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*Mathematica Slovaca*, Vol. 54 (2004), No. 1, 23--42

Persistent URL: http://dml.cz/dmlcz/131798

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LORENZEN’S THEOREM
FOR PSEUDO-EFFECT ALGEBRAS

ANATOLIJ DVUREČENSKIJ

(Communicated by Gejza Wimmer)

ABSTRACT. We present a variation of the Lorenzen theorem for pseudo-effect algebras satisfying a kind of the Riesz decomposition property. We show that the representability of pseudo-effect algebras as a subdirect product of antilattice pseudo-effect algebras depends on the notion of the polar of a pseudo-effect algebra.

1. Introduction

The famous Lorenzen theorem ([Lor], [Gla]) says that an \( \ell \)-group \( G \) is representable, i.e., it is a subdirect product of linearly ordered groups if and only if the polars of \( G^+ \) are \( \ell \)-ideals.

Recently, new partial algebraic structures, called pseudo-effect algebras and pseudo MV-algebras (as total algebraic structures), were introduced in [DvVe1], [DvVe2] and [GeLo]. They are a non-commutative generalization of effect algebras and MV-algebras, respectively, which are studied in many branches of mathematics and its applications. For example, such structures serve as models of quantum structures ([DvPu]) as well as in mathematical logic. Under some natural conditions, supposing a kind of Riesz decomposition property, they are always intervals in unital po-groups, see [DvVe1], [DvVe2]. Moreover, every pseudo MV-algebra is an interval in a unital \( \ell \)-group, see [Dvu1].

2000 Mathematics Subject Classification: Primary 06F20, 03G12, 03B50.

Keywords: pseudo-effect algebra, pseudo MV-algebra, ideal, polar, \( C \)-polar, carrier, representability, unital po-group, unital \( \ell \)-group.

The paper has been supported by the grant VEGA 2/3163/23 SAV, Bratislava, Slovakia.
A generalization of the Lorenzen theorem for directed interpolation groups was presented by Glass [Gla; Theorem 42]; however in its proof, there are some unclear points. The Lorenzen theorem for pseudo MV-algebras was proved in [GeIo].

Inspired by these results, we present a variation of the Lorenzen theorem for pseudo-effect algebras satisfying a kind of the Riesz decomposition property. For this aim we introduce the notion of a polar and of a $C$-polar. The paper is organized as follows. In Section 2, we introduce elements of pseudo-effect algebras and pseudo MV-algebras. In Section 3, the polars for pseudo-effect algebras are presented and some results are proved. $C$-polars, where $C$ is an ideal, are studied in Section 4. $C$-carriers are investigated in Section 5. Section 6 defines representable pseudo-effect algebras. Finally, the main result is given in Section 7, showing when a pseudo-effect algebra is a subdirect product of antilattice pseudo-effect algebras.

2. Pseudo-effect algebras

A partial algebra $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a pseudo-effect algebra ([DvVe1], [DvVe2]) if, for all $a, b, c \in E$, the following hold

(i) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$;

(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$;

(iii) if $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$;

(iv) if $1 + a$ or $a + 1$ exists, then $a = 0$.

If we define $a \leq b$ if and only if there exists an element $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering on $E$ such that $0 \leq a \leq 1$ for any $a \in E$. It is possible to show that $a \leq b$ if and only if $b = a + c = d + a$ for some $c, d \in E$. We write $c = a \backslash b$ and $d = b \setminus a$. Then

$$(b \setminus a) + a = a + (a \backslash b) = b,$$

and we write $a^- = 1 \setminus a$ and $a^\sim = a \backslash 1$ for any $a \in E$.

For basic properties of pseudo-effect algebras see [DvVe1], [DvVe2]. We recall that if $+$ is commutative, $E$ is said to be an effect algebra. For properties of effect algebras see [DvPu].
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For example, if \((G, u)\) is a unital (not necessarily Abelian) po-group with strong unit \(u\) (in fact it is sufficient to take a positive element \(u\) in \(G\)),

\[\Gamma(G, u) := \{g \in G : 0 \leq g \leq u\},\]

then \((\Gamma(G, u); +, 0, u)\) is a pseudo-effect algebra if we restrict the group addition + to \(\Gamma(G, u)\).

