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NONLINEAR BOUNDARY VALUE PROBLEM FOR SECOND ORDER DIFFERENTIAL EQUATIONS DEPENDING ON A PARAMETER

SVATOSLAV STANĚK

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ABSTRACT. By means of the Leray-Schauder degree theory, sufficient conditions are given for the existence and uniqueness of solutions of the boundary value problem $x'' = f(t,x,x',\lambda)$, $\alpha(x) = A$, $x'(0) = B$, $x'(1) = C$, depending on the parameter λ. Here $f \in C^0([0,1]\times\mathbb{R}^3)$, $\alpha\colon X \to \mathbb{R}$ is continuous increasing, $\operatorname{Im}\alpha = \mathbb{R}$, X is the Banach space of C^0-functions on $[0,1]$ and $A,B,C \in \mathbb{R}$.

1. Introduction

Let X be the Banach space of C^0-functions on $[0,1]$ with the norm $\|x\| = \max\{|x(t)|;\ 0 \le t \le 1\}$.

Consider the boundary value problem (BVP for short)

$$x'' = f(t,x,x',\lambda)\,, \tag{1}$$

$$\alpha(x) = A\,, \quad x'(0) = B\,, \quad x'(1) = C\,, \tag{2}$$

depending on the parameter λ. Here $f \in C^0([0,1]\times\mathbb{R}^3)$, $\alpha\colon X \to \mathbb{R}$ is continuous increasing (i.e. $x,y \in X$, $x(t) < y(t)$ on $[0,1]$ $\implies$ $\alpha(x) < \alpha(y)$), $\operatorname{Im}\alpha = \mathbb{R}$, where $\operatorname{Im}\alpha$ is the range of α, and $A,B,C \in \mathbb{R}$.

We say that the pair $(x,\lambda_0) \in C^2([0,1]) \times \mathbb{R}$ is a *solution* of the BVP (1), (2) if x is a solution of (1) for $\lambda = \lambda_0$ satisfying (2).

In this paper, sufficient conditions are given for the existence and uniqueness of solutions of the BVP (1), (2). The existence theorem is proved using the invariance of the Leray-Schauder degree with respect to a homotopy (see, e.g., [2]).

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The proof of the uniqueness of solutions is based on methods of classical mathematical analysis. We note that the BVP (1), (2) for $A = B = C = 0$ was studied in [9] under the assumptions that f satisfies sign conditions and at the same time conditions of monotonicity. The results were proved using a combination of the coincidence degree theory and the shooting method. Our results generalize those in [9].

We observe that second-order (ordinary and functional) differential equations depending on a parameter were studied under various boundary conditions, e.g., in [1], [3] and in [5] – [9], usually under linear boundary conditions. The existence results were proved using the Schauder linearization and quasilinearization technique, the technique of Green's functions, the Schauder fixed point theorem, a surjectivity result in $\mathbb{R}^n$, the Leray-Schauder degree method and a suitable combination of the above methods.

2. Lemmas

Remark 1. Let $A \in \mathbb{R}$ and $\alpha(b) = A$ for some $b \in X$. If $\alpha(x+b) = A$ for $x \in X$, then there exists $\xi \in [0,1]$ such that $x(\xi) = 0$. Otherwise, $x(t) + b(t) \neq b(t)$ on $[0,1]$, and then $\alpha(x+b) \neq \alpha(x)$ since α is increasing.

Remark 2. One can easily verify that the functionals

$$\max\{x(t);\ 0 \le t \le 1\}, \qquad \min\{x(t);\ 0 \le t \le 1\},$$

$$\int_a^b x^3(s)\,\mathrm{ds} \quad (0 \le a < b \le 1),$$

$$\sum_{k=1}^{n} a_k x^5(t_k) \qquad (a_k > 0,\ 0 \le t_k < t_{k+1} \le 1)$$

defined on X have the same properties as the functional α.

Let $A \in \mathbb{R}$ and $\alpha(b) = A$ for some $b \in X$. Let $h \in C^0([0,1] \times \mathbb{R}^3)$, and consider the BVP

$$x'' = h(t, x, x', \lambda), \tag{3}$$

$$\alpha(x+b) = A, \quad x'(0) = 0, \quad x'(1) = 0 \tag{4}$$

depending on the parameter λ. We shall assume that h satisfies the following assumptions:

There exist constants $M > 0$, $\mu > 0$ and a nondecreasing function

$$w_1 \colon [0,\infty) \to (0,\infty)$$

such that

(A_1) $h(t,x,0,\mu) > 0$ for $(t,x) \in [0,1] \times [0,M]$,
$h(t,x,0,-\mu) < 0$ for $(t,x) \in [0,1] \times [-M,0]$;

