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Tatiana Lutterová; Sylvia Pulmannová

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AN INDIVIDUAL ERGODIC THEOREM ON THE HILBERT SPACE LOGIC

TATIANA LUTTEROVÁ—SYLVIA PULMANNOVÁ

1. Introduction

The classical individual ergodic theorem of G. Birkhoff states that if $(X, \mathcal{S}, \mu)$ is a probability measure space, T is a measure-preserving transformation of X and f is an integrable real (or complex) valued function on X, then the averages

$$s_n(f) = \frac{1}{n}(f + T\circ f + T^2\circ f + \ldots + T^{n-1}\circ f)$$

converge almost everywhere to an T-invariant function $\tilde{f}$ (where $T\circ f$ is the function defined by $T\circ f(x) = f(Tx)$).

In the quantum logic approach, probability measure space is replaced by the couple (L, m), where L is a logic and m is a state on L. Measure-preserving transformation T is replaced by a σ-homomorphism of L preserving the state m, and instead of an integrable function we consider an observable x on L. Individual ergodic theorem on a logic was formulated and proved in [1] for the case when the σ-homomorphism τ of L is x-measurable, and this result was generalized in [2] for the case when the observalbes x, $\tau\circ x$, $\tau^2\circ x$, ... are mutually compatible. In this paper, we shall prove the individual ergodic theorem on the Hilbert-space logic. We shall replace the condition of compatibility by a weaker condition of the existence of a joint distribution of x, $\tau\circ x$, $\tau^2\circ x$, ... in the state m. We shall also give simplified proofs of some theorems on joint distributions which were proved in [3].

2. Preliminaries

Let $(L, \leq, \perp, 0, 1)$ be a logic (= an orthomodular σ-lattice). Two elements a, b of L are said to be orthogonal if $a \leq b^\perp$ (we write $a \perp b$), and they are said to be compatible, written $a \leftrightarrow b$, if $a = (a\wedge b)\vee(a\wedge b^\perp)$, $b = (a\wedge b)\vee(a^\perp\wedge b)$. A state m on L is a map $m\colon L \to [0, 1]$ such that (i) $m(1) = 1$, (ii) $m(\vee a_i) = \Sigma m(a_i)$ for any sequence $\{a_i\}_i$ of mutually orthogonal elements of L.

Let L_1, L_2 be two logics. A map $\tau\colon L_1 \to L_2$ is a σ-homomorphism if (i) $\tau(1) = 1$,

(ii) $\tau(a^\perp)=\tau(a)^\perp$ for any $a\in L_1$, (iii) $\tau(\vee a_i)=\vee(\tau(a_i))$ for any sequence $\{a_i\}_i$ in L_1. A σ-homomorphism τ is an isomorphism if it is one-to-one and onto. An isomorphism $\tau\colon L\to L$ is an automorphism.

Let us denote by $\mathcal{B}(R^n)$ the σ-algebra of Borel subsets of R^n. Clearly, $\mathcal{B}(R^n)$ with the ordering defined by set-theoretical inclusion and with the set-theoretical complementation is a logic. An observable on L is a σ-homomorphism x from $\mathcal{B}(R^1)$ into L. If x is an observable and f is a Borel measurable function on R^1, then $f(x)=x\circ f^{-1}$ is also an observable. Two observables x, y are said to be compatible ($x\leftrightarrow y$) if $x(E)\leftrightarrow y(F)$ holds for any E, $F\in\mathcal{B}(R)$. If x is an observable and m is a state on L, then $m_x\colon \mathcal{B}(R^1)\to[0,1]$

$$E\mapsto m(x(E))$$

is a probability measure on $\mathcal{B}(R^1)$. This m_x is called the probability distribution of the observable x in the state m. The expectation of x in the state m is defined by $m(x)=\int\lambda m_x(d\lambda)$ if the latter integral exists. For a Borel Function f we have $m(f(x))=\int f(\lambda)m_xd(\lambda)$. If x is an observable and $\tau\colon L\to L$ is a σ-homomorphism, then $\tau\circ x\colon \mathcal{B}(R^1)\to L$

$$E\mapsto \tau(x(E))$$

is also an observable. A σ-homomorphism τ is said to be (i) x-measurable if $\tau(R(x))\subset R(x)$, where $R(x)=\{x(E)\colon E\in\mathcal{B}(R^1)\}$ is the range of x, (ii) m-preserving if $m(\tau(a))=m(a)$ for all $a\in L$, (iii) ergodic in m if it is m-preserving and $\tau(a)=a$ implies $m(a)\in\{0,1\}$. We put $\tau^0\circ x=x$, $\tau^{n+1}\circ x=\tau\circ\tau^n\circ x$, $n\geq 1$.

