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Herbert Fleischner; Wilfried Imrich

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TRANSITIVE PLANAR GRAPHS

HERBERT FLEISCHNER—WILFRIED IMRICH

A graph whose group of automorphisms acts transitively on its set of vertices is called *vertex-transitive,* whereas it is called *edge-transitive* if its automorphism group is transitive on the set of edges. In this paper all edge-transitive finite planar graphs are determined. The triply connected vertex-transitive graphs turn out to be the nets of the uniform convex polyhedra and the triply connected edge-transitive graphs are the nets of the Platonic bodies, the cuboctahedron, the icosidodecahedron, the rhombic octahedron and the rhombic triacontahedron.

The arguments in this paper are similar to those used to determine the Platonic and Archimedian solids. Mani [5] has shown that one can find a convex polyhedron P to every finite planar three-connected graph G such that the net of P is isomorphic to G and such that the symmetry group of P is isomorphic to the automorphism group of G. Theorem 1 and 3 are immediate consequences of Mani's result. However, his proof is long and it may be worthwhile to give a more direct proof of these results.

We should also like to draw the reader's attention to the papers [8] and [9] by Zelinka, whose results are closely related to Theorem 3.

The *uniform convex polyhedra* are the convex polyhedra with vertex-transitive symmetry group and regular faces. Their symmetry groups, considered as permutation groups on the set of vertices, are identical with the automorphism groups of their nets. However, if one wants to represent such a net as a polyhedron whose symmetry group is the same as that of its net the result is unique only if one requires all edges of the polyhedron to be of the same length. Nevertheless one can combinatorially characterize the uniform convex polyhedra as those polyhedra whose net is vertex-transitive, whose symmetry group is the group of its net and whose faces, considered as polygons, have the same groups as the faces of the net, considered as simple circuits. The Platonic bodies, the cuboctahedron, the icosidodecahedron and the duals of the latter two are completely described by the requirement that they be edge-transitive and that they have the same groups as their nets.

It should be noted that the Platonic bodies can be combinatorially characterized by the requirement that their nets and the duals of the nets be triply connected regular graphs, and that their symmetry groups be the same as the automorphism groups of their nets [6].

The main graph theoretical results used here are Whitney's theorem on the unique embeddability of triply connected planar graphs into the plane and Euler's polyhedral formula. By Whitney's theorem we can assume that every planar triply-connected graph is already given with an embedding into the plane. Every automorphism of the graph maps faces into faces and either it preserves the cyclic order of the edges on the boundary of the faces and the cyclic order of the edges with a common vertex, or it reverses all these orders [3]. In other words, every automorphism of a triply connected planar graph can be considered to be induced by a deformation of the identity mapping of the plane or by a deformation of an inversion of the plane. Results of Watkins [7] allow an easy determination of the connectivity of transitive graphs by the degrees of the vertices. We shall not go into geometric details of uniform polyhedra. In fact we shall only determine the modified Schläfli symbols of their nets and leave it up to the reader to show that the nets are uniquely determined by these characteristics and also their polyhedra, under the possible additional requirement of edges of equal length. Constructions of nets are exemplified in the book of Grünbaum [4] and there exist a number of books with excellent diagrams of the uniform polyhedra and their nets, for example the books by Fejes Tóth [2] and Cundy and Rollet [1].

Let G be a finite triply connected planar graph, and let v, e and p denote the number of vertices, edges and faces of G. Then Euler's formula read as follows:

$$p - e + v = 2.$$

Denoting the number of faces whose boundary consists of k edges with p_k and the number vertices of degree k with v_k one can deduce the following equation from Euler's formula:

$$p_3 + v_3 = 8 + \sum_{k \geq 5} (k-4)(v_k + p_k) \tag{1}$$

The deduction of this formula is simple and it can be found for example in [4] on page 237. It implies that every triply connected planar graph has at least 8 3-valent elements. We shall also use the fact [6] that every finite planar graph has at least one vertex of degree ≤ 5. By dualization this implies that every finite triply connected planar graph has at least one face whose circumference is at most 5.