(A_2) $h(t,-M,0,\lambda) < 0 < h(t,M,0,\lambda)$ for $(t,\lambda) \in [0,1] \times (-\mu,\mu)$;

(A_3) $|h(t,x,y,\lambda)| \le w_1(|y|)$ for $(t,x,\lambda) \in [0,1] \times [-M,M] \times [-\mu,\mu]$, $y \in \mathbb{R}$ and

$$\int_0^\infty \frac{s\,\mathrm{ds}}{w_1(s)} = \infty .$$

Lemma 1. *Let assumptions* (A_1)–(A_3) *be satisfied for positive constants* M, μ *and a nondecreasing function* $w_1\colon [0,\infty) \to (0,\infty)$. *Let* (x,λ_0) *be a solution of the BVP* (3), (4) *such that*

$$\|x\| \le M , \qquad |\lambda_0| \le \mu .$$

Then

$$\|x\| < M , \quad \|x'\| < T , \quad \|x''\| < w_1(T) + 1 , \quad |\lambda_0| < \mu , \tag{5}$$

where $T > 0$ *is a positive constant such that*

$$\int_0^T \frac{s\,\mathrm{ds}}{w_1(s)} > 2M . \tag{6}$$

P r o o f . By Remark 1, $x(\xi) = 0$ for some $\xi \in [0,1]$, hence

$$0 \le \max\{x(t)\,;\ 0 \le t \le 1\} = x(\tau) , \qquad 0 \ge \min\{x(t)\,;\ 0 \le t \le 1\} = x(\nu) ,$$

where $\tau,\nu \in [0,1]$. Assume $|\lambda_0| = \mu$, say for example, $\lambda_0 = -\mu$. Since $x(\nu) \in [-M,0]$ and $x'(\nu) = 0$, we have (cf. (A_1)) $x''(\nu) = h\big(\nu, x(\nu), 0, -\mu\big) < 0$, a contradiction.

Thus $|\lambda_0| < \mu$.

Assume $x(\varrho) = M$ for $\varrho \in [0,1]$. Then $x'(\varrho) = 0$ and (cf. (A_2)) $x''(\varrho) = h(\varrho, M, 0, \lambda_0) > 0$, a contradiction. Similarly, $x(\eta) = -M$ for $\eta \in [0,1]$ leads to a contradiction, and consequently, $\|x\| < M$.

Using (A_3), (6) and a standard procedure (see, e.g., [4]) we obtain $\|x'\| < T$ and then $|x''(t)| = \big|h\big(t, x(t), x'(t), \lambda_0\big)\big| \le w_1\big(|x'(t)|\big) \le w_1(T) < w_1(T) + 1$ on $[0,1]$. □

Lemma 2. *Let assumptions* (A_1)–(A_3) *be satisfied for positive constants* M, μ *and a nondecreasing function* $w_1\colon [0,\infty) \to (0,\infty)$. *Then there exists a solution of the BVP* (3), (4).

P r o o f . Let $k = M/\mu$ and consider the differential equation

$$x'' = c \cdot h(t,x,x',\lambda) + (1-c)(x + k\lambda) , \qquad c \in [0,1] . \tag{6_c}$$

Setting $p_c(t,x,y,\lambda) = c\cdot h(t,x,y,\lambda) + (1-c)(x+k\lambda)$ for $(t,x,y,\lambda) \in [0,1]\times\mathbb{R}^3$ and $c \in [0,1]$, then p_c is continuous and

$$\begin{aligned}
p_c(t,x,0,\mu) &= c\cdot h(t,x,0,\mu) + (1-c)(x+k\mu) > 0 && \text{for} \quad (t,x)\in[0,1]\times[0,M],\\
p_c(t,x,0,-\mu) &= c\cdot h(t,x,0,-\mu) + (1-c)(x-k\mu) < 0 && \text{for} \quad (t,x)\in[0,1]\times[-M,0],\\
p_c(t,-M,0,\lambda) &= c\cdot h(t,-M,0,\lambda) + (1-c)(-M+k\lambda) < 0 && \text{for} \quad (t,\lambda)\in[0,1]\times(-\mu,\mu),\\
p_c(t,M,0,\lambda) &= c\cdot h(t,M,0,\lambda) + (1-c)(M+k\lambda) > 0 && \text{for} \quad (t,\lambda)\in[0,1]\times(-\mu,\mu),
\end{aligned}$$

$$|p_c(t,x,y,\lambda)| \le c|h(t,x,y,\lambda)| + (1-c)|x+k\lambda| \le c\cdot w_1(|y|) + 2(1-c)M \le w_1(|y|) + 2M$$
$$\text{for} \quad (t,x,\lambda)\in[0,1]\times[-M,M]\times[-\mu,\mu], \quad y\in\mathbb{R}.$$