According to [DvVe1], we introduce for pseudo-effect algebras the following forms of the Riesz decomposition properties:

(a) For \(a, b \in E\), we write \(a \com b\) to mean that for all \(a_1 \leq a\) and \(b_1 \leq b\), \(a_1\) and \(b_1\) commute.

(b) We say that \(E\) fulfills the Riesz interpolation property, (RIP) for short, if for any \(a_1, a_2, b_1, b_2 \in E\) such that \(a_1, a_2 \leq b_1, b_2\), there is a \(c \in E\) such that \(a_1, a_2 \leq c \leq b_1, b_2\).

(c) We say that \(E\) fulfills the weak Riesz decomposition property, \((\text{RDP}_0)\) for short, if for any \(a, b_1, b_2 \in E\) such that \(a \leq b_1 + b_2\), there are \(d_1, d_2 \in E\) such that \(d_1 \leq b_1, d_2 \leq b_2\) and \(a = d_1 + d_2\).

(d) We say that \(E\) fulfills the Riesz decomposition property, \((\text{RDP})\) for short, if for any \(a_1, a_2, b_1, b_2 \in E\) such that \(a_1 + a_2 = b_1 + b_2\), there are \(d_1, d_2, d_3, d_4 \in E\) such that \(d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1, d_2 + d_4 = b_2\).

(e) We say that \(E\) fulfills the commutational Riesz decomposition property, \((\text{RDP}_1)\) for short, if for any \(a_1, a_2, b_1, b_2 \in E\) such that \(a_1 + a_2 = b_1 + b_2\), there are \(d_1, d_2, d_3, d_4 \in E\) such that

\[
\begin{align*}
(i) & \quad d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1, d_2 + d_4 = b_2, \\
(ii) & \quad d_2 \com d_3.
\end{align*}
\]

(f) We say that \(E\) fulfills the strong Riesz decomposition property, \((\text{RDP}_2)\) for short, if for any \(a_1, a_2, b_1, b_2 \in E\) such that \(a_1 + a_2 = b_1 + b_2\), there are \(d_1, d_2, d_3, d_4 \in E\) such that

\[
\begin{align*}
(i) & \quad d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1, d_2 + d_4 = b_2, \\
(ii) & \quad d_2 \land d_3 = 0.
\end{align*}
\]

We introduce analogical notions for po-groups. Let \(G\) be a po-group and for \(a, b \in G^+\), we write \(a \com b\) if and only if, for all \(a_1, b_1 \in G^+\) such that \(a_1 \leq a\) and \(b_1 \leq b\), we have \(a_1 + b_1 = b_1 + a_1\).

Let \((G; +, 0, \leq)\) be a directed po-group. According to [DvVe1], [DvVe2], we say that \(G\) fulfills (RIP), \((\text{RDP}_0)\), \((\text{RDP})\), \((\text{RDP}_1)\), and \((\text{RDP}_2)\), respectively, if

\[1\]We say that a positive element \(u\) of a po-group \(G\) is a strong unit if, for any \(g \in G\), there is an integer \(n \geq 1\) such that \(g \leq nu\).
anallogical properties as those for pseudo-effect algebras hold also for the positive cone $G^+$ of $G$.

A mapping $h: E \to F$, where $E$ and $F$ are pseudo-effect algebras, is said to be a homomorphism if

(i) $h(0) = 0$ and $h(1) = 1$, 
(ii) $h(a + b) = h(a) + h(b)$ whenever $a + b$ is defined in $E$.

If $h$ is injective and surjective such that also $h^{-1}$ is a homomorphism, then $h$ is said to be an isomorphism, and $E$ and $F$ are isomorphic. It is clear that a one-to-one homomorphism $f$ from $E$ onto $F$ is an isomorphism if and only if $f(a) \leq f(b)$ implies $a \leq b$.

According to [Gelo], a pseudo MV-algebra is an algebra $(M; \oplus, -, \sim, 0, 1)$ of type $(2,1,1,0,0)$ such that the following axioms hold for all $x,y,z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^- = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^- \oplus y^-)^\sim$;
(A6) $x \oplus x^- \odot y = y \oplus y^- \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^-) \odot y$;
(A8) $(x^-)^\sim = x$.

