George Grätzer; Harry Lakser

Freely adjoining a complement to a lattice

Mathematica Slovaca, Vol. 56 (2006), No. 1, 93--104

Persistent URL: http://dml.cz/dmlcz/131652

Math. Slovaca, **56** (2006), No. 1, 93–104

Mathematica Slovaca

Dedicated to Ján Jakubík on his 80th birthday

FREELY ADJOINING A COMPLEMENT TO A LATTICE

G. GRÄTZER* — H. LAKSER**

(*Communicated by Tibor Katriňák*)

ABSTRACT. For a bounded lattice K and an element a of $K - \{0,1\}$, we directly describe the structure of the lattice freely generated by K and an element u subject to the requirement that u be a complement of a. Earlier descriptions of this lattice used multi-step procedures.

As an application, we give a short and direct proof of the classical result of R. P. Dilworth (1945): *Every lattice can be embedded into a uniquely complemented lattice.* We prove it in the stronger form due to C. C. Chen and G. Grätzer (1969): *Every at most uniquely complemented bounded lattice has a* $\{0,1\}$*-embedding into a uniquely complemented lattice.*

1. Introduction

1.1. Background.

E. V. Huntington [12] in 1904 conjectured that a uniquely complemented lattice is Boolean. This was disproved in a real *tour de force* in R. P. Dilworth [6] in 1945, after many failed attempts by a number of mathematicians to verify the conjecture. (See [7; Chap. VI, Sec. "Further Topics and References"] for a detailed accounting up to 1975; see [7; Appendix A, Sec. 7.1] for the 1998 update.) Dilworth disproved the conjecture by verifying the following very strong result (almost the opposite of the conjecture):

Every lattice can be embedded into a uniquely complemented lattice.

2000 Mathematics Subject Classification: Primary 06B10; Secondary 06B15.
Keywords: relative complement, free lattice, uniquely complemented.

The research of both authors was supported by the NSERC of Canada.

Let K be a bounded lattice. Let $a \in K - \{0,1\}$, and let u be an element not in K. We extend the partial ordering $\leq$ of K to $Q = K \cup \{u\}$ as follows: $0 \leq u \leq 1$. We extend the lattice operations $\wedge$ and $\vee$ to Q as *commutative partial meet and join operations*. For $x \leq y$ in Q, define $x \wedge y = x$ and $x \vee y = y$. In addition, let $a \wedge u = 0$ and $a \vee u = 1$; see Figure 1.

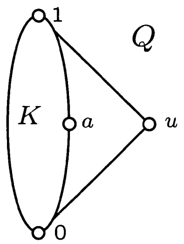

FIGURE 1. The partial lattice Q.

The proof of Dilworth was very complex, using free algebras that went way beyond lattices; however, the penultimate step was the description of $\mathrm{F}(Q)$, the lattice freely generated by Q and preserving the partial joins and meets of Q.

In C. C. Chen and G. Grätzer [3] (reproduced in G. Grätzer [7] and also in P. Crawley and R. P. Dilworth [4]), the description of $\mathrm{F}(Q)$ was reached in two steps.

1.2. New results.

As opposed to the approaches in [6] and [3], in this paper, we describe $\mathrm{F}(Q)$ directly.

To construct $\mathrm{F}(Q)$, we consider polynomials (words) A built from $Q = K \cup \{u\}$ with the operations $\wedge$ and $\vee$. A polynomial A naturally represents an element $\langle A \rangle$ of $\mathrm{F}(Q)$. We prove that with a polynomial A, we can associate its *lower cover* A_* and *upper cover* A^* in K. (Recursively computable upper and lower covers were introduced for free products in G. Grätzer, H. Lakser, and C. R. Platt [11].) The crucial result is Theorem 1, which presents a recursive algorithm to calculate A_* and A^* for any polynomial A.