An observable x is bounded if there is compact subset $C\subset R^1$ such that $x(C)=1$ and it is called simple if $x\{0,1\}=1$. To any $a\in L$ there is a (unique) simple observable x_a such that $x_a\{1\}=a$ and $x_a\{0\}=a^\perp$. If $E\in\mathcal{B}(R^1)$ is such that $x(E)=1$ for the observable x and τ is a σ-homomorphism, then $\tau\circ x(E)=\tau(x(E))=\tau(1)=1$. This implies that if x is bounded, $\tau\circ x$ is bounded, too.

3. Joint distributions of observables

Compatible observables have joint distributions in any state. Joint distributions for observables not necessarily compatible were introduced in [4] in the following form.

Definition 1. *We say that the observables $x_1, x_2, \ldots, x_n$ have a joint distribution in a state m if there is a measure μ on $\mathcal{B}(R^n)$ such that*

$$\mu(E_1\times E_2\times\ldots\times E_n)=m(x_1(E_1)\wedge x_2(E_2)\wedge\ldots\wedge x_n(E_n)) \tag{1}$$

for any measurable rectangle $E_1\times E_2\times\ldots\times E_n$.

It is easily seen that if the joint distribution exists, it is uniquely defined.

The following theorem has been proved in [5]. Here we shall give a more elementary proof.

Theorem 2. *Observables $x_1, x_2, \ldots, x_n$ have a joint distribution in the state m if and only if*

$$m(x_1(E_1)\wedge\ldots\wedge x_i(E_i)\wedge\ldots\wedge x_n(E_n))=$$
$$=\sum_{j=1}^{2} m(x_1(E_1)\wedge\ldots\wedge x_i(E_i^j)\wedge\ldots\wedge x_n(E_n)) \qquad (2)$$

for any $E_1, E_2, \ldots, E_i^j, \ldots, E_n \in \mathscr{B}(R^1)$, $E_i=E_i^1\cup E_i^2$, $E_i^1\cap E_i^2=\emptyset$, $1\leqslant i\leqslant n$.

Proof. If the joint distribution exists, then there is a measure μ satisfying (1). Condition (2) then follows from the σ-additivity of the measure μ.

Now let (2) hold. Let us define

$$F(t_1, t_2, \ldots, t_n)=m(x_1(-\infty, t_1)\wedge x_2(-\infty, t_2)\wedge\ldots\wedge x_n(-\infty, t_n)) \qquad (3)$$

We shall show that $F(t_1, \ldots, t_n)$ is a distribution function.

(i) Let $t_i\leqslant s_i$, $i=1, 2, \ldots, n$. Then $(-\infty, t_i)\cup\langle t_i, s_i)=(-\infty, s_i)$. Using (2), we get

$$F(s_1, \ldots, s_n)=F(t_1, \ldots, t_n)+\sum_{i=1}^{n} m(x_1(-\infty, t_1)\wedge\ldots\wedge x_{i-1}(-\infty, t_{i-1})\wedge$$
$$\wedge x_i(\langle t_i, s_i))\wedge x_{i+1}(-\infty, s_{i+1})\wedge\ldots\wedge x_n(-\infty, s_n)),$$

and therefore $F(t_1, \ldots, t_n)\leqslant F(s_1, \ldots, s_n)$.