Theorem 1. *The finite, simple, planar edge-transitive triply connected graphs are the nets of the Platonic solids and the nets of the rhombic dodecahedron, the rhombic triacontahedron, the cuboctahedron and the icosidodecahedron.*

Proof. Let G be a finite, planar edge-transitive triply connected graph. We can assume that G is already given with an embedding into the plane. By Whitney's theorem every automorphism of G maps faces of G into faces of G. Now we partition the vertex set $V(G)$ of G and its set of faces $F(G)$ into classes on which the automorphism group $A(G)$ of G acts transitively. Since every edge is incident with two vertices and two faces we get at most two classes each, because of the edge-transitivity of G.

In case $A(G)$ acts transitively on the set of vertices and faces G is regular and also its dual. As G is triply connected any two faces of G have at most one edge in common. There are exactly five graphs with the property that G and its dual are regular and that two faces have at most one edge in common, namely the five Platonic solids. A short proof of this can be found in [6] on page 113. The five Platonic solids are indeed edge-transitive.

Let us assume now that $A(G)$ does not act transitively on the set of vertices of G. Then there are two classes of the set of vertices, say V_1 and V_2, on which $A(G)$ acts transitively. As every edge connects a vertex in V_1 with one in V_2, we can consider the edges of G as directed arcs with the initial vertex in V_1 and the terminal one in V_2. The automorphism of G preserves the direction of these arcs, otherwise V_1 and V_2 would coincide. Thus G is bipartite and has only even circuits, hence, the boundary of each face has even length. If $A(G)$ does not act transitively on the set of faces, it follows by dualization of the above construction that all vertices have even degrees.

It is therefore impossible that $A(G)$ is neither vertex-transitive nor transitive on the faces of G, otherwise G would have no 3-valent elements. As the dual of a threeconnected simple graph is three-connected we can restrict attention to the case, where $A(G)$ is not vertex-transitive, but acts transitively on the set of faces. In this case the number c of edges on the boundary of each face has to be even. As $3 \leqslant c \leqslant 5$ we have $c = 4$. Further $3 \leqslant d_1 \leqslant 5$ and we can assume $d_1 < d_2$, otherwise G and its dual would be regular and G would be a Platonic solid.

From $e = d_1\beta_1 = d_2\beta_2$ we get $\beta_1 = e/d_1$ and $\beta_2 = e/d_2$. As $v = \beta_1 + \beta_2$ this implies

$$v = e\,\frac{d_1 + d_2}{d_1 d_2}.$$

Now $2e = cp$ and $c = 4$ yield $p = e/2$. Substitung for v and p in Euler's formula and solving for e we get

$$e = \frac{4d_1 d_2}{2(d_1 + d_2) - d_1 d_2}.$$

As the denominator has to be larger than zero

$$\frac{2d_1}{d_1 - 2} > d_2.$$

For $d_1 = 5$ we would have therefore $d_2 < 10/3$ and for $d_1 = 4$ we would get $4 > d_2$, which is impossible. The case $d_1=3$ gives $6>d_2$, allowing the possibilities $d_2=4$ and $d_2=5$. By an easy calculation one arrives at the two solutions

$$v=14, \quad e=24, \quad p=12, \quad \beta_1=8, \quad \beta_2=6$$

in case $d_1=3$, $d_2=4$ and at

$$v=32, \quad e=60, \quad p=30, \quad \beta_1=20, \quad \beta_2=12$$

in case $d_1=3$, $d_2=5$. The first solution is the rhombic triacontahedron. Together with their duals, namely the cuboctahedron and the icosidodecahedron, these are the four edge-transitive polyhedral graphs different from the Platonic solids.