By identifying $x \in K$ with $\langle x \rangle$, we can view K as a sublattice of $\mathrm{F}(Q)$. We apply Theorem 1 to describe which pairs of elements are complementary in $\mathrm{F}(Q)$ — see Theorem 2 — provided that K contain no spanning N_5. The embedding theorem of Dilworth and its sharper form due to Chen and Grätzer immediately follow.

Another application of Theorem 1 is the solution to the "word problem" in $\mathrm{F}(Q)$: $\langle A \rangle \leq \langle B \rangle$ in $\mathrm{F}(Q)$ if and only if one of the Whitman Conditions (implicit in P. M. Whitman [15] to characterize $\langle A \rangle \leq \langle B \rangle$ in a free lattice) hold or $A^* \leq B_*$.

1.3. Alternative approaches.

There are alternative, purely lattice theoretic approaches to proving the theorem of Dilworth: The $\mathcal{C}$-reduced free products of G. Grätzer [8] and its generalization, the $\mathcal{R}$-reduced free products of M. E. Adams and J. Sichler [1] and [2] (reproduced, in part, in V. N. Saliĭ [14]). $\mathcal{R}$-reduced free products extend $\mathcal{C}$-reduced free products in two important ways:

(i) An $\mathcal{R}$-reduction is not necessarily determined by a $\mathcal{C}$-relation (a relation imposing complementarity on pairs of elements from distinct components of a free product).

(ii) An $\mathcal{R}$-reduction can be done in many lattice varieties not only in the variety of all lattices.

1.4. Future directions.

The new technique introduced in this paper (the direct description of $\mathrm{F}(Q)$ by the mutually recursive definition of $\leq$ and the lower and upper cover) have many other applications. We made a start in exploring these in [9] and [10].

Here is a sample result from [9]:

THEOREM. *Let K be a lattice, and let $[a,b]$ be an interval of K with $a < b$. If the lattice $[a,b]$ is at most uniquely complemented, then there is a lattice extension L of K such that the interval $[a,b]_L$ of L is uniquely complemented.*

The methods discussed in Section 1.1 and Section 1.3 cannot be utilized to prove this result.

Here is a sample result from [10]. Let $\mathfrak{m} \geq 1$ be a cardinal. Let us call a lattice L *transitively* (*at most*) $\mathfrak{m}$*-complemented* if every element of L has (at most) $\mathfrak{m}$ complements and the following (transitivity) property holds:

If b is a complement of a and c is a complement of b, then $a = c$ or c is a complement of a.

THEOREM. *Let K be a transitively at most $\mathfrak{m}$-complemented lattice. Then there is a transitively $\mathfrak{m}$-complemented lattice extension L.*

Note that $\mathfrak{m} = 1$ is the uniquely complemented case.

1.5. Summary.

The purpose of this paper is to introduce the new direct description of $\mathrm{F}(Q)$, and make the first short and elementary proof of the Dilworth theorem available. Equally importantly, we present the new technique in a very simple setup, easily accessible to algebraists. The more general results we obtain in [9] and [10] generalize our present results, but at the cost of very long, technical, and tedious proofs.

2. The relational system Q

Let K be a bounded lattice. Let $a \in K - \{0,1\}$, and let u be an element not in K. We extend the partial ordering $\leq$ of K to $Q = K \cup \{u\}$ as follows: $0 \leq u \leq 1$.

We extend the lattice operations $\wedge$ and $\vee$ to Q as *commutative partial meet and join operations*. For $x \leq y$ in Q, define $x \wedge y = x$ and $x \vee y = y$. In addition, let $a \wedge u = 0$ and $a \vee u = 1$; see Figure 1. In this section, we state a number of easy results on $Q = \langle Q; \wedge, \vee, \leq \rangle$. See, for instance, [7; Sec. I.5] for the basic concepts and facts.

The relational system Q has the property that for any $x, y \in Q$, if $x \wedge y$ is defined, then it is the greatest lower bound of x and y in $Q = \langle Q; \leq \rangle$, and dually. This property is sufficient for us to apply Dean's Theorem ([5]) to Q in the next section (while the result is due to Dean, our presentation here follows that in H. Lakser [13]).