If we define $x \leq y$ if and only if $x^- \oplus y = 1$, then $\leq$ is a partial order such that $M$ is a distributive lattice with $x \vee y = x \oplus (x^- \odot y)$ and $x \wedge y = x \odot (x^- \oplus y)$. For basic properties of pseudo MV-algebras see [Gelo] or [DvPu].

If we define a partial binary operation $+$ on $M$ via: $x + y$ is defined if and only if $x \leq y^-$, and in this case $x + y := x \oplus y$, then $(M; +, 0, 1)$ is a pseudo-effect algebra. Moreover, a pseudo-effect algebra $E$ can be converted into a pseudo MV-algebra such that the $+$ derived from $\oplus$ and the original $+$ coincide if and only if $E$ satisfies (RDP$_2$) ([DvVe2]).

For example, if $u$ is a strong unit of a (not necessarily Abelian) $\ell$-group $G$,

$$\Gamma(G, u) := [0, u]$$

and

$$x \oplus y := (x + y) \wedge u,$$

$$x^- := u - x,$$

$$x^- := u - x,$$

$$x \odot y := (x - u + y) \vee 0,$$

$2 \odot$ has a higher priority than $\oplus$. 

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then \((\Gamma(G, u); \oplus, - , \sim, 0, u)\) is a pseudo MV-algebra ([GeIo]).

The basic representation theorem for pseudo-effect algebras is the following result [DvVe1], [DvVe2], and for pseudo MV-algebras see also [Dvu1].

**Theorem 2.1.** For a pseudo-effect algebra \(E\) fulfilling (RDP\(_1\)), there is a unique (up to isomorphism of unital po-groups) unital po-group \((G, u)\) fulfilling (RDP\(_1\)) such that \(E \cong \Gamma(G, u)\).

If \(M\) is a pseudo MV-algebra, there is a unique (up to isomorphism of unital \(\ell\)-groups) unital \(\ell\)-group \((G, u)\) such that \(M \cong \Gamma(G, u)\).

A non-empty subset \(I\) of a pseudo-effect algebra \(E\) is said to be an **ideal** of \(E\) if

(i) \(x + y \in I\) whenever \(x, y \in I\) and if \(x + y\) is defined in \(E\),

(ii) if \(x \leq y\) for \(x \in E\) and \(y \in I\), then \(x \in I\).

Then \(E\) as well as \(\{0\}\) are ideals of \(E\).

Let \(\mathcal{I}(E)\) denote the set of all ideals of a pseudo-effect algebra \(E\). According to [Dvu3] if \(E\) satisfies (RDP), then \(\mathcal{I}(E)\) is a lattice with respect to the set-theoretical inclusion with meets and joins denoted simply by \(\wedge\) and \(\vee\).

An ideal \(I\) of \(E\) is

(i) **normal** if \(a + I = I + a\) for all \(a \in E\),

(ii) **maximal** if \(I\) is a proper subset of \(E\) and it is not included in any proper ideal of \(E\) as a proper subset,

(iii) **prime** if \(I_0(a) \cap I_0(b) \subseteq I\) implies \(a \in I\) or \(b \in I\) for all \(a, b \in E\).

We denote by \(\mathcal{N}(E)\), \(\mathcal{M}(E)\), and \(\mathcal{P}(E)\) the set of all normal ideals, maximal ideals, and prime ideals, respectively, of \(E\). Using the Zorn lemma, we see that \(\mathcal{M}(E)\) is non-void. Under some conditions on \(E\), [Dvu3], we can prove that \(\mathcal{M}(E) \subseteq \mathcal{P}(E)\).

We recall that if \(E\) satisfies (RDP), then an ideal \(I\) is prime if and only if \(E/I\) is an antilattice, see [Dvu3; Proposition 4.6].

### 3. Polars and pseudo-effect algebras

For \(\emptyset \neq A \subseteq E\), we set \(A^\perp := \{x \in E : x \wedge a = 0\text{ for all }a \in A\}\), and we refer to \(A^\perp\) as the **polar** of \(A\). We define \(a^\perp := \{a\}^\perp\) for \(a \in E\). Then

\[
a^\perp \cap a^\perp = \{0\}, \quad a \in E,
\]

\(^3\)If \(A\) is a non-empty subset of \(E\), then \(a + A := \{a + x : x \in A\text{ and }a + x\text{ is defined in }E\}\).