(ii) Let $(t_1^i, t_2, \ldots, t_n)\nearrow(t_1, t_2, \ldots, t_n)$. Then

$$|F(t_1, t_2, \ldots, t_n)-F(t_1^i, t_2, \ldots, t_n)|=|m(x_1(-\infty, t_1)\wedge x_2(-\infty, t_2)\wedge\ldots\wedge x_n(-\infty, t_n))-$$
$$-m(x_1(-\infty, t_1^i)\wedge x_2(-\infty, t_2)\wedge\ldots\wedge x_n(-\infty, t_n))|=$$
$$=m(x_1<t_1^i, t_1)\wedge x_2(-\infty, t_2)\wedge\ldots\wedge x_n(-\infty, t_n))\leqslant m(x_1<t_1^i, t_1))\to 0$$

as $i\to\infty$, because $m(x_1(-\infty, t))$ is a distribution function.

(iii) Evidently, $F(-\infty, t_2, \ldots, t_n)=0$, $F(\infty, \ldots, \infty)=1$.

(iv) We have to show that for non-negative $h_1, h_2, \ldots, h_n$

$$F(t_1+h_1, t_2+h_2, \ldots, t_n+h_n)-\sum_{i=1}^{n} F(t_1+h_1, t_2+h_2, \ldots,$$
$$t_{i-1}+h_{i-1}, t_i, t_{i+1}+h_{i+1}, \ldots, t_n+h_n)+\sum_{\substack{i,j=1\\ i<j}}^{n} F(t_1+h_1, \ldots,$$
$$t_{i-1}+h_{i-1}, t_i, t_{i+1}+h_{i+1}, \ldots, t_{j-1}+h_{j-1}, t_j, t_{j+1}+h_{j+1},$$
$$\ldots, t_n+h_n)+\ldots+(-1)^n F(t_1, t_2, \ldots, t_n)\geqslant 0.$$

We shall proceed by induction. For $n=2$ we obtain

$$F(t_1+h_1, t_2+h_2)-F(t_1, t_2+h_2)-F(t_1+h_1, t_2)+F(t_1, t_2)=$$
$$=m(x_1<t_1, t_1+h_1)\wedge x_2<t_2, t_2+h_2)).$$

This can be obtained by direct computation from (2). Now let us suppose that for $n = k$,

$$F(t_1 + h_1, \ldots, t_k + h_k) + \ldots + (-1)^k F(t_1, \ldots, t_k) = \\ = m(x_1 \langle t_1, t_1 + h_1) \wedge x_2 \langle t_2, t_2 + h_2) \wedge \ldots \wedge x_k \langle t_k, t_k + h_k)).$$

For $n = k + 1$ we get

$$F(t_1 + h_1, t_2 + h_2, \ldots, t_k + h_k, t_{k+1} + h_{k+1}) - \sum_{i=1}^{k+1} F(t_1 + h_1, \\ \ldots, t_i, \ldots, t_k + h_k, t_{k+1} + h_{k+1}) + \ldots + (-1)^{k+1} F(t_1, \ldots, t_k, t_{k+1}) = \\ = m(x_1(-\infty, t_1 + h_1) \wedge \ldots \wedge x_k(-\infty, t_k + h_k) \wedge x_{k+1}(-\infty, t_{k+1} + h_{k+1})) - \\ - \sum_{i=1}^{k+1} m(x_1(-\infty, t_1 + h_1) \wedge \ldots \wedge x_i(-\infty, t_i) \wedge \ldots \wedge x_k(-\infty, t_k + h_k) \wedge x_{k+1}(-\infty, t_{k+1} + h_{k+1})) + \ldots + \\ + (-1)^{k+1} m(x_1(-\infty, t_1) \wedge \ldots \wedge x_k(-\infty, t_k) \wedge x_{k+1}(-\infty, t_{k+1})).$$

We shall divide the right-hand side into two parts. In the first part we assemble the members with the interval $(-\infty, t_{k+1} + h_{k+1})$ on the $(k+1)$-th place, in the second part we assemble the members with the interval $(-\infty, t_{k+1})$ on the $(k+1)$-th place. We obtain in both parts the same number of members which differ only on the $(k+1)$-th place and have opposite signs. If we omit in both parts the $(k+1)$-th place, we get the same expressions as for $n = k$. Using (2), we have for the first member

$$m(x_1 \langle t_1, t_1 + h_1) \wedge x_2 \langle t_2, t_2 + h_2) \wedge \ldots \wedge x_k \langle t_k, t_k + h_k) \wedge x_{k+1}(-\infty, t_{k+1} + h_{k+1})),$$

and for the second member

$$-m(x_1 \langle t_1, t_1 + h_1) \wedge x_2 \langle t_2, t_2 + h_2) \wedge \ldots \wedge x_k \langle t_k, t_k + h_k) \wedge x_{k+1}(-\infty, t_{k+1})).$$