A subset I of Q is an *ideal* if it is hereditary and it is closed under the joins defined. *Dual ideals* are defined dually. Observe that a proper ideal I of Q is either a proper ideal of K or it is of the form $I \cup \{u\}$, where I is an ideal of K with $a \notin I$. For ideals I and J of Q, the meet is given by

$$I \wedge J = I \cap J\,.$$

The join is described by the rule:

$$I \vee J = \begin{cases} I \vee_K J & \text{if } I, J \subseteq K \text{ and } I \vee_K J \subset K; \\ \big((I \cap K) \vee_K (J \cap K)\big) \cup \{u\} & \text{if } u \in I \cup J \text{ and} \\ & a \notin (I \cap K) \vee_K (J \cap K); \\ Q, & \text{otherwise.} \end{cases} \tag{1}$$

In this formula, we use the convention that if I, J are ideals of K, then $I \vee_K J$ denotes the join of the two ideals in K, while $I \vee J$ denotes the join of the two ideals in Q. Similarly, for $x \in K$, we denote by $(x]_K$ the principal ideal generated by x in K, while $(x]$ denotes the principal ideal generated by x in Q. Note that $(x] = (x]_K$, unless $x = 1$.

If $x, y \in K$, then $(x] \wedge (y] = (x \wedge y]$. If $x \in K$, then

$$(x] \wedge (u] = \begin{cases} (u] = \{u, 0\} & \text{if } x = 1; \\ \{0\} & \text{if } x < 1. \end{cases} \tag{2}$$

If $x, y \in K$, then

$$(x] \vee (y] = (x \vee y] = \begin{cases} (x \vee y]_K & \text{if } x \vee y < 1; \\ Q & \text{if } x \vee y = 1. \end{cases} \tag{3}$$

If $x \in K$, then

$$(x] \vee (u] = \begin{cases} Q & \text{if } a \leq x\,; \\ (x] \cup \{u\} & \text{if } a \nleq x\,. \end{cases} \tag{4}$$

So for x and y in Q, the ideal $(x] \wedge (y]$ of Q is principal; the ideal $(x] \vee (y]$ of Q is principal unless $\{x, y\} = \{z, u\}$, with $z \in K$, and $a \nleq z$, in which case, $(x] \vee (y] = (z] \cup \{u\}$. Now an easy induction proves the following statement:

Lemma 1. *A finitely generated ideal of Q is either principal or of the form*

$$(x] \vee (u] = (x] \cup \{u\} \qquad \textit{with} \quad x \in K\,, \ \ 0 < x\,, \ \ \textit{and } a \nleq x\,.$$

3. The free lattice $\mathrm{F}(Q)$

We now discuss the lattice $\mathrm{F}(Q)$, the lattice freely generated by Q and preserving the partial joins and meets of Q. Note that Q is a $\{0,1\}$-extension of K, so $\mathrm{F}(Q)$ is a $\{0,1\}$-extension of Q.

We consider the set $\mathbf{P}(Q)$ of polynomials on the elements of Q formed with the operations $\wedge$ and $\vee$. Each polynomial A determines an element $\langle A \rangle$ of $\mathrm{F}(Q)$ if we interpret $\wedge$ as the meet operation in $\mathrm{F}(Q)$ and $\vee$ as the join operation. Given $A, B \in \mathbf{P}(Q)$, we set $A \equiv B$ if $\langle A \rangle = \langle B \rangle$ in $\mathrm{F}(Q)$. Let $A \leq B$ if $\langle A \rangle \leq \langle B \rangle$ in $\mathrm{F}(Q)$; $\leq$ is a quasi-ordering on $\mathbf{P}(Q)$.

We now recall the solution to the "word problem" in $\mathrm{F}(Q)$, which is a set of rules that determine when $A \leq B$ in $\mathbf{P}(Q)$ for polynomials A and B.