\(^4\)By \(I_0(a)\) and \(N_0(a)\) we define any ideal and any normal ideal generated by \(a \in E\).
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and, for \( \emptyset \neq A \subseteq E \),
\[
A^\perp \cap A^\perp = \{0\}, \quad A \subseteq A^\perp, \quad A^\perp = A^\perp^\perp, \quad (3.2)
\]
\( A^\perp = \bigcap \{a^\perp : a \in A\} \), \( B^\perp \subseteq A^\perp \) if \( A \subseteq B \subseteq E \), and \( b^\perp \subseteq a^\perp \) if \( a \leq b \), \( a, b \in E \).

We recall that if \( E \) satisfies \( (\text{RDP}_0) \) and \( I_0(a) \) is the ideal of \( E \) generated by an element \( a \in E \), and \( A \) is a non-void subset of \( E \), then
\[
a^\perp = I_0(a)^\perp \quad \text{and} \quad A^\perp = I_0(A)^\perp,
\]
where \( I_0(A) \) is the ideal of \( E \) generated by \( A \).

**Proposition 3.1.** Let \( E \) be a pseudo-effect algebra with \( (\text{RDP}_0) \). If \( \emptyset \neq A \in E \), then \( A^\perp \) is an ideal of \( E \). In addition, if \( a + b \in E \), then
\[
(a + b)^\perp = a^\perp \cap b^\perp.
\]

**Proof.** \( 0 \in A^\perp \). If \( x, y \in E \) and \( x \leq y \in A^\perp \), then \( x \in A^\perp \). Assume now \( x, y \in A^\perp \) and let \( x + y \in E \). Fix \( a \in A \). If \( z \leq x + y \) and \( z \leq a \), then \( z = x_1 + y_1 \), where \( x_1 \leq x, y_1 \leq y \), and \( x_1, y_1 \in a^\perp \). While \( x_1, y_1 \leq a \), we have \( x_1 = x_1 \wedge a = 0 = y_1 \wedge a = y_1 \), which proves \( z = 0 \).

In a similar way we prove the equation. \( \square \)

**Proposition 3.2.** If \( A \) is an ideal of a pseudo-effect algebra \( E \) with \( (\text{RDP}_0) \), then \( A \cap A^\perp = \{0\} \) and \( A^\perp \) is the greatest ideal of \( E \) whose intersection with \( A \) is the null ideal.

**Proof.** The first statement follows from (3.2). Assume that \( I \) is an ideal of \( E \) such that \( I \cap A = \{0\} \). Let \( x \in I \) and \( a \in A \), then \( x \wedge a = 0 \), which yields \( x \in A^\perp \). \( \square \)

**Proposition 3.3.** Let \( E \) be a pseudo-effect algebra with \( (\text{RDP}_0) \). If \( A \) and \( B \) are ideals of \( E \), then
\[
(A \cap B)^\perp = A^\perp \cap B^\perp^\perp. \quad (3.3)
\]

In particular, if \( a, b \in E \), then
\[
(I_0(a) \cap I_0(b))^\perp = a^\perp \cap b^\perp^\perp.
\]

**Proof.** It is necessary to verify that \( A^\perp \cap B^\perp^\perp \subseteq (A \cap B)^\perp \). Choose \( x \in A^\perp \cap B^\perp^\perp \), \( y \in (A \cap B)^\perp \), and \( a \in A \), \( b \in B \). Assume \( w \leq x, y, a, b \). Then \( w \in A \cap B \), and since \( w \leq w, y, a, b \), we have \( w = 0 \). So if \( g \leq x, y, a, b, \) then \( g \in b^\perp \), therefore, \( g \in B^\perp \). Since \( x \in B^\perp^\perp \) and \( 0 \leq g \leq g, x \), we have \( g = 0 \). Hence, if \( v \leq x, y \) and \( w \leq v, a \), then \( w = 0 \), i.e., \( v \in a^\perp \) and \( v \in A^\perp \). But \( v \leq x \in A^\perp \), which by (3.1) gives \( v = 0 \), consequently, \( x \in (A \cap B)^\perp \). \( \square \)
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PROPOSITION 3.4. Let \( A \) and \( B \) be two ideals of a pseudo-effect algebra \( E \) with \((\text{RDP}_0)\). Then
\[
(A \cap B)\perp = (A\perp \cup B\perp)^\perp.
\]

