^k(y) = x$ and $f^{-k}(M) = \bigcup\limits_{x \in M} f^{-k}(x)$.

An element $x \in V$ is referred to as *cyclic*, if there exists a non-negative integer n such that $f^n(x) = x$.

A partial monounary algebra (V, f) is called *connected* if for arbitrary elements $x, y \in V$ there are non-negative integers u, v such that $f^u(x) = f^v(y)$. A maximal connected subalgebra of a partial monounary algebra is called a (*connected*) *component.*

Given two partial monounary algebras (V_1, f_1), (V_2, f_2) and a mapping $\varphi\colon V_1 \to V_2$. The mapping φ is called a *homomorphism* of the partial monounary algebra (V_1, f_1) into the partial monounary algebra (V_2, f_2) if for all elements $x \in \operatorname{dom} f_1$ the condition $\varphi\big(f_1(x)\big) = f_2\big(\varphi(x)\big)$ is satisfied.

The set $\operatorname{Im}\varphi$ is the set of $\varphi(x)$ for all $x \in V_1$ or expressed in another way $\operatorname{Im}\varphi = \big\{y \in V_2 :\ \big(\exists x \in V_1\big)\big(\varphi(x) = y\big)\big\}$. The set $\operatorname{Im}\varphi$ is called the *image* of the mapping φ.

Let V be a nonempty set, E a binary relation on V. Then the corresponding relational structure (V, E) is called a (*directed*) *graph.* We admit the existence of loops in a graph.

1.1. Definition. Let (V_1, E_1), (V_2, E_2) be graphs, φ be a homomorphism of (V_1, E_1) into (V_2, E_2). Then φ is called a *strong homomorphism of* (V_1, E_1) *into* (V_2, E_2) if for any $x, y \in V_1$ the condition $(x, y) \in E_1$ is satisfied if and only if $\bigl(\varphi(x), \varphi(y)\bigr) \in E_2$ holds.

Remark. The terminology concerning strong homomorphisms is not uniform in literature. For example in [5], a strong homomorphism of a graph (V_1, E_1) into a graph (V_2, E_2) is defined as a homomorphism φ of (V_1, E_1) into (V_2, E_2) which satisfies the following condition: if $\bigl(\varphi(x), \varphi(y)\bigr) \in E_2$ for any $x, y \in V_1$, then there are elements $\bar{x}, \bar{y} \in V_1$ such that $(\bar{x}, \bar{y}) \in E_1$, $\varphi(\bar{x}) = \varphi(x)$ and $\varphi(y) = \varphi(y)$. For other notion of a strong homomorphism cf. also [8] and [9].

1.2. Definition. Let (V_1, f_1), (V_2, f_2) be partial monounary algebras, $\mathcal{G}(V_1, f_1)$, $\mathcal{G}(V_2, f_2)$ be their graphs and let φ be a mapping of V_1 into V_2. The mapping φ is denoted as a *$\mathcal{G}$-homomorphism (graph homomorphism) of the partial monounary algebra (V_1, f_1) into the partial monounary algebra (V_2, f_2)* if φ is a strong homomorphism of the graph $\mathcal{G}(V_1, f_1)$ into the graph $\mathcal{G}(V_2, f_2)$.

2. Remark on graph algebras

We should not avoid the mention of graph (or Shallon) algebras that inspired us to deal with graph homomorphisms of partial monounary algebras.

Let (V, E) be a graph. To the graph (V, E) an algebra $\mathcal{A}(V, E) = \bigl(V \cup \{\infty\}, \infty, \circ\bigr)$ is assigned in the following way: $\infty \notin V$ is a nullary operation and for any $x, y \in V \cup \{\infty\}$, the binary operation $\circ$ is defined by the following formula

$$x \circ y = \begin{cases} x & \text{if } (x, y) \in E\,, \\ \infty & \text{otherwise.} \end{cases}$$

Then $\mathcal{A}(V, E)$ is called the *graph algebra corresponding to* (V, E). Graph algebras were introduced by R. C. Shallon [14] and were studied, e.g. in [4], [10] [13].

The proof of the following proposition on graph algebras will be omitted since in what follows, we do not refer to graph algebras themselves.

2.1. Proposition. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras, $(V_1, E_1) = \mathcal{G}(V_1, f_1)$, $(V_2, E_2) = \mathcal{G}(V_2, E_2)$ be their graphs and $\mathcal{A}_1 = \mathcal{A}(V_1, E_1)$, $\mathcal{A}_2 = \mathcal{A}(V_2, E_2)$ be graph algebras corresponding to graphs (V_1, E_1) and (V_2, E_2) respectively. Next suppose that Φ is a mapping of $V_1 \cup \{\infty_1\}$ into $V_2 \cup \{\infty_2\}$ such that $\Phi(\infty_1) = \infty_2$ and $\varphi = \Phi \restriction V_1$ is a mapping of the set V_1 into V_2. Then the following conditions are equivalent:*

(i) *Φ is a homomorphism of $\mathcal{A}_1$ into $\mathcal{A}_2$,*
(ii) *φ is a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2),*
(iii) *φ is a strong homomorphism of (V_1, E_1) into (V_2, E_2).*

3. Graph homomorphisms

In this part we look into the basic properties of $\mathcal{G}$-homomorphisms.