We associate, with each polynomial A, a finitely generated ideal $\underline{A}$ of Q, its *lower cover ideal* in Q, and a finitely generated dual ideal $\overline{A}$ of Q, its *upper cover dual ideal* in Q, as follows.

If $x \in Q$, then $\underline{x} = (x]$. Inductively,

$$\begin{aligned} \underline{A \wedge B} &= \underline{A} \wedge \underline{B} = \underline{A} \cap \underline{B}\,, \\ \underline{A \vee B} &= \underline{A} \vee \underline{B}\,. \end{aligned}$$

Clearly, $\underline{A}$ is a finitely generated ideal of Q. Dually, we define $\overline{A}$, a finitely generated dual ideal of Q.

The solution to the word problem in $\mathrm{F}(Q)$ is given by the following result.

Dean's Theorem (for Q). *Let $A, B \in \mathbf{P}(Q)$. Then $A \leq B$ if and only if it follows from the following rules:*

(E) $A = B$.
$({}_\wedge\mathrm{W})$ $A = A_0 \wedge A_1$ *with* $A_0 \leq B$ *or* $A_1 \leq B$.
$({}_\vee\mathrm{W})$ $A = A_0 \vee A_1$ *with* $A_0 \leq B$ *and* $A_1 \leq B$.
$(\mathrm{W}_\wedge)$ $B = B_0 \wedge B_1$ *with* $A \leq B_0$ *and* $A \leq B_1$.
$(\mathrm{W}_\vee)$ $B = B_0 \vee B_1$ *with* $A \leq B_0$ *or* $A \leq B_1$.
(C_Q) $\overline{A} \cap \underline{B} \neq \emptyset$.

Conditions (E), $({}_\wedge\mathrm{W})$, $({}_\vee\mathrm{W})$, $(\mathrm{W}_\wedge)$, $(\mathrm{W}_\vee)$ are called *the Whitman Conditions*, while (C_Q) is the *covering condition for* Q. The following statements follow from this result:

Corollary.

(i) *Q is a subposet of* $\mathrm{F}(Q)$.
(ii) $\underline{A} = \{x \in Q : x \leq A\}$ (*and dually for* $\overline{A}$).
(iii) $A \leq B$ *implies that* $\underline{A} \subseteq \underline{B}$.

It follows from Lemma 1 that, given any polynomial $A \in \mathbf{P}(Q)$, there are uniquely defined elements A_* and A^* of K with $\underline{A} \cap K = (A_*]_K$ and $\overline{A} \cap K = [A^*)_K$. So we have:

Lemma 2. *$x \leq A$ if and only if $x \leq A_*$ for any $x \in K$. If $A \leq B$, then $A_* \leq B_*$.*

The most important properties of A_* and of $u \leq A$ are summarized as follows:

Theorem 1. *The following statements hold:*

(i) $u \leq u$. *If $x \in K$, then $u \leq x$ if and only if $x = 1$.*
(ii) $u_* = 0$. *If $x \in K$, then $x_* = x$.*
(iii) $u \leq A \wedge B$ *if and only if* $u \leq A$ *and* $u \leq B$.
(iv) $(A \wedge B)_* = A_* \wedge B_*$.
(v) $u \leq A \vee B$ *if and only if either* $u \leq A$, *or* $u \leq B$, *or* $A_* \vee B_* = 1$.
(vi)

$$(A \vee B)_* = \begin{cases} 1 & \text{if } a \leq A_* \vee B_* \text{ and either } u \leq A \text{ or } u \leq B; \\ A_* \vee B_*, & \text{otherwise.} \end{cases}$$

Proof.

(i) This statement is contained in Statement (i) of the Corollary to Dean's Theorem.

(ii) $\underline{u} \cap K = \{0\}$, and so $u_* = 0$. If $x \in K$, then $\underline{x} \cap K = (x]_K$, implying that $x_* = x$.