Proof. Since \( A \cap B \subseteq A, B \), we have \( A\perp \cup B\perp \subseteq (A \cap B)^\perp \). Hence, \((A \cap B)^\perp \subseteq (A\perp \cup B\perp)^\perp \). By Proposition 3.3, \( A^{\perp \perp} \cap B^{\perp \perp} \subseteq (A\perp \cup B\perp)^\perp \). Hence, if \( x \in (A\perp \cup B\perp)^\perp \) and \( y \in A\perp \cup B\perp \), then \( x \land y = 0 \). If now \( y \in A\perp \), then \( x \in A^{\perp \perp} \); if \( y \in B\perp \), then \( x \in B^{\perp \perp} \), i.e., \( x \in A^{\perp \perp} \cap B^{\perp \perp} \).

4. C-polars in pseudo-effect algebras

According to [Gla], we generalize the notion of a polar as follows. Let \( C \) be an ideal of a pseudo-effect algebra \( E \). The C-polar of a non-void subset \( A \) of \( E \) is the set \( A^{\perp C} := \{g \in E : (\forall a \in A)(c \leq g, a \Rightarrow c \in C)\} \). We set \( g^{-c} := \{g\}^{\perp C} \) if \( g \in E \). We define \( A^{\perp C} \perp C = (A^{\perp C})^{\perp C} \). For example, if \( C = \{0\} \), then \( A^{\perp (0)} = A^{\perp} \).

Many analogical properties as those for polars hold also for C-polars. We recall that C-polars for interpolation groups were studied in [Gla].

PROPOSITION 4.1. Let \( E \) be a pseudo-effect algebra, \( \emptyset \neq A \subseteq E \), and \( C \in \mathcal{I}(E) \).

(o) \( A^{\perp C} = \bigcap\{a^{\perp C} : a \in A\} \).

(i) \( C \subseteq A^{\perp C} \).

(ii) \( B^{\perp C} \subseteq A^{\perp C} \) if \( A \subseteq B \subseteq E \).

(iii) \( A^{\perp C} \perp C = A^{\perp C} \).

(iv) \( A \subseteq A^{\perp C} \perp C \).

(v) \( A^{\perp C} \cap A^{\perp C} \perp C = C \).

Let \( E \) satisfy \((\text{RDP}_0)\).

(vi) \( A^{\perp C} \in \mathcal{I}(E) \).

(vii) \( (I_0(A))^{\perp C} = A^{\perp C} \).

(viii) If \( x + y \in E \), then \( (x + y)^{\perp C} = x^{\perp C} \cap y^{\perp C} \).

(ix) If \( C \subseteq A \in \mathcal{I}(E) \), then \( A \cap A^{\perp C} = C \), and \( A^{\perp C} \) is the largest ideal of \( E \) whose intersection with \( A \) is \( C \).

Proof. It follows the same ideas as those for polars.

PROPOSITION 4.2. If \( A \) is a non-void subset of a pseudo-effect algebra \( E \), the following statements are equivalent.

(i) \( A \subseteq C \).

(ii) \( A^{\perp C} = E \).

(iii) \( A \subseteq A^{\perp C} \).
Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are evident. Assume now (iii). Then $A \subseteq A^\perp_c$ and, for any $a \in A$, we have $a \in A^\perp_c \subseteq a^\perp_c$. Therefore, if $c \leq a$, then $c \in C$, i.e., $a \in C$. \qed

As a consequence, we have $g^\perp_c = E$ if and only if $g \in C$. The following statement is direct.

**Proposition 4.3.** Let $E$ be a pseudo-effect algebra and $A$ a non-void subset of $E$.

(i) If $C_1, C_2 \in \mathcal{I}(E)$, $C_1 \subseteq C_2$, then $A^\perp_{C_1} \subseteq A^\perp_{C_2}$.

(ii) If $C_1, C_2 \in \mathcal{I}(E)$, then $A^\perp_{C_1} \cap A^\perp_{C_2} = A^\perp_{(C_1 \cap C_2)}$.

(iii) If $A, C \in \mathcal{I}(E)$, then $A^\perp_c = A^\perp_{(A \cap C)}$.