By substracting the two members and using (2) again we obtain

$$0 \leq m(x_1 \langle t_1, t_1 + h_1) \wedge \ldots \wedge x_k \langle t_k, t_k + h_k) \wedge x_{k+1} \langle t_{k+1}, t_{k+1} + h_{k+1})).$$

We have thus shown that $F(t_1, t_2, \ldots, t_n)$ is a distribution function. Then there is a measure μ on $\mathscr{B}(R^n)$ such that

$$F(t_1, t_2, \ldots, t_n) = \mu((-\infty, t_1) \times (-\infty, t_2) \times \ldots \times (-\infty, t_n))$$

for any $(t_1, t_2, \ldots, t_n) \in R^n$. It is easily seen that μ satisfies (1), i.e. it is the required joint distribution.

Let us set $D = \{0, 1\}$, $d = (d_1, d_2, \ldots, d_n) \in D^n$, $a^0 = a^\perp$, $a^1 = a$ for $a \in L$. The following theorem has been proved in [3].

Theorem 3. *Condition (2) of Theorem 1 is equivalent to the condition*

$$1 = m\left(\bigvee_{d\in D^n} x_1(E_1)^{d_1} \wedge x_2(E_2)^{d_2} \wedge \ldots \wedge x_n(E_n)^{d_n}\right) =$$
$$= \sum_{d\in D^n} m(x_1(E_1)^{d_1} \wedge x_2(E_2)^{d_2} \wedge \ldots \wedge x_n(E_n)^{d_n}). \tag{4}$$

Definition 1 can be generalized to any set of observables as follows.

Definition 4. *Let $\{x_\alpha : \alpha \in A\}$ be any set of observables on a logic L. We shall say that $\{x_\alpha : \alpha \in A\}$ have a joint distribution in the state m if for any $n = 1, 2, \ldots$ and any $\alpha_1, \alpha_2, \ldots, \alpha_n$ the observables $x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}$ have a joint distribution in the state m.*

A logic L is said to be separable if any subset of mutually orthogonal elements of L is at most countable. We recall (see [17]) that if $\{a_\alpha : \alpha \in A\}$ is any subset of elements of a separable logic L, then there is a countable subset $I \subset A$ such that

$$\bigvee_{\alpha\in A} a_\alpha = \bigvee_{\alpha\in I} a_\alpha \left(\bigwedge_{\alpha\in A} a_\alpha = \bigwedge_{\alpha\in I} a_\alpha\right).$$

Let $\{x_\alpha : \alpha \in A\}$ be a set of observables on a separable logic L. For any finite subset $S = \{\alpha_1, \ldots, \alpha_n\}$ of A (with $\alpha_1, \ldots, \alpha_n$ not necessarily all different) let us set

$$a_S(E_1, E_2, \ldots, E_n) = \bigvee_{d\in D^n} x_{\alpha_1}(E_1)^{d_1} \wedge \ldots \wedge x_{\alpha_n}(E_n)^{d_n}, \tag{5}$$

where $E_1, E_2, \ldots, E_n \in \mathcal{B}(R^1)$, and

$$a_S = \bigwedge_{(E_1, E_2, \ldots, E_n)} a_S(E_1, E_2, \ldots, E_n) \tag{6}$$

where the infimum is to be taken over all E_1. $E_2, \ldots, E_n \in \mathcal{B}(R^1)$. Finally,

$$a = \bigwedge_{S\subset A} a_S \tag{7}$$

where the infimum is to be taken over all finite subsets S of A.

By Theorem 3, the observables $\{x_\alpha : \alpha \in A\}$ have a joint distribution in a state m if $m(a_S(E_1, \ldots, E_n)) = 1$ for any $S \subset A$ and any $E_1, \ldots, E_n \in \mathcal{B}(R^1)$.