Hence, by Lemma 1,

$$\|x_c\| < M, \qquad \|x_c'\| < T_1, \qquad \|x_c''\| < w_1(T_1) + 2M + 1, \qquad |\lambda_c| < \mu \tag{7}$$

for any solution (x_c, λ_c) of the BVP (6_c), (4) satisfying $\|x_c\| \le M$, $|\lambda_c| \le \mu$, where T_1 is a positive constant such that

$$\int_0^{T_1} \frac{s\,\mathrm{ds}}{w_1(s)+2M} > 2M.$$

Let $Y = C^1([0,1])$ and $Z = C^2([0,1])$ be the Banach spaces endowed with the norms $\|x\|_1 = \|x\| + \|x'\|$ and $\|x\|_2 = \|x\|_1 + \|x''\|$, respectively; $Y_0 = \{x;\ x\in Y,\ x'(0) = x'(1) = 0\}$, $Z_0 = Z\cap Y_0$. Let $X\times\mathbb{R} = \{(x,\lambda);\ x\in X,\ \lambda\in\mathbb{R}\}$, $Y_0\times\mathbb{R} = \{(x,\lambda);\ x\in Y_0,\ \lambda\in\mathbb{R}\}$ and $Z_0\times\mathbb{R} = \{(x,\lambda);\ x\in Z_0,\ \lambda\in\mathbb{R}\}$ be the Banach spaces with the norms $\|(x,\lambda)\| = \|x\| + |\lambda|$, $\|(x,\lambda)\|_1 = \|x\|_1 + |\lambda|$ and $\|(x,\lambda)\|_2 = \|x\|_2 + |\lambda|$, respectively. Define the operators $K, H, L\colon Z_0\times\mathbb{R}\to X\times\mathbb{R}$ by

$$\begin{aligned}
\big(K(x,\lambda)\big)(t) &= \big(x''(t) + x(t) + k\lambda,\ \alpha(x+b) - A - 2\lambda\big)\\
\big(H(x,\lambda)\big)(t) &= \big(h(t,x(t),x'(t),\lambda),\ -\lambda\big)\\
\big(L(x,\lambda)\big)(t) &= \big(x(t) + k\lambda,\ -\lambda\big).
\end{aligned}$$

Consider the operator equation

$$K(x,\lambda) = cH(x,\lambda) + (2-c)L(x,\lambda), \qquad c\in[0,1]. \tag{8_c}$$

We see that the BVP (3), (4) has a solution (x,λ_0) if and only if that is a solution of (8_1).

Now, we shall prove that $K\colon Z_0\times\mathbb{R}\to X\times\mathbb{R}$ is one to one and onto, and $K^{-1}\colon X\times\mathbb{R}\to Z_0\times\mathbb{R}$ is continuous. Let $(u,\tau)\in X\times\mathbb{R}$ and consider the operator equation

$$K(x,\lambda) = (u,\tau), \tag{9}$$

that is, the equations

$$x'' + x + k\lambda = u(t), \tag{10'}$$

$$\alpha(x+b) - A - 2\lambda = \tau, \tag{10''}$$

where $x \in Z_0$, $\lambda \in \mathbb{R}$. The function $x(t) = c_1 \sin(t) + c_2 \cos(t) - k\lambda + v(t)$ is general solution of (10′) with $v(t) = \int_0^t u(s) \sin(t-s)\, ds$. So, $\bar{x}(t) = v'(1)\cos(t)/\sin(1) - k\lambda + v(t)$ is the unique solution of (10′) in Z_0. Setting

$$\begin{aligned} p(\lambda) &= \alpha\big(v'(1)\cos(t)/\sin(1) - k\lambda + v(t) + b(t)\big) - A - 2\lambda \\ &\big(= \alpha(\bar{x} + b) - A - 2\lambda \big), \qquad \lambda \in \mathbb{R}, \end{aligned} \tag{11}$$

p is continuous decreasing, $\lim_{\lambda \to -\infty} p(\lambda) = \infty$, $\lim_{\lambda \to \infty} p(\lambda) = -\infty$. Therefore the equation $p(\lambda) = \tau$ has a unique solution, say $\lambda = \lambda_0$; hence