3.1. Lemma. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras and φ be a mapping of V_1 into V_2. Then the following conditions are equivalent:*
(i) *φ is a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2);*
(ii) *$f_1(x) = y$ if and only if $f_2(\varphi(x)) = \varphi(y)$ for any $x, y \in V_1$, i.e. $f_1(x)$ exists and is equal to y if and only if $f_2(\varphi(x))$ exists and is equal to $\varphi(y)$.*

Proof. By Definitions 1.1 and 1.2, the mapping φ is a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2) if and only if

$$(x, y) \in E_1 \iff (\varphi(x), \varphi(y)) \in E_2 ,$$

which is equivalent to the condition (ii). □

It is natural to ask a question concerning the relation between homomorphisms and $\mathcal{G}$-homomorphisms of partial monounary algebras.

3.2. Lemma. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras and φ be a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2). Then φ is the homomorphism of the algebra (V_1, f_1) into the algebra (V_2, f_2).*

Proof. By Lemma 3.1(ii) we have $f_2(\varphi(x)) = \varphi(f_1(x))$ for every $x \in \operatorname{dom} f_1$, i.e. φ is a homomorphism. □

Remark. The converse implication does not hold. Consider two simple monounary algebras:

$(\{a, b, c\}, f_1)$ where $f_1(a) = b$ and $f_1(b) = f_1(c) = c$,
$(\{d\}, f_2)$ where $f_2(d) = d$,

and a mapping $\varphi\colon \{a, b, c\} \to \{d\}$ such that $\varphi(a) = \varphi(b) = \varphi(c) = d$.

The mapping φ is a homomorphism, but is not a $\mathcal{G}$-homomorphism, since $f_2(\varphi(a)) = \varphi(c)$ and $f_1(a) \neq c$.

3.3. Lemma. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras, φ be a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2) and let n be a positive integer. Then*

$$(\forall x \in V_1)(\forall y \in V_1)\big(f_1^n(x) = y \iff f_2^n(\varphi(x)) = \varphi(y)\big) .$$

Proof. The proof immediately follows from Lemma 3.1(ii) by induction on n. □

3.4. Corollary. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras and φ be a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2). Further, let $x \in V_1$ and let m, n be minimal integers such that $f_1^m(x)$ and $f_2^n(\varphi(x))$ are cyclic elements. Then $m = n$.*

Now we will try to look into the way of how a $\mathcal{G}$-homomorphism assigns the images to those elements of a partial monounary algebra that do not belong to the domain of its operation.

3.5. Lemma. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras and φ be a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2). Further suppose that $x \in V_1 \setminus \operatorname{dom} f_1$ and set $z = \varphi(x)$. Then the element z has the following properties:*

(i) *either $z \notin \operatorname{dom} f_2$, or $f_2(z) \notin \operatorname{Im} \varphi$,*
(ii) *$f_1^{-1}(x) = \emptyset$ if and only if $f_2^{-1}(z) \cap \operatorname{Im} \varphi = \emptyset$.*

Proof. Let $x \in V_1 \setminus \operatorname{dom} f_1$ and $z = \varphi(x)$.

(i) Suppose that $z \in \operatorname{dom} f_2$ and $f_2(z) \in \operatorname{Im} \varphi$, thus there is an element $y \in V_1$ such that $\varphi(y) = f_2(z)$. Then $\varphi(y) = f_2(z) = f_2(\varphi(x))$ implies $y = f_1(x)$ (see Lemma 3.1), which is a contradiction with $x \notin \operatorname{dom} f_1$. Hence either $z \notin \operatorname{dom} f_2$ or $f_2(z) \notin \operatorname{Im} \varphi$.

(ii) If $f_1^{-1}(x) \neq \emptyset$, then there exists an element $y \in V_1$ such that $f_1(y) = x$. In view of Lemma 3.1 we have $z = \varphi(x) = f_2(\varphi(y))$, which implies that $f_2^{-1}(z) \cap \operatorname{Im} \varphi \neq \emptyset$.