**Proposition 4.4.** If $A, B, C \in \mathcal{I}(E)$, where $E$ is a pseudo-effect algebra with (RDP$_0$), then

\[(A \cap B)^\perp_c = A^\perp_c \cap B^\perp_c \cap C,\]
\[(A \cap B)^\perp_c = (A^\perp_c \cup B^\perp_c)^\perp_c \cap C.\]

Proof. It follows the proof of (3.3), where we change $w = 0$ and $v = 0$ to $w \in C$ and $v \in C$, respectively. \qed

**Proposition 4.5.** Let $\{A_t\}_t$ be a non-void system of ideals of a pseudo-effect algebra $E$ satisfying (RDP$_0$). If $A = \bigcup_t A_t$, then $A^\perp_c = \bigcap_t A^\perp_{c_t}$.

Proof. Since $A \supseteq A_t$ for any $t$, we have $A^\perp_{c_t} \subseteq A_t^\perp_c$, i.e., $A^\perp_c \subseteq \bigcap_t A^\perp_{c_t}$.

Choose now $x \in \bigcap_t A^\perp_{c_t}$ and $a \in A$, and assume $w \leq x, a$. Then $w \in A^\perp_{c_t}$ for any $t$ and simultaneously $w \in A_{t_0}$ for some $t_0$. Hence, $w \in C$ proving $x \in A^\perp_c$. \qed

Let $C$ be an ideal of $E$. We denote by

\[
\Pol_C(E) := \{ A \subseteq E : A = A^\perp_c \perp_c \}.\]

By (i) of Proposition 4.1, we have $C \subseteq A \subseteq E$ for any $A \in \Pol_C(E)$.

**Theorem 4.6.** Let $E$ be a pseudo-effect algebra with (RDP). Then $(\Pol_C(E); \subseteq, \perp_c, C, E)$ is a complete Boolean algebra such that for the corresponding meets and joins we have $\bigwedge_t C A_t = \bigcap_t A_t$, $\bigvee_t C A_t = \bigcup_t A_t \perp_c \perp_c$, and $A \perp_c \perp_c \bigwedge_t C A_t = \bigvee_t C(A \perp_c \perp_c A_t)$.
In addition, the mapping $\pi_C : \mathcal{I}(E) \to \text{Pol}_C(E)$ given by $\pi_C(A) := A^{\perp_C \perp_C}$, $A \in \mathcal{I}(E)$, is a lattice homomorphism of $\mathcal{I}(E)$ onto $\text{Pol}_C(E)$, and $C$ is the largest element of the set $\{A \in \mathcal{I}(E) : \pi_C(A) = C \}$. If $\phi$ is a lattice homomorphism of $\mathcal{I}(E)$ into a lattice $\mathcal{X}$ with $0$ such that $C$ is the largest element in the set $\{A \in \mathcal{I}(E) : \phi(A) = 0 \}$, then $\phi(I_1) = \phi(I_2)$ implies $\pi_C(I_1) = \pi_C(I_2)$.

Proof. According to Proposition 4.4, $\text{Pol}_C(E)$ is a de Morgan lattice with $A \wedge^C B = A \cap B$ and $A \vee^C B = (A \cup B)^{\perp_C \perp_C}$, and $A \wedge^C A^{\perp_C} = C$ and $A \vee^C A^{\perp_C} = E$. In view of Proposition 4.5, $\bigvee_t A_t = \left( \bigcup_t A_t \right)^{\perp_C \perp_C} \in \text{Pol}_C(E)$ and $\bigwedge_t A_t = \bigcap_t (A_t^{\perp_C})^{\perp_C} \in \text{Pol}_C(E)$. Hence, $\bigwedge_t A_t = \bigcap_t A_t$.

Further, $A \wedge^C \left( \bigvee_t A_t \right) = A \cap \left( \bigcup_t A_t \right)^{\perp_C \perp_C} = A^{\perp_C \perp_C} \cap \left( I_0 \left( \bigcup_t A_t \right) \right)^{\perp_C \perp_C} = \left( A \cap \left( \bigvee_t A_t \right) \right)^{\perp_C \perp_C} = \left( \bigvee_t (A \cap A_t) \right)^{\perp_C \perp_C} = \left( I_0 \left( \bigcup_t (A \cap A_t) \right) \right)^{\perp_C \perp_C} = \left( \bigcup_t (A \cap A_t) \right)^{\perp_C \perp_C} = \bigvee_t (A \wedge^C A_t)$, where we have used distributivity in the lattice $\mathcal{I}(E)$, see [Dvu3; Proposition 3.2].