$$\big(v'(1)\cos(t)/\sin(1) - k\lambda_0 + v(t),\ \lambda_0\big)$$

is the unique solution of (9). This proves that K^{-1} exists and $K^{-1}(u,\tau) = (\tilde{x}, \lambda_0)$, where $\tilde{x}(t) = v'(1)\cos(t)/\sin(1) - k\lambda_0 + v(t)$, $v(t) = \int_0^t u(s)\sin(t-s)\, ds$ and $\alpha(\tilde{x} + b) - A - 2\lambda_0 = \tau$. To prove the continuity of K^{-1}, we assume that $\{(u_n, \tau_n)\} \subset X \times \mathbb{R}$ is a convergent sequence, $\lim_{n\to\infty} (u_n, \tau_n) = (u, \tau_0)$. Let $K^{-1}(u_n, \tau_n) = (x_n, \lambda_n)$, $n \in \mathbb{N}$, and $K^{-1}(u, \tau_0) = (x, \lambda_0)$.

Then

$$\begin{gathered} x_n(t) = v_n'(1)\cos(t)/\sin(1) - k\lambda_n + v_n(t), \\ x(t) = v'(1)\cos(t)/\sin(1) - k\lambda_0 + v(t), \\ \alpha(x_n + b) - A - 2\lambda_n = \tau_n, \qquad \alpha(x + b) - A - 2\lambda_0 = \tau_0 \end{gathered}$$

for $t \in [0,1]$ and $n \in \mathbb{N}$, where

$$v_n(t) = \int_0^t u_n(s)\sin(t-s)\, ds, \qquad v(t) = \int_0^t u(s)\sin(t-s)\, ds.$$

Evidently, $\lim_{n\to\infty} v_n^{(i)}(t) = v^{(i)}(t)$ uniformly on $[0,1]$ for $i = 0,1$, and $\{\lambda_n\}$ is a bounded sequence. Assume, on the contrary, that $\{\lambda_n\}$ is not convergent. Then there exist convergent subsequences $\{\lambda_{k_n}\}$ and $\{\lambda_{l_n}\}$ of $\{\lambda_n\}$ such that $\lim_{n\to\infty} \lambda_{k_n} = \varrho_1$, $\lim_{n\to\infty} \lambda_{l_n} = \varrho_2$, $\varrho_1 < \varrho_2$, and consequently

$$\begin{gathered} \lim_{n\to\infty} x_{k_n}(t) = v'(1)\cos(t)/\sin(1) - k\varrho_1 + v(t), \\ \lim_{n\to\infty} x_{l_n}(t) = v'(1)\cos(t)/\sin(1) - k\varrho_2 + v(t) \end{gathered}$$

uniformly on $[0,1]$. Therefore $\alpha\big(v'(1)\cos(t)/\sin(1) - k\varrho_1 + v(t) + b(t)\big) - A - 2\varrho_1 = \tau_0$, $\alpha\big(v'(1)\cos(t)/\sin(1) - k\varrho_2 + v(t) + b(t)\big) - A - 2\varrho_2 = \tau_0$, and then

$p(\varrho_1) = p(\varrho_2)$ with p defined by (11) which contradicts the fact that p is decreasing on $\mathbb{R}$; hence $\{\lambda_n\}$ is convergent, $\lim_{n\to\infty} \lambda_n = \mu_0$. Since

$$\lim_{n\to\infty} x_n(t) = v'(1)\cos(t)/\sin(1) - k\mu_0 + v(t)$$

uniformly on $[0,1]$ and $\alpha\big(v'(1)\cos(t)/\sin(1) - k\mu_0 + v(t) + b(t)\big) - A - 2\mu_0 = \tau_0$, we have $\mu_0 = \lambda_0$, $\lim_{n\to\infty} x_n = x$, and consequently, $\lim_{n\to\infty} K^{-1}(u_n, \tau_n) = (x, \lambda_0) = K^{-1}(u, \tau_0)$.

Equation (8_c) can be written in the equivalent form

$$(x, \lambda) = K^{-1}\big(cHj(x,\lambda) + (2-c)Lj(x,\lambda)\big), \qquad c \in [0,1], \tag{12_c}$$

where $j\colon Z_0 \times \mathbb{R} \to Y_0 \times \mathbb{R}$ is the natural embedding, which is completely continuous by the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem.