Conversely if $f_2^{-1}(z) \cap \operatorname{Im} \varphi \neq \emptyset$, then there is an element $y \in V_1$ such that $\varphi(y) \in f_2^{-1}(z)$. Hence $\varphi(x) = z = f_2(\varphi(y))$. Using Lemma 3.1 we obtain $x = f_1(y)$. Thus $f_1^{-1}(x) \neq \emptyset$. □

3.6. Lemma. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras and φ be a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2). If x, y are distinct elements of V_1 such that $\varphi(x) = \varphi(y)$, then $f_1^{-1}(x) = \emptyset$.*

Proof. Let $x, y \in V_1$, $x \neq y$ and $\varphi(x) = \varphi(y)$. Suppose that $f_1^{-1}(x) \neq \emptyset$. Then there is an element $\bar{x} \in V_1$ such that $f_1(\bar{x}) = x$. Then, using Lemmas 3.1 and 3.2, the equalities

$$\varphi(y) = \varphi(x) = \varphi(f_1(\bar{x})) = f_2(\varphi(\bar{x}))$$

entail $y = f_1(\bar{x})$. Hence $y = f_1(\bar{x}) = x$, which contradicts the assumption $x \neq y$. Therefore $f_1^{-1}(x) = \emptyset$. □

3.7. Lemma. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras and φ be a $\mathcal{G}$-homomorphism. If x, y are distinct elements of V_1 such that $\varphi(x) = \varphi(y)$, then*

(i) $x \in \operatorname{dom} f_1 \iff y \in \operatorname{dom} f_1$,
(ii) $x \in \operatorname{dom} f_1 \implies f_1(x) = f_1(y) \notin \{x, y\}$.

P r o o f . Let $x, y \in V_1$, $x \neq y$, and $\varphi(x) = \varphi(y)$. Suppose that $x \in \operatorname{dom} f_1$ and put $z = f_1(x)$. Then by Lemma 3.1 we obtain $\varphi(z) = f_2(\varphi(x)) = f_2(\varphi(y))$, which entails $z = f_1(y)$, i.e. $y \in \operatorname{dom} f_1$ as well. The converse implication of (i) can be proved analogously. Furthermore, we showed that $f_1(x) = z = f_1(y)$ and by Lemma 3.6 we have $z \notin \{x, y\}$, which concludes the proof of (ii). □

3.8. NOTATION. Let (V, f) be a partial monounary algebra. $K(V, f)$ will be a binary relation on V such that a pair $(a, b) \in V \times V$ belongs to $K(V, f)$ if and only if $f^{-1}(a) = f^{-1}(b) = \emptyset$ and either $\{a, b\} \cap \operatorname{dom} f = \emptyset$ or $f(a) = f(b)$. Obviously, $K(V, f)$ is an equivalence relation. Further, $P(V, f)$ will denote the union of all elements in those blocks of the equivalence $K(V, f)$ that contain more than one element, i.e.

$$P(V, f) = \{x \in V : [x]_{K(V,f)} \neq \{x\}\}.$$

The following lemmas and example deal with the connectivity of a partial monounary algebra and its relation to a $\mathcal{G}$-homomorphism.

3.9. LEMMA. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras and φ be a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2). If elements $x, y \in V_1$ belong to the same connected component of (V, f_1), then so do $\varphi(x), \varphi(y)$ in (V_2, f_2).*

P r o o f . Let x, y belong to the same connected component of (V_1, f_1) hence there are non-negative integers u, v such that $f_1^u(x) = f_1^v(y)$. In view of Lemmas 3.2 and 3.3 we obtain $f_2^u(\varphi(x)) = \varphi(f_1^u(x)) = \varphi(f_1^v(y)) = f_2^v(\varphi(y))$, therefore the elements $\varphi(x)$ and $\varphi(y)$ belong to the same connected component of (V_2, f_2). □

Remark. Lemma 3.9 states that a connected component does not split under $\mathcal{G}$-homomorphisms. On the other hand, a $\mathcal{G}$-homomorphism can join two connected component, i.e. the converse of 3.9 does not hold as the following example shows.

3.10. EXAMPLE. Consider the partial monounary algebras $(\{a, b\}, f_1)$ with $\operatorname{dom} f_1 = \emptyset$ and $(\mathbb{N}, f_2)$ where $f_2(i) = i + 1$ for all $i \in \mathbb{N}$. The mapping $\varphi\colon \{a, b\} \to \mathbb{N}$ satisfying $\varphi(a) = 4$ and $\varphi(b) = 15$ is a $\mathcal{G}$-homomorphism (see Lemma 3.1). Notice, that elements a, b belong to different connected components of $(\{a, b\}, f_1)$ but $\varphi(a)$ and $\varphi(b)$ belong to the same component of $(\mathbb{N}, f_2)$.

This example suggests that the elements that do not belong to the domain of the operation spoil the converse implication in Lemma 3.9.