Let $0 \neq a \in L$. The set $L_{[0,a]} = \{b \in L : b \leqslant a\}$ is a logic with the partial ordering inherited from L, with the greatest element a and with the relative orthocomplementation $b' = b^\perp \wedge a$. If x is an observable on L such that $x \leftrightarrow a$ (i.e. $x(E) \leftrightarrow a$ for any $E \in \mathcal{B}(R^1)$), then the map $x \wedge a$ defined by $x \wedge a(E) = x(E) \wedge a$, $E \in \mathcal{B}(R^1)$, is an observable on the logic $L_{[0,a]}$.

Proposition 5. *Let $K = \{a_\alpha : \alpha \in A\}$ be any set of elements of a separable logic L. For any finite subset $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset A$ we put*

$$a_S = \bigvee_{d\in D^n} a_{\alpha_1}^{d_1} \wedge a_{\alpha_2}^{d_2} \wedge \ldots \wedge a_{\alpha_n}^{d_n}, \tag{8}$$

$$a = \bigwedge_{S \subset A} a_S. \tag{9}$$

Then (i) $a_\alpha \leftrightarrow a$ *for all* $\alpha \in A$, (ii) *if* $m(a_S) = 1$ *for any* $S \subset A$ (m *is a state on* L) *then* $m(a) = 1$, (iii) $\{a_\alpha \wedge a\colon \alpha \in A\}$ *are mutually compatible.*

For the proof, see [6].

We shall call the element a defined by (9) the commutator of the set $\{a_\alpha\colon \alpha \in A\}$. It is easily seen that the element a defined by (7) is the commutator for the set $\bigcup_{\alpha \in A} R(x_\alpha)$ where $R(x_\alpha)$ is the range of x_α.

Proposition 5 gives rise to the following theorem.

Theorem 6. *Let* $\{x_\alpha\colon \alpha \in A\}$ *be a set of observables on a separable logic L. Let m be a state on L, and let a be the commutator of* $\bigcup_{\alpha \in A} R(x_\alpha)$. *Then*

(i) $\{x_\alpha\colon \alpha \in A\}$ *have a joint distribution in the state m if and only if* $m(a) = 1$,

(ii) *for any* $\alpha \in A$, $x_\alpha \leftrightarrow a$ *and the observables* $\{x_\alpha \wedge a\colon \alpha \in A\}$ *on* $L_{[0,a]}$ *are mutually compatible.*

4. Hilbert space logic

A very important example of a logic is the lattice of all closed linear subspaces of a Hilbert space H (real or complex). Let H be a complex Hilbert space $3 \leqslant \dim H \leqslant \aleph_0$. We denote by $L(H)$ the set of all closed linear subspaces of H ordered by the inclusion and with the orthocomplementation defined by $M^\perp = \{u \in H\colon (u, v) = 0 \text{ for all } v \in M\}$. Obviously, $L(H)$ is a separable logic. The lattice operations on $L(H)$ are $M_1 \wedge M_2 = M_1 \cap M_2$, and $M_1 \cup M_2 = (M_1 + M_2)^-$ (the closure of the linear envelope of M_1 and M_2). The elements of $L(H)$ are in one-to-one correspondence with the orthogonal projections. We shall write P^M for the projector corresponding to the subspace M. Due to the spectral theorem [7], the observables are in one-to-one correspondence with self-adjoint operators on $L(H)$. If A is a self-adjoint operator, we shall write $P^{A(\,)}$ for the corresponding spectral measure, i.e. the observable corresponding to A. Due to the Gleasson theorem [8] any state on $L(H)$ can be written in the form

$$m = \sum_{i=1}^{\infty} w_i m_{\varphi_i}, \quad m_{\varphi_i}\colon M \mapsto (P^M \varphi_i, \varphi_i)$$

where $\{\varphi_i\}_i$ is a sequence of mutually orthogonal unit vectors in H.