Define

$$\begin{aligned}\Omega = \big\{(x,\lambda);\ (x,\lambda) \in Z_0 \times \mathbb{R},\ \|x\| < M,\ \|x'\| < T_1,\\ \|x''\| < w_1(T_1) + 2M + 1,\ |\lambda| < \mu\big\}.\end{aligned}$$

Then Ω is a bounded open convex subset of $Z_0 \times \mathbb{R}$ which is symmetric with respect to $0 \in \Omega$. Let $V\colon [0,1] \times \bar{\Omega} \to Z_0 \times \mathbb{R}$ be given by $V(c,x,\lambda) = K^{-1}\big(cHj(x,\lambda) + (2-c)Lj(x,\lambda)\big)$. Then V is a compact operator and (cf. (7)) $V(c,x,\lambda) \neq (x,\lambda)$ for all $(x,\lambda) \in \partial\Omega$ and $c \in [0,1]$, hence (cf., e.g., [2]) $D\big(I - K^{-1}(Hj + Lj), \Omega, 0\big) = D\big(I - K^{-1}(2Lj), \Omega, 0\big)$, where D denotes the Leray-Schauder degree. In order to prove our lemma, it is sufficient to show that $D\big(I - K^{-1}(2Lj), \Omega, 0\big) \neq 0$. Let $P = I - K^{-1}(2Lj)$.

Assume $P(-x_0, -\varepsilon_0) = aP(x_0, \varepsilon_0)$ for some $a \geq 1$ and $(x_0, \varepsilon_0) \in \partial\Omega$. Then

$$(-x_0, -\varepsilon_0) - K^{-1}(-2x_0 - 2k\varepsilon_0,\ 2\varepsilon_0) = a(x_0, \varepsilon_0) - aK^{-1}(2x_0 + 2k\varepsilon_0,\ -2\varepsilon_0)$$

and

$$(a+1)(x_0, \varepsilon_0) = aK^{-1}(2x_0 + 2k\varepsilon_0, -2\varepsilon_0) - K^{-1}(-2x_0 - 2k\varepsilon_0, 2\varepsilon_0). \tag{13}$$

So, since

$$\begin{aligned}K^{-1}(2x_0 + 2k\varepsilon_0, -2\varepsilon_0) &= \big(w'(1)\cos(t)/\sin(1) - k\lambda_0 + w(t), \lambda_0\big),\\ K^{-1}(-2x_0 - 2k\varepsilon_0, 2\varepsilon_0) &= \big(-w'(1)\cos(t)/\sin(1) - k\mu_0 - w(t), \mu_0\big),\end{aligned}$$

where $w(t) = 2\int_0^t \big(x_0(s) + k\varepsilon_0\big)\sin(t-s)\,\mathrm{d}s$, and λ_0, μ_0 are (unique) constants such that

$$\alpha\big(w'(1)\cos(t)/\sin(1) - k\lambda_0 + w(t) + b(t)\big) - A - 2\lambda_0 = -2\varepsilon_0, \tag{14'}$$

$$\alpha\big(-w'(1)\cos(t)/\sin(1) - k\mu_0 - w(t) + b(t)\big) - A - 2\mu_0 = 2\varepsilon_0, \tag{14''}$$

we obtain (cf. (13))

$$x_0(t) = w'(1)\cos(t)/\sin(1) + w(t) + k(\mu_0 - a\lambda_0)/(1+a)\,,$$
$$\varepsilon_0 = (a\lambda_0 - \mu_0)/(1+a)\,,$$

and therefore

$$x_0(t) = \frac{2\cos(t)}{\sin(1)}\int_0^1 x_0(s)\cos(1-s)\ \mathrm{d}s + 2\int_0^t x_0(s)\sin(t-s)\ \mathrm{d}s + \frac{k}{1+a}(a\lambda_0 - \mu_0)$$

because of

$$\begin{aligned} x_0(t) &= w'(1)\cos(t)/\sin(1) + w(t) + \frac{k}{1+a}(\mu_0 - a\lambda_0) \\ &= \frac{2\cos(t)}{\sin(1)}\int_0^1 x_0(t)\cos(1-s)\ \mathrm{d}s + 2\int_0^t x_0(s)\sin(t-s)\ \mathrm{d}s + 2k\varepsilon_0 \\ &\qquad + \frac{k}{1+a}(\mu_0 - a\lambda_0) \\ &= \frac{2\cos(t)}{\sin(1)}\int_0^1 x_0(t)\cos(1-s)\ \mathrm{d}s + 2\int_0^t x_0(s)\sin(t-s)\ \mathrm{d}s \\ &\qquad + \frac{k}{1+a}(a\lambda_0 - \mu_0)\,. \end{aligned}$$