3.11. LEMMA. *Let (V_1, f_1), (V_2, f_2) be total monounary algebras, φ be a $\mathcal{G}$-homomorphism and let $x, y \in V_1$. Then the elements x, y belong to the same connected component of (V_1, f_1) if and only if do so $\varphi(x)$, $\varphi(y)$ in (V_2, f_2).*

P r o o f . Let $\operatorname{dom} f_1 = V_1$. The first part of the proof follows from Lemma 3.9. Conversely, suppose that $\varphi(x)$, $\varphi(y)$ belong to the same connected component of the algebra (V_2, f_2), i.e. there are $u, v \in \mathbb{N}_0$ such that $f_2^u(\varphi(x)) = f_2^v(\varphi(y))$. Then using the assumption of Lemma 3.2 we have $\varphi(f_1^u(x)) = \varphi(f_1^v(y))$. If $f_1^u(x) \neq f_1^v(y)$, then by Lemma 3.7 we have $f_1(f_1^u(x)) = f_1(f_1^v(y))$ which means that x, y belong to the same connected component. If $f_1^u(x) = f_1^v(y)$, we immediately come to the same conclusion. $\square$

4. Characterization result

Let (V_1, f_1), (V_2, f_2) be partial monounary algebras, φ be a mapping of V_1 into V_2. For $x, y \in V_1$ we put $(x, y) \in \ker\varphi$ if and only if $\varphi(x) = \varphi(y)$. Binary relation $\ker\varphi$ is the equivalence and is called the *kernel* of the mapping φ. We denote by $P(\varphi)$ the system of all elements that belong to those blocks of the equivalence $\ker\varphi$ which contain more than one element, i.e.

$$P(\varphi) = \{x \in V : [x]_{\ker\varphi} \neq \{x\}\}.$$

Let (V, f) be a partial monounary algebra, (V, E) be its graph and let M be a set such that $\emptyset \neq M \subseteq V$. A partial algebra (M, g) will be called a *relative subalgebra of* (V, f) if its graph $\mathcal{G}(M, g)$ is exactly $(M, E \cap (M \times M))$. The partial operation f is thus "strongly" reduced to M, therefore instead of g we will write $f \restriction_s M$. By notation $f \restriction_s M$ we thus understand that if for $x \in M$ the element $f(x)$ does not belong to M, then $x \notin \operatorname{dom}(f \restriction_s M)$. The relative subalgebra $(M, f\restriction_s M)$ is called an *absolute subalgebra of* (V, f) (cf. [2]) if the condition

$$E \cap (M \times M) = E \cap (M \times V)$$

is valid.

4.1. PROPOSITION. *Let φ be a $\mathcal{G}$-homomorphism of a partial monounary algebra (V_1, f_1) into a partial monounary algebra (V_2, f_2). Set $M_1 = V_1 \setminus P(\varphi)$ and suppose that $M_1 \neq \emptyset$. Then the partial algebra $(M_1, f_1\restriction_s M_1)$ is an absolute subalgebra of (V_1, f_1) and $\varphi \restriction M_1$ is an isomorphism of the partial algebra $(M_1, f_1\restriction_s M_1)$ onto a relative subalgebra of (V_2, f_2).*

P r o o f .

(a) If $x \in \operatorname{dom} f_1$, then $f_1(x) \notin P(\varphi)$, which is a consequence of Lemma 3.6. It implies that for any $x \in M_1 \cap \operatorname{dom} f_1 = (V_1 \setminus P(\varphi)) \cap \operatorname{dom} f_1$, we obtain

$f_1(x) \in V_1 \setminus P(\varphi) = M_1$. Thus, $(M_1, f_1\restriction_s M_1)$ is an absolute subalgebra of (V_1, f_1).

(b) Next, let $M_2 = \operatorname{Im}(\varphi \restriction M_1)$. Then $(M_2, f_2\restriction_s M_2)$ is a relative subalgebra of (V_2, f_2).

(c) Lemma 3.2 implies that φ is a homomorphism of (V_1, f_1) into (V_2, f_2). Then $\varphi \restriction M_1$ is a homomorphism of $(M_1, f_1\restriction_s M_1)$ onto $(M_2, f_2\restriction_s M_2)$.

(d) The equivalence $\ker\varphi$ has only one-element blocks on the set M_1, which yields that the mapping $\psi = \varphi \restriction M_1$ is injective. Thus, ψ is a bijection of M_1 onto M_2. By (c), ψ is an isomorphism of $(M_1, f_1\restriction_s M_1)$ onto $(M_2, f_2\restriction_s M_2)$. □

NOTATION. Let (V_1, f_1), (V_2, f_2) be partial monounary algebras and φ be a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2). For $A \in V_1/\ker\varphi$ we set

$$F_1(A) = \{f_1(a) : \ a \in A \cap \operatorname{dom} f_1\},$$

if $A \cap \operatorname{dom} f_1 \neq \emptyset$; otherwise $F_1(A)$ is not defined. Notice that F_1 is a partial unary operation on $V_1/\ker\varphi$. If there is no danger of confusion, we will use the symbol f_1 instead of F_1.

4.2. LEMMA. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras and φ be a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2). Then there exists an absolute subalgebra $(U_1, f_1\restriction_s U_1)$ of (V_1, f_1) such that*

(i) *$A \cap U_1 \neq \emptyset$ for each $A \in V_1/\ker\varphi$,*

(ii) $(U_1, f_1\restriction_s U_1) \cong (V_1/\ker\varphi, f_1) \cong (\operatorname{Im}\varphi, f_2\restriction_s \operatorname{Im}\varphi)$.