Finally assume that $\mathcal{X}$ is a lattice with $0$ and that $\phi : \mathcal{I}(E) \to \mathcal{X}$ is a lattice homomorphism with $C$ the largest element of the set $\{A \in \mathcal{I}(E) : \phi(A) = 0 \}$. Let $I$ be an ideal of $E$ and define $\hat{I} = \{ M \in \mathcal{I}(E) : \phi(M) \wedge^C \phi(I) = \phi(C) \}$. If $M \in \hat{I}$, then $M \cap I \subseteq C$, which yields $M \subseteq I^{\perp \perp (C \cap I)} = I^{\perp_C}$ by (iii) of Proposition 4.3. In addition, $\phi(I^{\perp_C} \cap I) = \phi(I^{\perp \perp (C \cap I)} \cap I) = \phi(I \cap C) = \phi(C)$. Hence, $I^{\perp_C} \in \hat{I}$, and so is the largest element of $\hat{I}$. Consequently, if $\phi(I_1) = \phi(I_2)$, $I_1^{\perp_C} = I_2^{\perp_C}$ yielding $\pi_C(I_1) = \pi_C(I_2)$. \hfill $\square$

In the rest of the present section, we show the relation among prime ideals and $C$-polars.

We say that an ideal $C$ of a pseudo-effect algebra $E$ is prime in an ideal $A$ of $E$ if

(i) $C \subseteq A$,
(ii) for $a, b \in A$, $I_0(a) \cap I_0(b) \subseteq C$ implies $a \in C$ or $b \in C$.

Using ideas from [Dvu3], we have that an ideal $C$ of a pseudo-effect algebra $E$ with (RDP) is prime in $A$ ($C \subseteq A$) if and only if $I \cap J \subseteq C$ for $I, J \subseteq A$, $I, J \in \mathcal{I}(E)$, implies $I \subseteq C$ or $J \subseteq C$ or if and only if $I \cap J = C$ for $I, J \subseteq A$, $I, J \in \mathcal{I}(E)$, implies $I = C$ or $J = C$. 

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THEOREM 4.7. Let $C$ and $A$, $C \subseteq A$, be ideals of a pseudo-effect algebra $E$ with (RDP). The following statements are equivalent.

(i) $C$ is prime in $A^{\perp C \perp C}$.
(ii) $C$ is prime in $A$.
(iii) $A^{\perp C}$ is a prime ideal of $E$.
(iv) $A^{\perp C} = a^{\perp C}$ for all $a \in A \setminus C$.
(v) $A^{\perp C}$ is a maximal C-polar of an ideal containing $C$.
(vi) $A^{\perp C \perp C}$ is a minimal C-polar of an ideal containing $C$.
(vii) $A^{\perp C \perp C}$ is an ideal maximal with respect to the property of being $C$ prime in it.

Proof.

(i) $\implies$ (ii). Since $C \subseteq A \subseteq A^{\perp C \perp C}$, the implication is evident.

(ii) $\implies$ (iii). Let $I, J \in \mathcal{I}(E)$ be such that $I \cap J = A^{\perp C}$. Then $(A \cap I) \cap (A \cap J) = C$. Therefore, $A \cap I = C$ or $A \cap J = C$. Hence, $I \subseteq A^{\perp C}$ or $J \subseteq A^{\perp C}$ (by (ix) of Proposition 4.1), which proves $A^{\perp C}$ is a prime ideal of $E$.

(iii) $\implies$ (ii). Let $A^{\perp C}$ be a prime ideal of $E$ and let $I, J \in \mathcal{I}(E)$ be subsets of $A$ such that $I \cap J = C$. Then $(I \cup A^{\perp C}) \cap (J \cup A^{\perp C}) = A^{\perp C}$, where $\cup$ denotes the join in the lattice $\mathcal{I}(E)$, which yields $I \cup A^{\perp C} \subseteq A^{\perp C}$ or $J \cup A^{\perp C} \subseteq A^{\perp C}$. Hence, $I \subseteq A^{\perp C}$ and in view of hypothesis $I \subseteq A$, we have $I \subseteq A^{\perp C} \cap A = C$. In a similar way we proceed in the second case.