Then $x_0''(t) = x_0(t) + k(a\lambda_0 - \mu_0)/(1+a)$ on $[0,1]$, hence $x_0(t) = c_1e^t + c_2e^{-t} - k(a\lambda_0 - \mu_0)/(1+a)$, where c_1, c_2 are suitable constants. Since $x_0 \in Z_0$, $c_1 = 0 = c_2$, and therefore $x_0(t) = -k\varepsilon_0$ on $[0,1]$, which implies (cf. assumption $(x_0, \varepsilon_0) \in \partial\Omega$) that $|\lambda_0| + |\mu_0| > 0$. Next, we have (cf. (14))

$$\alpha(-k\lambda_0 + b) - A = 2\lambda_0 - 2\varepsilon_0 = \frac{2(\lambda_0 + \mu_0)}{1+a}\,,$$
$$\alpha(-k\mu_0 + b) - A = 2\mu_0 + 2\varepsilon_0 = \frac{2a(\lambda_0 + \mu_0)}{1+a}\,,$$

thus

$$0 > \lambda_0\big(\alpha(-k\lambda_0 + b) - \alpha(b)\big) = \lambda_0\big(\alpha(-k\lambda_0 + b) - A\big) = \frac{2\lambda_0(\lambda_0 + \mu_0)}{1+a} \quad \text{for} \quad \lambda_0 \neq 0\,,$$

$$0 > \mu_0\big(\alpha(-k\mu_0 + b) - \alpha(b)\big) = \mu_0\big(\alpha(-k\mu_0 + b) - A\big) = \frac{2a\mu_0(\lambda_0 + \mu_0)}{1+a} \quad \text{for} \quad \mu_0 \neq 0\,,$$

and then

$$\mu_0\lambda_0 \geq 0, \qquad 0 > \frac{2\lambda_0(\lambda_0+\mu_0)}{1+a} + \frac{2a\mu_0(\lambda_0+\mu_0)}{1+a} = \frac{2(a\mu_0+\lambda_0)(\lambda_0+\mu_0)}{1+a}.$$

Since $(a\mu_0+\lambda_0)(\lambda_0+\mu_0) = \left(a^{1/2}\mu_0+\lambda_0\right)^2 + \mu_0\lambda_0\left(1+a-2a^{1/2}\right)$ and $\mu_0\lambda_0\left(1+a-2a^{1/2}\right) \geq 0$ for $a \geq 1$, we obtain $(a\mu_0+\lambda_0)(\lambda_0+\mu_0) \geq 0$, a contradiction. Therefore $P(-x,-\varepsilon) \neq aP(x,\varepsilon)$ for all $(x,\varepsilon) \in \partial\Omega$ and $a \geq 1$, and hence $D\left(I - K^{-1}(2Lj), \Omega, 0\right)$ is an odd integer by [2; p. 58, Theorem 8.3]. □

Remark 3. Let $A, B, C \in \mathbb{R}$. Then $a(t) = a_0 + Bt + (C-B)t^2/2$ $(t \in [0,1])$ is a function satisfying the boundary conditions (2), where $a_0 \in \mathbb{R}$ is the unique solution of the equation

$$\alpha\left(a + Bt + (C-B)t^2/2\right) = A, \qquad a \in \mathbb{R}.$$

3. Existence theorem

THEOREM 1. *Assume that the following assumptions are satisfied:*

(H_1) *For each positive constant E there exist constants $K > 0$ and $\Lambda > 0$ such that*

$$\begin{aligned}
f(t,x,y,\Lambda) &> E \\
&\quad \text{for} \quad (t,x,y) \in [0,1] \times [-E, K+E] \times [-E,E], \\
f(t,x,y,-\Lambda) &< -E \\
&\quad \text{for} \quad (t,x,y) \in [0,1] \times [-K-E, E] \times [-E,E], \\
f(t,x,y,\lambda) &< -E \\
&\quad \text{for} \quad (t,x,y,\lambda) \in [0,1] \times [-K-E, -K+E] \times [-E,E] \times (-\Lambda,\Lambda), \\
f(t,x,y,\lambda) &> E \\
&\quad \text{for} \quad (t,x,y,\lambda) \in [0,1] \times [K-E, K+E] \times [-E,E] \times (-\Lambda,\Lambda);
\end{aligned}$$

(H_2) *A nondecreasing function $w(\cdot, \mathcal{D}_0)\colon [0,\infty) \to (0,\infty)$ exists for any bounded subset $\mathcal{D}_0$ of $\mathbb{R}^2$ such that*

$$|f(t,x,y,\lambda)| \leq w\left(|y|; \mathcal{D}_0\right) \qquad \text{for} \quad (t,x,\lambda) \in [0,1] \times \mathcal{D}_0, \quad y \in \mathbb{R},$$

and

$$\int_0^\infty \frac{s\, \mathrm{ds}}{w(s;\mathcal{D}_0)} = \infty.$$

Then the BVP (1), (2) has a solution for each $A, B, C \in \mathbb{R}$.