Theorem 2. *The finite, simple, connected edge-transitive planar graphs which are not triply connected are the single vertex, the single edge, stars, simple circuits and the graphs obtained from the single edge, simple circuits and the nets of the Platonic solids, the cuboctahedron, and the icosidodecahedron by replacing every edge by $K_{2,n}$, where n is a fixed positive integer.*

Remark. By *replacing an edge* $[a, b]$ *by* $K_{2,n}$ we mean the deletion of $[a, b]$ followed by the identification of the vertices of degree n in $K_{2,n}$ with a and b, respectively. This procedure is unique unless $n=2$, in which case we identify any pair of nonadjacent vertices with the pair a, b.

Proof. As has been shown by Watkins [7] the connectivity of connected edge-transitive graphs G is the minimal degree of the graph. Let $[a, b]$ be an edge of G, and denote the degree of a by d_1 and the degree of b by d_2. We can choose the notation so that $d_1 \leqslant d_2$. If d_1 is one G is a single edge or a star. If d_1 and d_2 are two G is a circuit. So we only have to consider the case where $d_1=2$ and $d_2>2$. Clearly $A(G)$ acts transitively on the vertices of degree d_2 and on the vertices of degree 2. Every vertex of degree 2 is connected with two vertices of degree d_2. If we replace every such pair of adjoining edges by a single one, we get a regular (multi-)graph of degree $d_2 \geqslant 3$ whose group acts transitively on the set of edges and on the set of vertices. We identify multiple edges to obtain a simple edge- and vertex-transitive graph G. By the above and Theorem 1 G is a single edge, a simple circuit, the net of a Platonic body, the cuboctahedron or the icosidodecahedron. This proves the theorem.

To characterize the triply connected vertex-transitive planar graphs we use the fact that automorphisms either preserve or reverse the cyclic order of the faces incident with every vertex. Given such a cyclic order for an arbitrary vertex v we therefore list the circumferences of the faces incident with v in that order. Any two such listings, e.g.

$$\{3, 6, 8\}, \quad \{6, 8, 3\} \quad \text{or} \quad \{8, 6, 3\},$$

are considered the same if one arises from the other by a cyclic transformation or by inversion. These symbols are called *modified Schläfli symbols* and are used to characterize uniform convex polyhedra. We do not show that they uniquely determine their planar, triply connected graphs, although this is important because we want to characterize the uniform convex polyhedra combinatorially.

Theorem 3. *The connected, simple, planar vertex-transitive graphs are the single vertex, the single edge, simple circuits and the nets of the uniform convex polyhedra, namely the nets of regular prisms and antiprisms, the Platonic and the Archimedian bodies.*

Proof. Every vertex-transitive graph is regular. Trivially the single vertex, the single edge and simple circuits are the connected planar regular graphs whose degree is at most two. Regular connected vertex-transitive graphs of at least third degree are three-connected [7], and we can apply Whitney's results to them. As every planar graph has vertices of degree smaller than or equal to five, we only have to consider the degrees 3, 4 and 5.

We treat the case $d=3$ first. Let G be a vertex-transitive planar graph of degree three. We choose a vertex g of G and denote the faces of G incident with g by α, β, γ in positive cyclic order. Further we denote by $[\alpha]$, $[\beta]$ and $[\gamma]$ the set of faces of G which are (transitively) equivalent with α, β and γ respectively. Of course they need not be different. If β is for example equivalent to γ, symbolically $\beta\sim\gamma$, the classes $[\beta]$ and $[\gamma]$ coincide. Without loss of generality we can choose the notation such that the circumference $c(\alpha)$ of α is minimal. As every triply connected planar graph has faces whose circumference is at most five there are only three possibilities for $c(\alpha)$, namely $c(\alpha)=3$, 4 or 5.