Proof. It is easy to show that the mapping $\psi\colon [x]_{\ker\varphi} \mapsto \varphi(x)$, whenever $x \in V_1$, is an isomorphism of $(V_1/\ker\varphi, f_1)$ onto $(\operatorname{Im}\varphi, f_2\restriction_s \operatorname{Im}\varphi)$.

If $A \in V_1/\ker\varphi$, then let $\nu(A)$ be a fixed element of A. Put

$$U_1 = \{\nu(A) : \ A \in V_1/\ker\varphi\}.$$

It is obvious that (i) is valid.

Let us show that $(U_1, f_1\restriction_s U_1)$ is an absolute subalgebra of (V_1, f_1). Let $x \in U_1 \cap \operatorname{dom} f_1$. If $f_1(x) = x$, then $f_1(x) \in U_1$. If $f_1(x) = y \neq x$, then $f_1^{-1}(y) \neq \emptyset$ and in view of Lemma 3.6 we obtain $y \notin P(\varphi)$, hence $[y]_{\ker\varphi} = \{y\}$, therefore $\nu([y]_{\ker\varphi}) = y$ and $y \in U_1$.

Clearly, ν is a bijection of $V_1/\ker\varphi$ onto U_1, since $\ker\varphi$ is an equivalence. Let $A \in V_1/\ker\varphi$ and $a = \nu(A)$. Then $\nu(f_1(A)) = \nu(\{f_1(x) : \ x \in A\}) \in \{f_1(x) : \ x \in A\}$, thus $\nu(f_1(A)) = f_1(b)$ for some $b \in A$. Since $(a, b) \in \ker\varphi$, Lemma 3.7 yields that $f_1(a) = f_1(b)$. Hence

$$\nu(f_1(A)) = f_1(a) = f_1(\nu(A)),$$

i.e. ν is an isomorphism of $(V_1/\ker\varphi, f_1)$ onto $(U_1, f_1\restriction_s U_1)$. □

Our aim is to prove the "characterization theorem" 4.3, out of which a way of looking for all $\mathcal{G}$-homomorphisms of (V_1, f_1) into (V_2, f_2) could be concluded: first we take an absolute subalgebra $(U_1, f_1 \upharpoonright U_1)$ of (V_1, f_1) with the property that $A \in V_1/K(V_1, f_1)$ implies $A \cap U_1 \neq \emptyset$, and an isomorphism ψ of $(U_1, f_1 \upharpoonright U_1)$ into (V_2, f_2) satisfying (4). Further, we extend ψ into a mapping φ in such a way that φ satisfies the conditions (5) and (6).

4.3. THEOREM. *Let (V_1, f_1), (V_2, f_2) be partial monounary algebras, $\varphi\colon V_1 \to V_2$ be a mapping. Then φ is a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2) if and only if there exist an absolute subalgebra $(U_1, f_1 \upharpoonright_s U_1)$ of (V_1, f_1) and a mapping $\psi\colon U_1 \to V_2$ such that the following conditions are satisfied:*

(1) *if $A \in V_1/K(V_1, f_1)$, then $A \cap U_1 \neq \emptyset$;*
(2) *ψ is an isomorphism of $(U_1, f_1 \upharpoonright_s U_1)$ into (V_2, f_2);*
(3) *$\varphi \upharpoonright U_1 = \psi$;*
(4) *$\psi(U_1 \setminus \operatorname{dom} f_1) \cap f_2^{-1}(\operatorname{Im} \psi) = \emptyset$;*
(5) *if $a \in (V_1 \setminus U_1) \cap \operatorname{dom} f_1$, then there is $u \in U_1 \cap P(V_1, f_1)$ such that $f_1(a) = f_1(u)$, $\varphi(a) = \psi(u)$;*
(6) *if $a \in (V_1 \setminus U_1) \setminus \operatorname{dom} f_1$, then there is $u \in \big(U_1 \cap P(V_1, f_1)\big) \setminus \operatorname{dom} f_1$ such that $\varphi(a) = \psi(u)$.*

Proof.

I. First, let φ be a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2). By Lemma 4.2 there exists an absolute subalgebra $(U_1, f_1 \upharpoonright_s U_1)$ of (V_1, f_1) and let ψ be a mapping of $(U_1, f_1 \upharpoonright_s U_1)$ into (V_2, f_2) described in proof of Lemma 4.2.

Then $(U_1, f_1 \upharpoonright_s U_1)$ and ψ satisfy the conditions (2) and (3) respectively. We shall show that $(U_1, f_1 \upharpoonright_s U_1)$ also satisfies other conditions of the theorem.

Assume that (4) fails to hold. Then there are $x, y \in V_1$ such that $x \in U_1 \setminus \operatorname{dom} f_1$, $\psi(x) \in f_2^{-1}\big(\psi(y)\big)$, thus $f_2\big(\psi(x)\big) = \psi(y)$, i.e. $f_2\big(\varphi(x)\big) = \varphi(y)$. The mapping φ is a $\mathcal{G}$-homomorphism, hence Lemma 3.1 yields $f_1(x) = y$, $x \in \operatorname{dom} f_1$, which is a contradiction.