P r o o f . Let $A, B, C \in \mathbb{R}$, and let $a \in C^2([0,1])$ satisfy boundary conditions (2) (see Remark 3). Set $E_1 = \max\{\|a\|, \|a'\|, \|a''\|\}$ and

$$h(t,x,y,\lambda) = f\big(t, x + a(t), y + a'(t), \lambda\big) - a''(t) \qquad \text{for} \quad (t,x,y,\lambda) \in [0,1] \times \mathbb{R}^3 .$$

We see that (x_0, λ_0) is a solution of the BVP (3), (4) (with $b = a$) if and only if $(x_0 + a, \lambda_0)$ is a solution of the BVP (1), (2). Hence to prove Theorem 1, it is sufficient to show that the BVP (3), (4) (with $b = a$) has a solution, which occurs if h satisfies assumptions of Lemma 2. Let $K > 0$, $\Lambda > 0$ be constants corresponding to $E = E_1$ in (H_1). Then

$$\begin{aligned}
h(t,x,0,\Lambda) &= f\big(t,\, x + a(t),\, a'(t),\, \Lambda\big) - a''(t) \; > E_1 - a''(t) \ge 0 \\
&\qquad\qquad \text{for} \quad (t,x) \in [0,1] \times [0,K], \\
h(t,x,0,-\Lambda) &= f\big(t, x + a(t), a'(t), -\Lambda\big) - a''(t) \; < -E_1 - a''(t) \le 0 \\
&\qquad\qquad \text{for} \quad (t,x) \in [0,1] \times [-K,0], \\
h(t,-K,0,\lambda) &= f\big(t, -K + a(t), a'(t), \lambda\big) - a''(t) < -E_1 - a''(t) \le 0 \\
&\qquad\qquad \text{for} \quad (t,\lambda) \in [0,1] \times (-\Lambda,\Lambda), \\
h(t,K,0,\lambda) &= f\big(t,\, K + a(t),\, a'(t),\, \lambda\big) - a''(t) \; > E_1 - a''(t) \ge 0 \\
&\qquad\qquad \text{for} \quad (t,\lambda) \in [0,1] \times (-\Lambda,\Lambda).
\end{aligned}$$

Set $\mathcal{D}_1 = [-K - E_1, K + E_1] \times [-\Lambda, \Lambda]$. By (H_2), there exists a nondecreasing function $(\cdot\,, \mathcal{D}_1)\colon [0,\infty) \to (0,\infty)$ such that $\int_0^\infty \frac{s\,\mathrm{d}s}{w(s;\mathcal{D}_1)} = \infty$ and

$$|f(t,x,y,\lambda)| \le w\big(|y|;\mathcal{D}_1\big) \qquad \text{for} \quad (t,x,\lambda) \in [0,1] \times \mathcal{D}_1\,, \quad y \in \mathbb{R};$$

hence

$$\begin{aligned}
|h(t,x,y,\lambda)| &= \big|f\big(t, x + a(t), y + a'(t), \lambda\big) - a''(t)\big| \\
&\le w\big(|y + a'(t)|;\mathcal{D}_1\big) + E_1 \le w\big(|y| + E_1;\mathcal{D}_1\big) + E_1 \\
&\text{for} \quad (t,x,\lambda) \in [0,1] \times [-K,K] \times [-\Lambda,\Lambda], \quad y \in \mathbb{R}.
\end{aligned}$$

The function h satisfies the assumptions of Lemma 2 with $M = K$, $\mu = \Lambda$ and $w_1(u) = w(u + E_1;\mathcal{D}_1) + E_1$ on $[0,\infty)$. □

EXAMPLE 1. Theorem 1 can be applied to the differential equation

$$x'' = p(t,x) + q(t,x,x') + k(t,x,x')\lambda\,, \tag{15}$$

with $p \in C^0([0,1] \times \mathbb{R})$, $q,k \in C^0([0,1] \times \mathbb{R}^2)$, $\liminf\limits_{|x|\to\infty} \operatorname{sign}(x) \cdot p(t,x) = \infty$ uniformly on $[0,1]$, $\limsup\limits_{|x|\to\infty} \dfrac{|q(t,x,y)|}{y^2 + 1} < \infty$ uniformly on $[0,1] \times \mathbb{R}$, $a \le$