(iii) $A \wedge B \leq A$ and $A \wedge B \leq B$ by (E) and $({}_{\wedge}\mathrm{W})$. Therefore, if $u \leq A \wedge B$, then $u \leq A$ and $u \leq B$ by the transitivity of $\leq$. The converse follows from $(W_{\wedge})$.

(iv) Since

$$((A \wedge B)_*]_K = \underline{A \wedge B} \cap K = \underline{A} \cap \underline{B} \cap K = (A_*]_K \cap (B_*]_K = (A_* \wedge B_*]_K,$$

the generators are equal.

(v) $A \leq A \vee B$ and $B \leq A \vee B$ by (E) and $(\mathrm{W}_{\vee})$. Therefore, if $u \leq A$ or $u \leq B$, then $u \leq A \vee B$, by the transitivity of $\leq$. Also, if $1 = A_* \vee B_*$, then $1 \in \overline{u} \cap \underline{A \vee B}$, and so, by (C_Q), $u \leq A \vee B$.

Conversely, by Dean's Theorem, $u \leq A \vee B$ if and only if either $u \leq A$, or $u \leq B$, or $\overline{u} \cap \underline{A \vee B} \neq \emptyset$ — since $(\mathrm{W}_{\vee})$ or (C_Q) applies. The last condition is equivalent to $u \in \underline{A \vee B}$, because $\overline{u} = \{u, 1\}$, so if $\overline{u} \cap \underline{A \vee B} \neq \emptyset$, then $1 \in \underline{A \vee B}$ or $u \in \underline{A \vee B}$, and both imply that $u \in \underline{A \vee B}$. If $u \leq A$ or $u \leq B$, then we are done. So assume that $u \nleq A$ and $u \nleq B$. Then $\underline{A} = (A_*]_K$ and $\underline{B} = (B_*]_K$ by Lemma 1. Thus if $A_* \vee B_* < 1$, then by the first equality in (3),

$$\underline{A \vee B} = (A_*]_K \vee (B_*]_K = (A_* \vee B_*]_K,$$

contradicting that $u \in \underline{A \vee B}$. We conclude that $A_* \vee B_* = 1$.

(vi) First, assume that $u \nleq A$ and $u \nleq B$. If $A_* \vee B_* < 1$, then as above, $\underline{A \vee B} = (A_* \vee B_*]_K$, and so $(A \vee B)_* = A_* \vee B_*$. If $A_* \vee B_* = 1$, then $\underline{A \vee B} = Q = (A_* \vee B_*]$, so again $(A \vee B)_* = A_* \vee B_*$.

Second, assume that $u \leq A$ or $u \leq B$; say, $u \leq A$. Then $u \in \underline{A}$, and so $\underline{A} = (A_*]_K \cup \{u\}$. Since $\underline{B} \subseteq (B_*]_K \cup \{u\}$, we have that

$$\underline{A}, \underline{B} \subseteq (A_* \vee B_*]_K \cup \{u\} \subseteq \underline{A \vee B},$$

the last containment since A_*, B_*, u are all elements of $\underline{A \vee B}$.

As the first subcase, assume that $a \nleq A_* \vee B_*$. Then $(A_* \vee B_*]_K \cup \{u\}$ is an ideal in Q. Thus $\underline{A \vee B} = (A_* \vee B_*]_K \cup \{u\}$, and so $(A \vee B)_* = A_* \vee B_*$.

As the second subcase, assume that $a \leq A_* \vee B_*$. Since $u \leq A$, it follows that $u, a \in \underline{A \vee B}$, and so $1 = u \vee a \in \underline{A \vee B}$. Thus $\underline{A \vee B} = Q$. Therefore, $(A \vee B)_* = 1$.

This concludes the proof of the theorem. □

Note that this theorem gives a mutually recursive definition of $u \leq A$ and A_*.