Let $A \in V_1/K(V_1, f_1)$. In view of Lemmas 3.6 and 3.7, for the equivalence relations $\ker\varphi$ and $K(V_1, f_1)$ we have $\ker\varphi \subseteq K(V_1, f_1)$. Thus 4.2(i) implies that $A \cap U_1 \neq \emptyset$, hence (1) is valid.

Eventually, suppose that $a \in V_1 \setminus U_1$. Then $[a]_{\ker\varphi} \neq \{a\}$ and since $[a]_{\ker\varphi} \cap U_1 \neq \emptyset$, we get that there is $u \in [a]_{\ker\varphi} \cap U_1$. Hence $\varphi(a) = \varphi(u) = \psi(u)$. If $a \in \operatorname{dom} f_1$, then Lemma 3.7 implies that $f_1(a) = f_1(u)$. If $a \notin \operatorname{dom} f_1$, then again by Lemma 3.7, $u \notin \operatorname{dom} f_1$. Moreover, $[u]_{\ker\varphi} \neq \{u\}$, thus Lemma 3.7 yields that $u \in P(V_1, f_1)$. Therefore (5) and (6) hold.

II. Conversely, suppose that there exist an absolute subalgebra $(U_1, f_1 \upharpoonright U_1)$ of (V_1, f_1) and a mapping $\psi\colon U_1 \to V_2$ such that the conditions (1) (6) are satisfied. By (1), for each $a \in V_1 \setminus U_1$, the block $[a]_{K(V_1, f_1)}$ contains more than one element, thus the element $a \in P(V_1, f_1)$. Let $x, y \in V_1$. Now we will apply

Lemma 3.1 to prove that the mapping φ is a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2).

(a) Assume that $f_1(x) = y$. Then $[y]_{K(V_1,f_1)} = \{y\}$, thus (1) implies $y \in U_1$ and by (3), $\varphi(y) = \psi(y)$. In view of (5) there is $u \in U_1$ with $f_1(x) = f_1(u)$ and $\varphi(x) = \psi(u)$. Using (2) and (3) we get

$$\varphi(y) = \psi(y) = \psi\big(f_1(x)\big) = \psi\big(f_1(u)\big) = f_2\big(\psi(u)\big) = f_2\big(\varphi(x)\big).$$

(b) Now suppose that $f_2\big(\varphi(x)\big) = \varphi(y)$.

First, let $x \in U_1 \cap \operatorname{dom} f_1$. Since $(U_1, f_1\restriction_s U_1)$ is an absolute subalgebra of (V_1, f_1), $f_1(x) \in U_1$. If $y \notin U_1$, by (5) or (6) there is $v \in U_1 \cap P(V_1, f_1)$ with $\varphi(y) = \psi(v)$ and then

$$f_2\big(\psi(x)\big) = f_2\big(\varphi(x)\big) = \varphi(y) = \psi(v).$$

By (2) the mapping ψ is an isomorphism of $(U_1, f\restriction_s U_1)$ into (V_2, f_2), therefore $\psi\big(f_1(x)\big) = \psi(v)$ and $f_1(x) = v$. Then $v \notin P(V_1, f_1)$, which is a contradiction. If $y \in U_1$, then $\varphi(y) = \psi(y)$, $f_2\big(\psi(x)\big) = \psi(y)$ and the condition (2) yields $y = f_1(x)$.

Next, let $x \in U_1 \setminus \operatorname{dom} f_1$. As above, there is $v \in U_1$ with $\varphi(y) = \psi(v)$. Then $f_2\big(\psi(x)\big) = \psi(v)$, which implies $\psi(x) \in \psi(U_1 \setminus \operatorname{dom} f_1) \cap f_2^{-1}\big(\psi(v)\big)$, a contradiction to (4).

Finally, let $x \notin U_1$. In view of (5) and (6), there is $u \in U_1 \cap P(V_1, f_1)$ such that $\varphi(x) = \psi(u)$. If $y \in U_1$, then $\varphi(y) = \psi(y)$ and we have

$$\psi(y) = f_2\big(\psi(u)\big),$$

which, in view of (2), implies $f_1(u) = y$; therefore $u \in \operatorname{dom} f_1$. Now, using (5), we get $f_1(x) = f_1(u) = y$. Let $y \notin U_1$. Then there is $v \in U_1 \cap P(V_1, f_1)$ with $\varphi(y) = \psi(v)$. This implies $\psi(v) = f_2\big(\psi(u)\big)$ and according to (2), $f_1(u) = v$, which is a contradiction to $v \in P(V_1, f_1)$.