The elements M_1, M_2 of $L(H)$ are compatible ($M_1 \leftrightarrow M_2$) if and only if the corresponding projectors commute, i.e. $P^{M_1} P^{M_2} = P^{M_2} P^{M_1}$. We shall write in this case $P^{M_1} \leftrightarrow P^{M_2}$. Two observables $x = P^A(\cdot)$, $y = P^B(\cdot)$ are compatible if $P^{A(E)} \leftrightarrow P^{B(F)}$ for any $E, F \in B(R^1)$. If x and y are bounded, then they are compatible if and only if the corresponding self-adjoint operators commute, i.e. if $AB = BA$.

Let M be a subspace of H and A a self-adjoint operator. The subspace M reduces the operator A, i.e. $AM \subset M$ if and only if $P^M \leftrightarrow A$. In this case A can be considered as a self-adjoint operator on the Hilbert space M; the logic $L(M)$ corresponds to $L_{[0,M]}$. The operator A reduced to M, written A/M, corresponds to the observable $P^{A(\cdot)} \wedge P^M = P^{A(\cdot)}P^M$.

If A and B are bounded self-adjoint operators on $L(H)$, the sum $A+B$ is also a self-adjoint operator. It is natural to consider the corresponding observable $P^{(A+B)(\cdot)}$ as the sum of the observables A and B. Clearly, if $A \leftrightarrow P$ and $B \leftrightarrow P$, where P is a projector, then also $(A+B) \leftrightarrow P$, so that if A and B reduce a subspace $M \in L(H)$, then also $A+B$ reduces M. Moreover, $A/M + B/M = (A+B)/M$, i.e. $P^{(A/M+B/M)(\cdot)} = P^{(A+B)(\cdot)} \wedge P^M$.

In the logic $L(H)$ we can introduce the convergences of observables analogically to the measure theoretical convergences (see [9]). We shall need only the almost everywhere convergence.

Definition 7. *We shall say that the sequence of bounded observables $\{x_i\}_i$ on the logic $L(H)$ converges to the observable x a.e. in a state m if*

$$m\left(\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} (x_n - x)(-\varepsilon, \varepsilon)\right) = 1 \tag{10}$$

for any $\varepsilon \geqslant 0$.

5. Individual ergodic theorem on the logic $L(H)$

In [2] the following individual ergodic theorem was proved.

Theorem 8. *Let m be a state on a logic L, τ be an m-preserving σ-homomorphism of L and x be an observable on L such that $m(x) < \infty$ and $\{\tau^i \circ x\}_{i=0}^{\infty}$ be pairwise compatible. Then there is an observable $\tilde{x}$ on L such that*

(i) $\tau \circ \tilde{x} = \tilde{x}$ a.e. $[m]$, i.e. $m((\tau \circ \tilde{x})\{0\}) = 1$

(ii) $m(\tilde{x}) = m(x)$

(iii) $\frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x \to \tilde{x}$ a.e. $[m]$.

We are now in the position to prove the main result of this paper, an individual ergodic theorem on the Hilbert space logic.

Theorem 9. *Let H be a complex Hilbert space, $3 \leqslant \dim H \leqslant \aleph_0$. Let $L(H)$ be the logic of all closed subspaces of H. Let A be a bounded self-adjoint operator on H, τ: $L(H) \to L(H)$ be a σ-homomorphism and m be a τ-invariant state on $L(H)$. Let P_0 be the commutator for the observables $\{\tau^i \circ A\}_{i=0}^{\infty}$, and let $\tau(P_0) = P_0$ and $m(P_0) = 1$. Then there is an observable $\tilde{A}$ on $L(H)$ such that*

(i) $\tau \circ \tilde{A} = \tilde{A}$ a.e. $[m]$,

(ii) $m(\tilde{A}) = m(A)$,

(iii) $\frac{1}{n}\sum_{i=0}^{n-1}\tau^i \circ A \to \tilde{A}$ a.e. $[m]$.