4. Complements

In this section, we shall investigate complements in $\mathrm{F}(Q)$. We want a result that describes all complemented pairs $\langle A\rangle$, $\langle B\rangle$. Obviously, we cannot get such a result if K contains a spanning N_5, that is, if K has a sublattice $\{0,p,q,r,0\}$ with $p<q$ and $q\wedge r=0$, $p\vee r=1$, isomorphic to the five-element nonmodular lattice N_5. Indeed, in this case, for almost any polynomial A, the element $\bigl(p\vee\langle A\rangle\bigr)\wedge q$ would be a complement of r in $\mathrm{F}(Q)$.

THEOREM 2.

(i) *The only complement of u in $\mathrm{F}(Q)$ is a.*
(ii) *Let K contain no spanning N_5. Let $\langle A\rangle$, $\langle B\rangle$ be complementary in $\mathrm{F}(Q)$. Then either*

$$\bigl\{\langle A\rangle,\langle B\rangle\bigr\}\subseteq K$$

or

$$\bigl\{\langle A\rangle,\langle B\rangle\bigr\}=\{u,a\}\,.$$

Proof.
(i) Let $A\in\mathbf{P}(Q)$ be such that $\langle A\rangle$ is a complement of u in $\mathrm{F}(Q)$, that is,

$$A\wedge u\equiv 0 \qquad\text{and}\qquad A\vee u\equiv 1\,.$$

By Statement (vi) of Theorem 1,

$$1=(A\vee u)_*=\begin{cases}1 & \text{if } a\le A_*\vee u_*=A_*\,;\\ A_*, & \text{otherwise.}\end{cases}$$

So either $a\le A_*$ or $1=A_*$; in either case, $a\le A_*$. Dually, $a\ge A^*$. Thus

$$A\le A^*\le a\le A_*\le A\,,$$

and so $A\equiv a$.

(ii) We have, by assumption,

$$A\wedge B\equiv 0 \qquad\text{and}\qquad A\vee B\equiv 1\,.$$

By Statements (ii) and (iv) of Theorem 1,

$$A_*\wedge B_*=0\,, \tag{5}$$

and, dually,

$$A^*\vee B^*=1\,. \tag{6}$$

Since $u \leq A \vee B$, we conclude, by Statement (v) of Theorem 1, that either

$$A_* \vee B_* = 1\,, \tag{7}$$

or

$$u \leq A\,, \tag{8}$$

or

$$u \leq B\,. \tag{9}$$

Dually, since $u \geq A \wedge B$, either

$$A^* \wedge B^* = 0\,, \tag{10}$$

or

$$u \geq A\,, \tag{11}$$

or

$$u \geq B\,. \tag{12}$$

First case: (7) holds.
If (10) holds, then

$$\begin{aligned} A^* \vee B^* &= 1 = A_* \vee B_*\,, \\ A_* \wedge B_* &= 0 = A^* \wedge B^*\,. \end{aligned}$$

Since $A_* \leq A^*$ and $B_* \leq B^*$, and since K contains no spanning N_5, we conclude that $A^* = A_*$ and $B^* = B_*$, that is, that $\langle A \rangle$, $\langle B \rangle \in K$.

If (11) holds, then $0 = u_* = A_*$, and so, by (7), $1 = B_*$.

Thus $B^* \leq 1 = B_*$, that is, $B \equiv 1$. Then $A \equiv 0$, and so $\{\langle A \rangle, \langle B \rangle\} = \{0, 1\}$.

Similarly, if (12) holds, then $\{\langle A \rangle, \langle B \rangle\} = \{0, 1\}$.

Thus, in this case, $\{\langle A \rangle, \langle B \rangle\} \subseteq K$.

Second case: (10) holds.
By duality, we conclude that $\{\langle A \rangle, \langle B \rangle\} \subseteq K$.

Third case: One of (8) or (9) holds, and one of (11) or (12) holds.
If (8) and (11) hold, then $A \equiv u$, and, by Statement (i) of our theorem, $B \equiv a$, that is $\{\langle A \rangle, \langle B \rangle\} = \{u, a\}$.