Hence we have proved that φ is a $\mathcal{G}$-homomorphism of (V_1, f_1) into (V_2, f_2), which concludes the proof. □

4.4. Definition. Let (V, f) be a partial monounary algebra. An absolute subalgebra $(T, f\restriction_s T)$ will be referred to as a *reduced subalgebra of* (V, f) if

(i) $P(T, f\restriction_s T) = \emptyset$,
(ii) $T \subset T' \subseteq V \implies P(T', f\restriction_s T') \neq \emptyset$.

4.5. Remark. The absolute subalgebra $(T, f\restriction_s T)$ satisfying the property above is uniquely determined up to isomorphism, hence it will be denoted $\big(\mathcal{R}(V), f\restriction_s \mathcal{R}(V)\big)$. It is easy to see that

$$\big(V/K(V,f), F\big) \cong \big(\mathcal{R}(V), f\restriction_s \mathcal{R}(V)\big),$$

where $F(A) = [f(a)]_{K(V,f)}$ for arbitrary $A \in V/K(V,f)$ and arbitrary $a \in A$.

Among absolute subalgebras of a partial algebra (V,f) the reduced subalgebra is minimal. Every $\mathcal{G}$-homomorphism can be characterized by applying a reduced subalgebra $(\mathcal{R}(V), f\upharpoonright_s \mathcal{R}(V))$ and the corresponding mapping ψ:

4.6. Corollary. *Let (V_1,f_1), (V_2,f_2) be partial monounary algebras, $\varphi\colon V_1 \to V_2$ be a mapping and let $(\mathcal{R}(V_1), f_1\upharpoonright_s \mathcal{R}(V_1))$ be the reduced subalgebra of (V_1,f_1). Then φ is a $\mathcal{G}$-homomorphism of (V_1,f_1) into (V_2,f_2) if and only if there exists a mapping $\psi\colon \mathcal{R}(V_1) \to V_2$ such that the following conditions are satisfied:*

(2) *ψ is an isomorphism of $(\mathcal{R}(V_1), f\upharpoonright_s \mathcal{R}(V_1))$ into (V_2,f_2);*
(3) *$\varphi \upharpoonright \mathcal{R}(V_1) = \psi$;*
(4) *$\psi(\mathcal{R}(V_1) \setminus \operatorname{dom} f_1) \cap f_2^{-1}(\operatorname{Im}\psi) = \emptyset$;*
(5) *if $a \in (V_1 \setminus \mathcal{R}(V_1)) \cap \operatorname{dom} f_1$, then there is $u \in \mathcal{R}(V_1) \cap P(V_1,f_1)$ such that $f_1(a) = f_1(u)$, $\varphi(a) = \psi(u)$;*
(6) *if $a \in (V_1 \setminus \mathcal{R}(V_1)) \setminus \operatorname{dom} f_1$, then there is $u \in (\mathcal{R}(V_1) \cap P(V_1,f_1)) \setminus \operatorname{dom} f_1$ such that $\varphi(a) = \psi(u)$.*

4.7. Remark. In the case $(\mathcal{R}(V_1), f_1\upharpoonright_s \mathcal{R}(V_1)) = (V_1,f_1)$, the mapping φ is a $\mathcal{G}$-homomorphism of (V_1,f_1) into (V_2,f_2) if and only if conditions (2) and (4) of 4.6 are satisfied.

5. On graphs without loops

Let (V,f) be a partial monounary algebra and let $\mathcal{G}(V,f)$ be the graph corresponding to (V,f) as it was defined above. We denote by $\mathcal{G}'(V,f) = (V,E')$ the graph such that

$$(\forall x \in V)(\forall y \in V)\big((x,y) \in E' \iff (x \neq y \;\&\; f(x) = y)\big).$$

Clearly, the graph $\mathcal{G}'(V,f)$ contains no loops, while in $\mathcal{G}(V,f)$ they are admitted.

We consider the two different structures (V,E) and (V,E'). The former one is more general and in spite of the fact that it comprises the other case as well, the latter structure is interesting, too because it deals with the graphs for which the elements of E are 2-element subsets of V (i.e. there are no loops or multi-edges) — a more common notion in the graph theory (cf.[1]). The other reason of mentioning both structures is an ambiguity with the notion of a graph of a monounary algebra by different authors. Similarly as $\mathcal{G}$-homomorphisms, $\mathcal{G}'$-homomorphisms of partial monounary algebras can be investigated as well. This case, of course, was not dealt separately, but follows from the former one.

Let (V, f) be a partial monounary algebra. It is easy to see that if x, y are the elements of V such that (x, y) is an edge of E' in the graph $\mathcal{G}'(V, f)$, then (x, y) belongs to E in the graph $\mathcal{G}(V, f)$. Thus $E' \subseteq E$ and therefore $\mathcal{G}'(V, f)$ is a subgraph of the graph $\mathcal{G}(V, f)$. In fact, the set $E \setminus E'$ contains only such pairs $(x, y) \in V \times V$, where $x = y$. Or in other words, the edges of $E \setminus E'$ are loops corresponding to one-element cycles in (V, f). If (V, f) contains no one-element cycles, then $\mathcal{G}(V, f) = \mathcal{G}'(V, f)$. Hence if we restrict the domain of f in such a way that the restricted operation (denote it f^*) contains no one-element cycles, i.e. if $\operatorname{dom} f^* = \operatorname{dom} f \setminus \{x \in V : f(x) = x\}$, we obtain a partial monounary algebra (V, f^*) with $\mathcal{G}'(V, f) = \mathcal{G}(V, f^*)$.