Proof. Let $H_0 = P_0 H$. By Theorem 6, $\tau^i \circ A \leftrightarrow P_0$, $i = 0, 1, 2, \ldots$, and so $\tau^i \circ A$ can be considered as self-adjoint operators on H_0. Moreover, again by Theorem 6, $\tau^i \circ A/H_0$ are mutually compatible. As $\tau(P_0) = P_0$, τ can be considered as a σ-homomorphism of the logic $L(H_0)$. Let $m = \sum_{i=1}^{\infty} w_i m_{\varphi_i}$. Since $m(P_0) = 1$ then $1 = \sum_{i=1}^{\infty} w_i m_{\varphi_i}(P_0)$, which implies $m_{\varphi_i}(P_0) = (P_0\varphi_i, \varphi_i) = \|P_0\varphi_i\| = 1$, so that $\varphi_i \in H_0$, $i = 1, 2, \ldots$. Hence m can be considered as a state on $L(H_0)$. We can apply Theorem 8 to obtain that there is a self-adjoint operator $\tilde{A}_0$ on $L(H_0)$ such that

(i′) $\tau \circ \tilde{A}_0 = \tilde{A}_0$ a.e. $[m]$,
(ii′) $m(\tilde{A}_0) = m(A/H_0)$,
(iii′) $\frac{1}{n}\sum_{i=0}^{n-1}\tau^i \circ A/H_0 \to \tilde{A}_0$ a.e. $[m]$.

Let us take a real number c and set

$$P^{\tilde{A}(\cdot)} = P^{\tilde{A}_0(\cdot)} \vee P^{C(\cdot)} \wedge P_0^{\perp} \tag{11}$$

where

$$P^{C(E)} = \begin{cases} 0 & \text{if} \quad c \notin E \\ 1 & \text{if} \quad c \in E, \end{cases}$$

$E \in \mathscr{B}(R^1)$.

It is easily checked that $P^{\tilde{A}(\cdot)}$ is an observable and the corresponding self-adjoint operator is

$$\tilde{A} = \tilde{A}_0 P_0 + c(1 - P_0). \tag{12}$$

Clearly, $\tilde{A} \leftrightarrow P_0$ and $\tilde{A}/H_0 = \tilde{A}_0$. We show that $\tilde{A}$ is the operator we looked for.

(i) $\tau(P^{\tilde{A}(E)}) = \tau(P^{\tilde{A}_0(E)}) \vee \tau(P^{C(E)}) \wedge \tau(P_0^{\perp}) = \tau(P^{\tilde{A}_0(E)}) \vee P^{C(E)} \wedge P_0^{\perp}$. Therefore we obtain that $(\tau\circ\tilde{A})/H_0 = \tau\circ\tilde{A}_0$. Thus

$$(\tau \circ \tilde{A} - \tilde{A})/H_0 = \tau \circ \tilde{A}/H_0 - \tilde{A}/H_0 = \tau \circ \tilde{A}_0 - \tilde{A}_0.$$

As $m(P_0) = 1$, we obtain $m(P^{(\tau\circ\tilde{A}-\tilde{A})(E)}) = m(P^{(\tau\circ\tilde{A}-\tilde{A})(E)} \wedge P_0) = m(P^{(\tau\circ\tilde{A}_0-\tilde{A}_0)(E)})$, $E \in \mathscr{B}(R^1)$, which implies

$$m(P^{(\tau\circ\tilde{A}-\tilde{A})\{0\}}) = m(P^{(\tau\circ\tilde{A}_0-\tilde{A}_0)\{0\}}) = 1$$

(ii) As $m(P_0) = 1$, we get $m(P^{\tilde{A}(E)}) = m(P^{\tilde{A}_0(E)})$, $E \in \mathscr{B}(R^1)$, which implies $m(\tilde{A}) = m(\tilde{A}_0)$. Similarly, $m(P^{A(E)}) = m(P^{A(E)}P_0)$ and thus $m(A) = m(A/H_0)$. By (ii′) we derive $m(\tilde{A}) = m(A)$.