Suppose one of the faces α, β, γ, say α, has odd circumference. We shall show first that the other two faces are equivalent. In case $c(\alpha)=c(\beta)=c(\gamma)$, the only solutions are the Platonic bodies $\{3, 3, 3\}$ and $\{5, 5, 5\}$ for which the assertion is true. Thus, we can assume $c(\alpha)\neq c(\beta)$. Clearly $c(\alpha)$ being odd implies $c(\beta)=c(\gamma)$ by vertex transitivity. Hence, every automorphism mapping a boundary point of α into a boundary point of α stabilizes α. This implies $\beta\sim\gamma$, for otherwise the faces having an edge in common with α would have to be alternatingly in $[\beta]$ and $[\gamma]$, in contradiction to $c(\alpha)$ being odd.

If $c(\alpha)=3$ we therefore have $\beta\sim\gamma$. For $\alpha\sim\beta$ we get the solution

$$\{3, 3, 3\},$$

the tetrahedron. In case α is not equivalent to β it is easily seen that the faces having a common edge with β are alternatingly in $[\alpha]$ and $[\beta]$. Consequently $c(\beta)$ has to be even. This gives the solutions

$$\{3, 4, 4\},\quad \{3, 6, 6\},\quad \{3, 8, 8\}\quad \text{and}\quad \{3, 10, 10\}.$$

These are the triangular prism, the truncated tetrahedron, the truncated cube and the truncated dodecahedron. That $\{3, m, m\}$ is not realizable for $m>10$ is shown as follows: By (1) we have

$$v+p_3=8+(m-4)p_m. \tag{2}$$

Further $v=3p_3=m\cdot p_m/2$. Substituting in (2) for v and p_3 and solving for p_m we get

$$p_m=\frac{24}{12-m},$$

and henceforth $12>m$.

In case $c(\alpha)=4$ we immediately have the solutions

$$\{4, 4, 4\} \quad \text{and} \quad \{4, 4, m\} \quad \text{with} \quad m\geqslant 5;$$

i.e., the cube and m-gonal prisms. If one of the numbers $c(\beta)$ and $c(\gamma)$ is odd the other has to be equal to $c(\alpha)$. Thus we have to investigate the case $\{4, k, m\}$ with even k, m and $4<k\leqslant m$. To be able to treat the cases $k<m$ and $k=m$ jointly we set $g_k=p_k$ and $q_m=p_m$ for $k<m$ and $p_k=2q_k$ for $k=m$. Then we have

$$v=4p_4=kq_k=mq_m$$

and equation (1) becomes

$$v=8+(k-4)q_k+(m-4)q_m.$$

Solving for p_4 we get

$$p_4=\frac{2km}{4(k+m)-km}.$$

The denominator has to be positive, and therefore

$$\frac{1}{k}+\frac{1}{m}>\frac{1}{4}. \tag{3}$$

From $k\leqslant m$ follows that $1/k>1/8$ and hence $8>k$. As k is larger than four, k can only be 6. Substituting again into (3) one can see that $12>m$. This allows the solutions

$$\{4, 6, 6\}, \quad \{4, 6, 8\} \quad \text{and} \quad \{4, 6, 10\},$$

the truncated octahedron, the truncated cuboctahedron and the truncated icosidodecahedron.

If $c(\alpha)=5$ the face β has to be equivalent to γ, for $\alpha\sim\beta$ we get the pentagondodecahedron

$$\{5, 5, 5\}.$$

In case $\alpha \nsim \beta$ we can show as we did for $c(\alpha)=3$, that $c(\beta)$ is even. Thus we have to investigate the case $\{5, m, m\}$ with even m. Using equation (1) and the relation

$$v=5p_5=mp_m/2$$

we can show by the same methods as before that $20>3m$. This gives the truncated icosahedron

$$\{5, 6, 6\}.$$

The next step is the investigation of regular graphs of degree four. Again we choose a vertex g of such a graph G and denote the faces of G incident with g by $\alpha, \beta, \gamma, \delta$ in positive cyclic order. Equivalence of faces is defined as before, and again we can assume $c(\alpha)$ to be minimal. As there are no vertices of degree three we have $c(\alpha)=3$ by (1).