Let (V_1, f_1), (V_2, f_2) be partial monounary algebras. Concerning the aforementioned familiarity of two types of graphs it proves to be useful to look into more general ones, i.e. $\mathcal{G}_1 = \mathcal{G}(V_1, f_1)$ and $\mathcal{G}_2 = \mathcal{G}(V_2, f_2)$ because the graphs $\mathcal{G}'(V_1, f_1)$ and $\mathcal{G}'(V_2, f_2)$ are treated as well. Therefore, we first considered $\mathcal{G}$-homomorphisms of (V_1, f_1) into (V_2, f_2).

In a similar way as in Definition 1.2 we introduce the notion of a $\mathcal{G}'$-homomorphism of the partial monounary algebra (V_1, f_1) into (V_2, f_2). In the above definition we consider $\mathcal{G}'(V_1, f_1)$ instead of $\mathcal{G}(V_1, f_1)$ and $\mathcal{G}(V_2, f_2)$ is replaced by $\mathcal{G}'(V_2, f_2)$. Hence we obtain the following result:

Let φ be a mapping of V_1 into V_2. Then the mapping φ is a $\mathcal{G}'$-homomorphism of (V_1, f_1) into (V_2, f_2) if and only if φ is a $\mathcal{G}$-homomorphism of (V_1, f_1^*) into (V_2, f_2^*).

Therefore Theorems 4.3 or 4.6 enable to describe $\mathcal{G}'$-homomorphisms as well.

REFERENCES

[1] *Graph Theory and Related Topics* (J. A. Bondy, U. S. R. Murty, eds.), Proceedings of the Conference Held in Honour of Professor W. T. Tutte on the Occasion of his Sixtieth Birthday, University of Waterloo, July 5–9, 1977, Academic Press, New York-San Francisco-London, 1979.

[2] GRÄTZER, G.: *Universal Algebra* (2nd ed.), Springer-Verlag, New York, 1979.

[3] JAKUBÍKOVÁ-STUDENOVSKÁ, D.: *Monounary algebras and bottleneck algebras*, Algebra Universalis **40** (1998), 59–72.

[4] KISS, E. W.—PÖSCHEL, R.—PRÖHLE, P.: *Subvarieties of varieties generated by graph algebras*, Acta Sci. Math. (Szeged) **54** (1990), 57–75.

[5] MALTSEV, A. I.: *Algebraic Systems*, Nauka, Moscow, 1970. (Russian)

[6] NELSON, E.: *Homomorphisms of monounary algebras*, Pacif. J. Math. **82** (1999), 427–429.

[7] NOVOTNÝ, M.: *Mono-unary algebras in the work of Czechoslovak mathematicians*, Arch. Math. (Brno) **26** (1990), 155–164.

[8] NOVOTNÝ, M.: *Homomorphisms and strong homomorphisms of relational structures*, Arch. Math. (Brno) **33** (1997), 57–64.

[9] NOVOTNÝ, M.: *Homomorphisms of heterogeneous algebras*, Czechoslovak Math. J. **52** (2002), 415 428.

[10] OATES MACDONALD, S.—VAUGHAN-LEE, M.: *Varieties that make one Cross*, J. Aust. Math. Soc. Ser. A **26** (1978), 368 382.

[11] OATES-WILLIAMS, S.: *On the variety generated by Murskii's algebra*, Algebra Universalis **18** (1984), 175 177.

[12] PÖSCHEL, R.: *Shallon algebras and varieties for graphs and relational systems.* In: Algebra und Graphentheorie, Jahrestagung Algebra und Grenzgebiete 28.10.-1.11.1985 in Siebenlehn (J. Machner, G. Schaar, eds.), Bergakademie Freiberg, Sekt. Math., Siebenlehn (GDR), 1986, pp. 53–56.

[13] PÖSCHEL, R.: *Graph algebras and graph varieties*, Algebra Universalis **27** (1990), 559 577.

[14] SHALLON, C. R.: *Nonfinitely Based Finite Algebras Derived from Lattices.* Ph.D. Dissertation, U.C.L.A, 1979.

Received February 10, 2003

Revised October 6, 2003

* *Department of Mathematics*
Faculty of Electrical Engineering
and Information Technology
Slovak University of Technology
Ilkovičova 3
SK–812 19 Bratislava
SLOVAK REPUBLIC

E-mail: karla.cipkova@stuba.sk

** *Institute of Mathematics*
Faculty of Science
P. J. Šafárik University
Jesenná 5
SK–041 54 Košice
SLOVAK REPUBLIC

E-mail: studenovska@science.upjs.sk