(iii) Let us denote $A_n = \frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ A$. Then

$$m\left(\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} P^{(A_n-\bar{A})(-\varepsilon,\varepsilon)}\right) = m\left(\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} P^{(A_n-\bar{A})(-\varepsilon,\varepsilon)} P_0\right) =$$

$$= m\left(\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} P^{((A_n-\bar{A})/H_0)(-\varepsilon,\varepsilon))}\right) = 1$$

for any $\varepsilon > 0$. The above equalities follow from the fact that $\tau^i \circ A \leftrightarrow P_0$, $i = 0, 1, \ldots$ implies $A_n \leftrightarrow P_0$, $n = 1, 2, \ldots$, i.e. $P^{(A_n-\bar{A})(-\varepsilon,\varepsilon)} \leftrightarrow P_0$ for $n = 1, 2, \ldots$ This implies by [10] that

$$\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} P^{(A_n-\bar{A})(-\varepsilon,\varepsilon)} \leftrightarrow P_0,$$

$$\left[\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} P^{(A_n-\bar{A})(-\varepsilon,\varepsilon)}\right] \wedge P_0 = \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} P^{(A_n-\bar{A})(-\varepsilon,\varepsilon)} \wedge P_0.$$

The setup of the latter theorem can be slightly simplified if τ is an automorphism.

Proposition 10. *Let τ be an automorphism of a separable logic L. Let a be the commutator of the set $M = \{\tau^i(a_\alpha) \colon \alpha \in A\}_{i=-\infty}^{\infty}$. Then $\tau(a) = a$.*

Proof. The case $a = 0$ is trivial. Let $a \neq 0$. For a finite subset $F = \{b_1, b_2, \ldots, b_n\}$ of M set

$$a(F) = \bigvee_{d \in D^n} b_1^{d_1} \wedge b_2^{d_2} \wedge \ldots \wedge b_n^{d_n}.$$

By the definition,

$$a = \bigwedge_{F \subset M} a(F).$$

Clearly,

$$\tau^j(a(F)) = \bigvee_{d \in D_n} \tau^j(b_1)^{d_1} \wedge \ldots \wedge \tau^j(b_n)^{d_n}, \quad j = \pm 1,$$

and $\{\tau^j(b_1), \tau^j(b_2), \ldots, \tau^j(b_n)\} \subset M$. As the logic is separable, there is a sequence $\{F_1, F_2, \ldots\}$ such that $a = \bigwedge_{i=1}^{\infty} a(F_i)$. Then $\tau(a) = \bigwedge_{i=1}^{\infty} \tau(a(F_i))$, but $\tau(a(F_i))$ is $a(G_i)$ for some finite subset G_i of M. This implies that $a \leq \tau(a(F_i))$, i.e. $a \leq \bigwedge_{i=1}^{\infty} \tau(a(F_i)) = \tau(a)$. Similarly, $a \leq \tau^{-1}(a)$, i.e. $\tau(a) = a$.

According to Proposition 10 if τ is an automorphism, then we can in Theorem 9 use the set $\{\tau^i \circ A\}_{i=-\infty}^{\infty}$ instead of the set $\{\tau^i \circ A\}_{i=0}^{\infty}$. If we have $m(P_0) = 1$ for its commutator P_0, then the individual ergodic theorem follows.

Let us make a final observation.

Lance [11] proved following individual ergodic theorem.

Theorem 11. *Let α be an automorphism of a von Neumann algebra $\mathscr{A}$ and let ϱ be a faithful normal α-ivariant state. For each A in $\mathscr{A}$ and $\varepsilon>0$ there is a projection E in $\mathscr{A}$ with $\varrho(E)>1-\varepsilon$ such that*

$$\left\|\left(\frac{1}{n}\sum_{i=0}^{n-1}\alpha^i\circ A-\tilde{A}\right)E\right\|\to 0 \quad \text{as} \quad n\to\infty.$$

It would be of some interest to compare Theorem 9 with Theorem 11. One can also look for the conditions under which an equivalent of Theorems 9 and 11 or other theorems on operator algebras [12], [13], [14] could be proved in so-called sum logics (introduced in [15] and [16]).

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Ústav systémového inžinierstva priemyslu
Hrachova 30
827 11 Bratislava

Matematický ústav SAV
Obrancov mieru 49
814 73 Bratislava

ИНДИВИДУАЛЬНАЯ ЭРГОДИЧЕСКАЯ ТЕОРЕМА НА ЛОГИКЕ ПРОСТРАНСТВА ГИЛЬБЕРТА

Tatiana Lutterová—Sylvia Pulmannová

Резюме

Индивидуальная эргодическая теорема на логике пространства Гильберта показана в случае, когда имеется совместное распределение вероятностей для исследованной последовательности наблюдаемых.