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CONVERGENCES AND COMPLETE DISTRIBUTIVITY  
OF LATTICE ORDERED GROUPS  

JÁN JAKUBÍK  

C. J. Everett and S. Ulam [2] investigated the order convergence of sequences in an abelian lattice ordered group $G$. Some other types of convergences in $G$ were studied by F. Papangelou [10]. An axiomatic treatment of sequential convergences on $G$ was performed by M. Harminc [3], [4], [5] (in some results of [5] the lattice ordered group $G$ need not be abelian).

Higher degrees of distributivity in lattice ordered groups (including complete distributivity) were studied by several authors (cf. e.g., Weinberg [8] and the author [6]).

Let $\text{Conv} G$ be the system of all sequential convergences on $G$ (for the definition, cf. below). The system $\text{Conv} G$ is partially ordered by inclusion. In [5] it was shown that $\text{Conv} G$ need not be a lattice and it was proved (without assuming the commutativity of $G$) that the following conditions are equivalent:

(i) $\text{Conv} G$ has a greatest element.
(ii) $\text{Conv} G$ is a lattice.
(iii) $\text{Conv} G$ is a complete lattice.

In the present paper it will be shown that each archimedean completely distributive lattice ordered group satisfies the condition (i).

1. Preliminaries

Throughout the paper, $G$ denotes a lattice ordered group. For denotations, cf. the monographs of P. Conrad [1] and V. M. Kopytov [7]. The group operation will be denoted additively.

Let $N$ be the set of all positive integers. The direct product $\prod_{n \in N} G_n$, where $G_n = G$ for each $n \in N$, will be denoted by $G^N$. The elements of $G^N$ will be denoted by $(g_n)_{n \in N}$, or simply $(g_n)$. If there exists $g \in G$ such that $g_n = g$ for each $n \in N$, then we denote $(g_n) = \text{const } g$.

$(g_n)$ is said to be a sequence in $G$. The notion of a subsequence has the usual meaning.
Let \( a \) be a convex normal subsemigroup of \((G^N)^+\) such that the following conditions are satisfied:

(I) If \((g_n) \in a\), then each subsequence of \((g_n)\) belongs to \(a\).

(II) Let \((g_n) \in (G^N)^+\). If each subsequence of \((g_n)\) has a subsequence belonging to \(a\), then \((g_n)\) belongs to \(a\).

(III) Let \( g \in G \). Then \( \text{const} g \) belongs to \(a\) if and only if \(|g_n - g| \in a\).

Under these assumptions \(a\) is said to be a convergence in \(G\). The system of all convergences in \(G\) will be denoted by \(\text{Conv} \ G\); this system is partially ordered by inclusion. (Cf. [5], Definition 1.4 and Lemma 1.9.)

For \((g_n) \in G^N\) and \(g \in G\) we put \(g_n \to_a g\) if and only if \(|g_n - g| \in a\).

Let \(A \subseteq (G^N)^+\). We denote by \(\delta A\) the system of all subsequences of sequences belonging to \(A\). The convex closure (in \(G^N\)) of the set \(A \cup \{\text{const} 0\}\) will be denoted by \([A]\). Next let \(\langle A \rangle\) be the subsemigroup of \(G^N\) generated by the set \(A\). The symbol \(A^*\) will denote the set of all sequences in \(G\) for which each subsequence has a subsequence belonging to \(A\).

1.1. **Proposition.** (Cf. [5], Theorem 1.18.) Let \(\emptyset \neq A \subseteq (G^N)^+\). Assume that \(G\) is abelian. Then the following conditions are equivalent.

(a) There exist \(a \in \text{Conv} \ G\) such that \(A \subseteq a\).

(b) If \(g \in G\), \(\text{const} g \in [\langle \delta A \rangle]\), then \(g = 0\).

2. **Complete distributivity**

For the notion of complete distributivity of lattice ordered groups cf. [8] or [6].

2.1. **Theorem.** (Cf. [8].) Let \(G\) be a completely distributive archimedean lattice ordered group. Then there exist linearly ordered groups \(G_i\) \((i \in I)\) and a complete isomorphism of \(G\) into \(\Pi_{i \in I} G_i\).