If (8) and (12) hold, then $B \leq A$, and so $A \equiv 1$ and $B \equiv 0$.

The two remaining cases are similar to the two immediately above, with the roles of A and B reversed. □

5. Applications

Now we state the result of C. C. Chen and G. Grätzer [3]:

THEOREM 3. *Let K be a bounded, at most uniquely complemented lattice (that is, a lattice with zero and unit, in which every element has at most one complement). Then K has a $\{0,1\}$-embedding into a uniquely complemented lattice L.*

Proof. Since K is at most uniquely complemented, it contains no spanning N_5. If K is uniquely complemented, there is nothing to do. If not, pick an $a \in K$ that has no complement, define $Q = K \cup \{u\}$, and form $L_1 = \mathrm{F}(Q)$. By Theorem 2, L_1 is an at most uniquely complemented $\{0,1\}$-extension of K, and a has a complement in L_1, namely, u. By transfinite induction, we obtain an at most uniquely complemented $\{0,1\}$-extension $\overline{L}$ of K in which every element of K has a complement. Repeating this construction ω-times, we obtain the lattice L of this theorem. □

The classical result of R. P. Dilworth [6] now easily follows.

THEOREM 4. *Every lattice can be embedded into a uniquely complemented lattice.*

Proof. Starting with an arbitrary lattice V, let K be the lattice we obtain by adjoining a new zero and unit to V. Then K is at most uniquely complemented, indeed, only the zero and the unit have complements. By Theorem 3, K has a $\{0,1\}$-embedding into a uniquely complemented lattice L. Of course, this L will do for V. □

But Theorem 3 says a lot more than its application to Theorem 4. If we start with a bounded, at most uniquely complemented lattice K, then in Theorem 3 we find an extension L of K preserving the bounds of K and preserving all existing complements.

We give one more application of Theorem 2. The reader should have no difficulty with coming up with many more variants.

Let $\mathfrak{m}$ be a cardinal number. A lattice K is called (*at most*) $\mathfrak{m}$*-complemented* if K has 0 and 1, and every $x \in K - \{0,1\}$ has (at most) $\mathfrak{m}$ complements.

THEOREM 5. *Let K be an at most $\mathfrak{m}$-complemented lattice with no spanning N_5. Then K has a $\{0,1\}$-embedding into an $\mathfrak{m}$-complemented lattice L.*

Proof. Follow the idea of the proof of Theorem 3. □

Dean's Theorem can be made considerably sharper for Q. By applying Theorem 1, we now show that Condition (C_Q) of Dean's Theorem involving the ideal $\underline{B}$ and the dual ideal $\overline{A}$ of Q can be replaced by a condition involving only the pair of elements B_* and A^* of K.

THEOREM 6. *Let $A, B \in \mathbf{P}(Q)$. Then $A \le B$ if and only if at least one of the following six conditions holds:*

(E) $A = B$;
(${}_\wedge$W) $A = A_0 \wedge A_1$ *with* $A_0 \le B$ *or* $A_1 \le B$;
(${}_\vee$W) $A = A_0 \vee A_1$ *with* $A_0 \le B$ *and* $A_1 \le B$;
(W${}_\wedge$) $B = B_0 \wedge B_1$ *with* $A \le B_0$ *and* $A \le B_1$;
(W${}_\vee$) $B = B_0 \vee B_1$ *with* $A \le B_0$ *or* $A \le B_1$;
(C${}_*$) $A^* \le B_*$.

P r o o f. The first five conditions are just the Whitman Conditions as in Dean's Theorem.

Our Condition (C${}_*$) is just the statement

$$\overline{A} \cap \underline{B} \cap K \neq \emptyset,$$

which trivially implies Condition (C${}_Q$) of Dean's Theorem. We thus need only show that if $A \le B$ and the Whitman Conditions fail, then $\overline{A} \cap \underline{B} \cap K \neq \emptyset$.