We assume first that $\alpha \nsim \beta, \gamma, \delta$. If $\beta \nsim \delta$ the faces having a common edge with α would have to be alternatingly in $[\beta]$ and $[\delta]$, which is impossible since $c(\alpha)=3$. Therefore $\beta \sim \delta$. As every face having a common boundary with γ is in $[\beta]$ the face γ cannot be equivalent to β. We observe that the faces around β are alternatingly in $[\alpha]$ and $[\gamma]$, and therefore $c(\beta)$ has to be even. For $c(\beta)=4$ we definitely get the solutions

$$\{3, 4, 4, 4\}, \quad \{3, 4, 5, 4\},$$

corresponding to the rhombicuboctahedron and the rhombicosidodecahedron, respectively. The numerical solution $\{3, 4, 3, 4\}$ corresponds to the cuboctahedron, for which $\alpha \sim \gamma$. Since we assumed $\alpha \nsim \beta, \gamma, \delta$ this case cannot be included here. It should also be noted that $c(\alpha)=c(\gamma)$ does not necessarily imply $\alpha \sim \gamma$.

There is no solution $\{3, 4, m, 4\}$ with $m>5$. Equation (1) becomes $p_3=8+(m-4)p_m$ in this case, and $v=3p_3=mp_m$. This implies $p_m(6-m)=12$ and therefore $m<6$.

Now we consider $\{3, k, m, k\}$, where k is even and larger than four. For $m=3,4$ or k the following relations hold:

$$m=3: 2v=3p_3=kp_k$$
$$m=4: v=3p_3=kp_k/2$$
$$m=k: v=3p_3=kp_k/3$$

In any case $3p_3 \geq kp_k/3$ and equation (1) assumes the form

$$p_3=8+(k-4)p_k,$$

from which follows $p_k(9-2k) \leq 18$, which is not possible for $k \geq 6$. Now let $m \neq 3$, 4, k. Then

$$v=3p_3=mp_m=kp_k/2$$

and by (1)

$$p_3 = 8 + (k-4)p_k + (m-4)p_m.$$

Solving for p_m we get

$$p_m = \frac{6k}{6m + 3k - 2km},$$

and hence $6/k + 3/m > 2$. As $1 \geq 6/k$ this implies $3 > m$, which is not possible.

The next case is the one, where α is equivalent to at least one of the faces β, γ, δ. By construction g is one of the vertices of α. Let e and f be the other ones, and let the notation be chosen so that α and β have the edge $[f, g]$ in common and α, δ the edge $[e, g]$. If $\alpha \sim \beta$ we consider the faces incident with e. They have to be α, β, γ, δ in positive or negative cyclic order. Hence $\delta \sim \alpha$ or f is incident with three faces equivalent to α, and therefore also g. In any case α has to be equivalent to at least one of the faces γ, δ. Without loss of generality let $\alpha \sim \gamma$. This gives the solutions

$$\{3, 3, 3, 3\} \quad \text{and} \quad \{3, 3, 3, m\} \quad \text{with} \quad m \geq 4,$$

namely the octahedron and the m-gonal antiprisms.

In the above we began with the assumption $\alpha \sim \beta$. We would have arrived at the same results from $\alpha \sim \delta$. Now let $\alpha \sim \gamma$ and $\alpha \not\sim \beta$, δ. Then $\beta \sim \delta$ since $c(\alpha)$ is odd and have to consider $\{3, k, 3, k\}$. For even k this has already been done. It should be noted that we did not need any assumptions about the equivalences of α, β, γ and δ to do this, only equation (1) and the relation $3p_3 = kp_k$. For odd k the case $k = 3$ is trivial. Thus we can assume $k \geq 5$. This gives $3p_3 = kp_k$ and $p_3 = 8 + (k-4)p_k$. Consequently, $p_k(12 - 2k) = 24$ and $k < 6$. Therefore the result is the icosidodecahedron