$k(t,x,y) \le b$ on $[0,1] \times \mathbb{R}^2$, $a, b \in \mathbb{R}$, $0 < a < b$. Indeed, let $E > 0$ be a positive constant. Set $A_1 = \inf\{p(t,x);\ 0 \le t \le 1,\ x \ge -E\}$ $(> -\infty)$, $B_1 = \sup\{p(t,x);\ 0 \le t \le 1,\ x \le E\}$ $(< \infty)$, $L = \sup\{|q(t,x,y)|;\ 0 \le t \le 1,\ x \in \mathbb{R},\ |y| \le E\}$ $(< \infty)$, $\Lambda = \frac{1}{a}\big(L + E + \max\{B_1, -A_1\} + 1\big)$, and let K be a positive constant such that

$$\begin{aligned} p(t,x) &> E + L + b\Lambda \qquad \text{for} \quad (t,x) \in [0,1] \times [K - E, \infty)\,,\\ p(t,x) &< -E - L - b\Lambda \qquad \text{for} \quad (t,x) \in [0,1] \times (-\infty, -K + E]\,. \end{aligned}$$

We see that (H_1) is satisfied, and (H_2) holds with $w(u;\mathcal{D}_0) = Au^2 + B$, where $A = A(\mathcal{D}_0)$, $B = B(\mathcal{D}_0)$ are suitable constants.

4. Uniqueness theorem

THEOREM 2. *Let the assumptions* (H_1), (H_2) *be satisfied, and, moreover, suppose that*

(H_3) $f(t,\cdot,y,\lambda)$ *is increasing on* $\mathbb{R}$ *for each fixed* $(t,y,\lambda) \in [0,1] \times \mathbb{R}^2$;
(H_4) $f(t,x,y,\cdot)$ *is increasing on* $\mathbb{R}$ *for each fixed* $(t,x,y) \in [0,1] \times \mathbb{R}^2$.

Then there exists a unique solution of the BVP (1), (2) *for each* $A, B, C \in \mathbb{R}$.

Proof. Let $A, B, C \in \mathbb{R}$. By Theorem 1, there exists a solution (x_1, λ_1) of the BVP (1), (2). Assume that (x_2, λ_2) is another solution of the BVP (1), (2), $\lambda_2 \ge \lambda_1$. Set $w = x_2 - x_1$. Then $w'(0) = w'(1) = 0$ and $w(\xi) = 0$ for a $\xi \in [0,1]$ since in the opposite case, $x_2(t) > x_1(t)$ or $x_2(t) < x_1(t)$ on $[0,1]$, and therefore $\alpha(x_2) > \alpha(x_1)$ or $\alpha(x_2) < \alpha(x_1)$, a contradiction. Hence $0 \le \max\{w(t);\ 0 \le t \le 1\} = w(\tau)$, $0 \ge \min\{w(t);\ 0 \le t \le 1\} = w(\nu)$ for some $\tau, \nu \in [0,1]$. Then $w'(\tau) = 0$, $w''(\tau) \le 0$; on the other hand (cf. (H_3), (H_4)),

$$w''(\tau) = f\big(\tau, x_2(\tau), x_2'(\tau), \lambda_2\big) - f\big(\tau, x_1(\tau), x_2'(\tau), \lambda_1\big) \ge 0\,,$$

and therefore $w''(\tau) = 0$, which occurs if and only if $w(\tau) = 0$ and $\lambda_2 = \lambda_1$. Next we see that $w'(\nu) = 0$, $w''(\nu) \ge 0$, and with respect to (H_3), $w''(\nu) = f\big(\nu, x_2(\nu), x_2'(\nu), \lambda_2\big) - f\big(\nu, x_1(\nu), x_2'(\nu), \lambda_2\big) \le 0$; hence $w''(\nu) = 0$ and then $w(\nu) = 0$. This proves $w = 0$; that is, $(x_1, \lambda_1) = (x_2, \lambda_2)$. □

EXAMPLE 2. Consider the differential equation (15), where p, q, k are as in Example 1, and, in addition, $p(t,\cdot)$, $q(t,\cdot,y)$ are increasing on $\mathbb{R}$ for each fixed $(t,y) \in [0,1] \times \mathbb{R}$, and $k(t,x,y) = k_1(t,y)$ does not depend on the variable x. Then, by Theorem 2, there exists a unique solution of the BVP (15), (2) for each $A, B, C \in \mathbb{R}$.

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