Assume, to the contrary, that

$$\overline{A} \cap \underline{B} \cap K = \emptyset. \tag{13}$$

Then, since $A \le B$ and the Whitman Conditions fail, it follows that $\overline{A} \cap \underline{B} = \{u\}$. Thus, since $u \in \underline{u}$ and $u \in \overline{u}$, we have that

$$A \le u \le B.$$

Since the Whitman Conditions fail for $A \le B$, it follows that B is not a meet. Thus either $B = C \vee D$ for polynomials C and D, or $B \in Q$, that is, $B \in K$ or $B = u$.

First case: $B = C \vee D$.

Then, by (v) of Theorem 1, either $u \le C$, or $u \le D$, or $C_* \vee D_* = 1$. But the first two of these possibilities imply that (W${}_\vee$) holds for $A \le B$. Thus

$$1 = C_* \vee D_* = (C \vee D)_* = B_*.$$

But then

$$A^* \le B_*,$$

contradicting (13).

Second case: $B \in K$.

Then

$$1 = u^* \le B^* = B = B_*,$$

thus again

$$A^* \le B_*,$$

contradicting (13).

Third case: $B = u$.

The dual of the above argument shows that $A = u$. Thus $A = B$, contradicting our assumption that (E) does not hold for $A \le B$. □

REFERENCES

[1] ADAMS, M. E.—SICHLER, J.: *Cover set lattices*, Canad. J. Math. **32** (1980), 1177–1205.

[2] ADAMS, M. E.—SICHLER, J.: *Lattices with unique complementation*, Pacific. J. Math. **92** (1981), 1-13.

[3] CHEN, C. C.—GRÄTZER, G.: *On the construction of complemented lattices*, J. Algebra **11** (1969), 56–63.

[4] CRAWLEY, P.—DILWORTH, R. P.: *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[5] DEAN, R. A.: *Free lattices generated by partially ordered sets and preserving bounds*, Canad. J. Math. **16** (1964), 136–148.

[6] DILWORTH, R. P.: *Lattices with unique complements*, Trans. Amer. Math. Soc. **57** (1945), 123–154.

[7] GRÄTZER, G.: *General Lattice Theory* (2nd ed.), Birkhäuser Verlag, Basel, 1998 (Softcover edition Birkhäuser Verlag, Basel-Boston-Berlin, 2003).

[8] GRÄTZER, G.: *A reduced free product of lattices*, Fund. Math. **73** (1971/72), 21–27.

[9] GRÄTZER, G.—LAKSER, H.: *Freely adjoining a relative complement to a lattice*, Algebra Universalis **53** (2005), 189–210.

[10] GRÄTZER, G.—LAKSER, H.: *Embedding lattices into* $\mathfrak{m}$*-transitively complemented lattices* (Manuscript).

[11] GRÄTZER, G.—LAKSER, H.—PLATT, C. R.: *Free products of lattices*, Fund. Math. **69** (1970), 233–240.

[12] HUNTINGTON, E. V.: *Sets of independent postulates for the algebra of logic*, Trans. Amer. Math. Soc. **79** (1904), 288-309.

[13] LAKSER, H.: *Free lattices generated by partially ordered sets*. Ph.D. Thesis, University of Manitoba, 1968.

[14] SALIĬ, V. N.: *Lattices with Unique Complements*. Transl. Math. Monogr. 69, Amer. Math. Soc., Providence, RI.

[15] WHITMAN, P. M.: *Free lattices. I; II*, Ann. of Math. (2) **42; 43** (1941; 1942), 325–330; 104–115.

Received August 25, 2003
Revised October 31, 2004

* *Department of Mathematics*
University of Manitoba
Winnipeg MB R3T 2N2
CANADA

URL: http://server.math.umanitoba.ca/homepages/gratzer/

E-mail: gratzer@ms.umanitoba.ca

** *Department of Mathematics*
University of Manitoba
Winnipeg MB R3T 2N2
CANADA

E-mail: hlakser@cc.umanitoba.ca