$$\{3, 5, 3, 5\}.$$

Finally we have to consider regular graphs of degree 5. As before we choose a regular graph of degree 5 and a vertex g. The faces incident with g we denote as usual by α, β, γ, δ, η in positive cyclic order, and we suppose $c(\alpha)$ to be minimal. Clearly $c(\alpha) = 3$. If one of the faces incident with g has a circumference $m \geq 5$ equation (1) gives

$$p_3 \geq 8 + v + (m-4)p_m. \tag{4}$$

Denoting the number of triangles incident with g by k_3 and the number of m-gons by k_m we get

$$3p_3/k_3 = mp_m/k_m = v$$

and $k_3 + k_m \leq 5$. Substituting for v and p_3 in (4) we obtain

$$p_m(12k_m - m(3 + 3k_m - k_3)) \geq 24k_m.$$

The second factor on the left side has to be positive, and this implies

$$12k_m > m(3 + 3k_m - k_3). \tag{5}$$

Taking into account that $k_3 \leqslant 5 - k_m$ it follows that

$$6k_m > m(2k_m - 1), \tag{6}$$

and therefore $6 > m$. Hence m has to be 5. Using equation (6) again we see that $5 > 4k_m$, and thus $k_m = 1$. Substituting into (5) this gives $12 > 5(6 - k_3)$, and consequently $k_3 > 3$. Since k_3 is smaller than 5 it has to be 4. The solution is the snub dodecahedron

$$\{3, 3, 3, 3, 5\}.$$

Henceforth we can assume that our graphs contain only triangles and quadrilaterals. Obviously $3p_3/k_3 = v$ and $p_3 = 8 + v$. Substituting for v we obtain

$$p_3(k_3 - 3) = 8k_3,$$

which implies $k_3 > 3$. This gives the remaining two solutions

$$\{3, 3, 3, 3, 4\} \quad \text{and} \quad \{3, 3, 3, 3, 3\},$$

namely the snub cube and the icosahedron.

REFERENCES

[1] CUNDY, H. M.—ROLLET, A. P.: Mathematical Models. Oxford University Press 1961.

[2] FEJES TÓTH, L.: Lagerungen auf der Kugel und im Raum. Berlin 1953.

[3] FLEISCHNER, H.: Über endliche, ebene Eulersche und paare kubische Graphen. Monatsh. Math., 74, 1970, 410—420.

[4] GRÜNBAUM, B.: Convex Polytops. John Wiley 1967.

[5] MANI, P.: Automorphismen von polyedrischen Graphen. Math. Ann., 192, 1971, 279—303.

[6] WAGNER, K.: Graphentheorie. Bibliographisches Institut, Mannheim 1970.

[7] WATKINS, M.: Connectivity of Transitive Graphs. J. Comb. Th., 8, 1970, 23—29.

[8] ZELINKA, B.: Finite vertex-transitive planar graphs of the regularity degree four or five. Mat. Čas., 25, 1975, 271—280.

[9] ZELINKA, B.: Finite vertex-transitive planar graphs of the regularity degree three. Časop. Pěstov. Mat., 102, 1, 1977, 1—9.

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Institut für Informationsverarbeitung
Österreichische Akademie der Wissenschaften
Fleischmarkt 20/I/1/4
A-1010 Wien
Österreich

Institut für Angewandte Mathematik
Montanistische Hochschule Leoben
A-8700 Leoben
Österreich

ТРАНЗИТИВНЫЕ ПЛАНАРНЫЕ ГРАФЫ

Герберт Флайшнер–Вильфрид Имрих

Резюме

Применяя результат Витны об однозначной вложимости планарных 3-связных графов в плоскость и формулу Эйлера, авторы находят все конечные 3-связные планарные графы с реберно-транзитивной или вершинно-транзитивной группой автоморфизмов. Это достигается число комбинаторным способом. Кроме того, находятся все конечные планарные графы с реберно-транзитивной группой автоморфизмов, которые не являются 3-связными.