Throughout this section we assume that \(G\) is a completely distributive archimedean lattice ordered group. In view of 2.1, we can suppose without loss of generality that \(G\) is an \(l\)-subgroup of a lattice ordered group \(\Pi_{i \in I} G_i\), where each \(G_i\) is linearly ordered and all joins and meets in \(G\) are performed component-wise. Moreover, we can assume that for each \(i \in I\) and each \(x' \in G_i\) there exists \(g \in G\) such that the \(i\)-th component of \(g\) is \(x'\).

2.2. **Lemma.** Let \(i \in I\). Let \(a_i\) be a non-discrete convergence on \(G_i\). Let \((x_n)\) be a sequence in \(G_i\), \(x_n \geq 0\) for \(n = 1, 2, \ldots\). Then the following conditions are equivalent:

(i) \(x_n \to_{a_i} 0\).

(ii) If \(0 < a' \in G_i\), then there exists a positive integer \(m\) such that \(x_n < a'\) for each \(n \geq m\).

(iii) The sequence \((x_n)\) \(o\)-converges to 0 in \(G_i\).
Proof. According to [5], Theorem 2.10, (i) $\Leftrightarrow$ (ii). The equivalence (ii) $\Leftrightarrow$ (iii) is obvious.

2.3. Lemma. Let $a \in \text{Conv } G$, $0 < a \in G$, $i \in I$. Assume that $a(i) > 0$. Let $(x_n)$ be a sequence in $G$ such that $x_n \to_a 0$. Then there is a positive integer $m$ such that $x_n(i) < a(i)$ for each $n \geq m$.

Proof. By way of contradiction, assume that the assertion to be proved fails to hold. Then there is a subsequence of $(x_n)$ such that the $i$-th component of each member of this sequence is greater than or equal to $a(i)$. For simplifying the notation, let us suppose that $(x_n)$ coincides with the subsequence under consideration. Put $y_n = x_n - A \cdot a$. Hence $y_n \to_a 0$ and $y_n(i) = a(i)$ for each positive integer $n$.

Denote $z_n = y_1 \wedge y_2 \wedge y_3 \wedge \ldots \wedge y_n$ for each positive integer $n$. Then $0 \leq z_n \leq y_n$, hence $z_n \to_a 0$. Moreover, $z_1 \geq z_2 \geq \ldots \geq z_n \geq \ldots$. Hence we must have $\bigwedge_{n=1}^{\infty} z_n = 0$. Since $G$ is a closed sublattice of $\Pi_{j \in I} G_j$ we infer that $\bigwedge_{n=1}^{\infty} z_n(i) = 0$. But $z_n(i) = a(i) > 0$ for each positive integer $n$, which is a contradiction.

Since for each $i \in I$ there exists $0 < a \in G$ with $a(i) > 0$, from 2.2 and 2.3 we obtain:

2.4. Corollary. Let $a \in \text{Conv } G$, $i \in I$. Let $(x_n)$ be a sequence in $G$ such that $x_n \to_a 0$. Then $(x_n(i))$ $o$-converges to $0$ in $G_i$.

Let us denote by $a_0$ the system of all sequences $(x_n)$ in $G^+$ such that for each $i \in I$, $(x_n(i))$ $o$-converges to $0$ in $G_i$.

2.5. Lemma. $a_0 \in \text{Conv } G$.

Proof. From the definition of $a_0$ we obtain that for $a_0 = A$ we have $[\langle \delta A \rangle]^* = A$ and that the condition (b) from 1.1 is satisfied. Hence according to [5], Thm. 1.18 we obtain $a_0 \in \text{Conv } G$.

Now, according to 2.4 we have $a \leq a_0$ for each $a \in \text{Conv } G$. Thus we have arrived at the following result:

2.6. Theorem. Let $G$ be an archimedean completely distributive lattice ordered group. Then the partially ordered set $\text{Conv } G$ possesses a greatest element.

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СХОДИМОСТЬ И ПОЛНАЯ ДИСТРИБУТИВНОСТЬ
РЕШЕТОЧНО УПОРЯДОЧЕННЫХ ГРУПП

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Резюме

В статье доказано, что упорядоченное множество Conv G всех сходимостей на вполне дистрибутивной архимедовой решеточно упорядоченной группе G является полной решеткой.