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MEET-PRESERVING FREE FUNCTORS

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Let K be a class of algebras closed with respect to subalgebras. Let $\mathcal{F}$ be the functor from the category of all sets to the category of all algebras of the class K which associates with every set X the K-free algebra on the set X and with every mapping $\varphi: X \to Y$ the homomorphism $\mathcal{F}(X) \to \mathcal{F}(Y)$ extending the mapping φ.

Let X be a fixed set. For $R, S \subseteq X$

$$\mathcal{F}(R \cup S) = \mathcal{F}(R) \vee \mathcal{F}(S)$$
$$\mathcal{F}(R \cap S) \subseteq \mathcal{F}(R) \cap \mathcal{F}(S)$$

always hold, where $\vee$ denotes the operation of join in the lattice of all subalgebras of the algebra $\mathcal{F}(X)$.

We shall investigate conditions for the functor $\mathcal{F}$ to satisfy also

$$\mathcal{F}(R) \cap \mathcal{F}(S) \subseteq \mathcal{F}(R \cap S),$$

where $R, S \subseteq X$.

Definition. *The identity $p = q$ is called totally irregular if the set of all variables occurring in p is disjoint with the set of all variables occurring in q.*

Theorem. *Let $R, S \subseteq X$. Then $\mathcal{F}(R) \cap \mathcal{F}(S) \subseteq \mathcal{F}(R \cap S)$ (and hence $\mathcal{F}(R \cap S) = \mathcal{F}(R) \cap \mathcal{F}(S)$) if and only if one of the following three conditions holds*

(i) $R \cap S \neq \emptyset$

(ii) *there are nullary operations in K*

(iii) *no totally irregular identity holds in K.*

Proof. If (i) holds, choose $t \in R \cap S$ and if (ii) holds, let e be the value of a nullary operation. In these two cases we shall show $\mathcal{F}(R) \cap \mathcal{F}(S) \subseteq \mathcal{F}(R \cap S)$. If $a \in \mathcal{F}(R) \cap \mathcal{F}(S)$, then $a = r(r_1, \ldots, r_n) = s(s_1, \ldots, s_m)$, where r and s are polynomials and $r_i \in R$ for $i = 1, \ldots, n$, $s_j \in S$ for $j = 1, \ldots, m$. Let $\varphi: \mathcal{F}(X) \to \mathcal{F}(X)$ be a homomorphism with the property

$$r_i\varphi = r_i \text{ for } i = 1, \ldots, n$$

$$s_j\varphi = \begin{cases} t \text{ if } R \cap S \neq \emptyset \text{ and } s_j \neq r_i \ (i = 1, \ldots, n) \\ e \text{ if } R \cap S = \emptyset \text{ and } s_j \neq r_i \ (i = 1, \ldots, n). \end{cases}$$

$r(r_1, \ldots, r_n) = s(s_1, \ldots, s_m)$ implies $a = r(r_1, \ldots, r_n) = s(t_1, \ldots, t_m)$ where $t_j = t$ or $t_j = e$ for $j = 1, \ldots, m$. Thus $a \in \mathcal{F}(R \cap S)$ holds. If neither (i) nor (ii) holds, then $\mathcal{F}(R \cap S) = \emptyset$ and if moreover $\mathcal{F}(R) \cap \mathcal{F}(S) \neq \emptyset$, then according to [1] (§ 26, Lemma 1) (iii) does not hold.

Conversely, if none of the conditions (i), (ii), (iii) is fulfilled then $\mathcal{F}(R \cap S) = \emptyset$, but $\mathcal{F}(R) \cap \mathcal{F}(S) \neq \emptyset$.

REFERENCES

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СВОБОДНЫЕ ФУНКТОРЫ СОХРАНЯЮЩИЕ ПЕРЕСЕЧЕНИЯ

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Резюме

Свободные функторы всегда сохраняют объединение, это значит, что имеет место $\mathcal{F}(R \cup S) = \mathcal{F}(R) \vee \mathcal{F}(S)$, где $\mathcal{F}$-свободный функтор, $R, S \subseteq X$ и $\vee$ обозначает операцию объединения в структуре всех подалгебр алгебры $\mathcal{F}(X)$. В работе показывается, что свободный функтор сохраняет пересечение множеств R и S тогда и только тогда, когда $R \cap S \neq \emptyset$ или существуют нулярные операции, или не имеет место никакое тождество содержающее на левой и правой стороне совсем разные переменные.