(ii) $\implies$ (iv). Assume that $C$ is a prime ideal of $A$. Then, for all $a \in A$, $A^{\perp C} \subseteq a^{\perp C}$. If there exists $a \in A \setminus C$ such that $A^{\perp C} \neq a^{\perp C}$, then we can choose an element $x \in a^{\perp C} \setminus A^{\perp C}$. Since $A^{\perp C} = \bigcap \{a^{\perp C} : a \in A\}$, there exists $a_0 \in A$ such that $x \notin a_0^{\perp C}$. Consequently, there exists $y \in E \setminus C$ such that $y \leq a_0, x$. Then $y \in a^{\perp C} \cap A$. But $C$ is prime in $A$, so we have by (v) of Proposition 4.1 $C = a^{\perp C} \cap a^{\perp C \perp C} = (a^{\perp C} \cap A) \cap (a^{\perp C \perp C} \cap A)$, so that $C = a^{\perp C} \cap A$ or $C = a^{\perp C \perp C} \cap A$. However, $y \in (a^{\perp C} \cap A) \setminus C$ and $a \in (a^{\perp C \perp C} \cap A) \setminus C$, which is absurd.

(iv) $\implies$ (ii). Suppose now that $A^{\perp C} = a^{\perp C}$ for all $a \in A \setminus C$, and let $x, y \in A \setminus C$ satisfy $I_0(x) \cap I_0(y) \subseteq C$. Then $y \in y^{\perp C \perp C}$ and $y \in x^{\perp C} = A^{\perp C} = y^{\perp C}$, which yields $y \in y^{\perp C} \cap y^{\perp C \perp C} = C$, a contradiction. Hence, $C$ is prime in $A$.

(iv) $\implies$ (v). Suppose $C \subseteq D \in \mathcal{I}(E)$ and let $A^{\perp C} \subseteq D^{\perp C}$. We claim $A^{\perp C} = D^{\perp C}$. We have $D \not\subseteq A^{\perp C}$, otherwise $D = D \cap A^{\perp C} \subseteq D \subseteq D^{\perp C} = C$, a contradiction. Hence, there exists $d \in D \setminus A^{\perp C}$ and by (o) of Proposition 4.1, there exists an element $u \in E \setminus C$ such that $u \leq a, d$. Consequently, $u \in (D \cap A) \setminus C$. By (iv), $D^{\perp C} \subseteq u^{\perp C} = A^{\perp C} \subseteq D^{\perp C}$.

(v) $\implies$ (vi) and (vii) $\implies$ (i). They are evident.
(vi) \implies (vii). First, we prove $C$ is prime in $A^\perp_{c^\perp\perp c}$. If not, there are two ideals $I$ and $J$ of $E$ such that $C \subseteq I, J \subseteq A^\perp_{c^\perp\perp c}$ and $C = I \subseteq J$. There exist two elements $a \in I \setminus C$ and $b \in J \setminus C$, and define $D = C \cap I_0(a)$. Then $A^\perp_{c^\perp\perp c} \subseteq D$ and $C \subseteq D$ while $a \in D^\perp_{c^\perp\perp c} = A^\perp_{c^\perp\perp c}$, i.e., $D^\perp_{c^\perp\perp c} = A^\perp_{c^\perp\perp c}$. Let $x \in D$, and as $b \in A^\perp_{c} \cap J \subseteq A^\perp_{c} \cap A^\perp_{c^\perp\perp c} = C$, we have a contradiction. Hence, $C$ is prime in $A^\perp_{c^\perp\perp c}$.

Second, assume there exists an ideal $B$ of $E$ such that $B \supseteq A^\perp_{c^\perp\perp c}$ and $C$ is prime in $B$. Therefore, for $C$ and $B$ the statement (vi) holds, i.e., $B^\perp_{c} = A^\perp_{c}$. and, consequently, $B \subseteq B^\perp_{c^\perp\perp c} = A^\perp_{c^\perp\perp c} \subseteq B$, which gives $B = A^\perp_{c^\perp\perp c}$. \(\square\)

**THEOREM 4.8.** Let $P$ be an ideal of a pseudo-effect algebra with (RDP). The following statements are equivalent.

(i) $P$ is prime.

(ii) $P = a^\perp_{\perp P}$ for all $a \in E \setminus P$.

(iii) $\text{Pol}_P(E) = \{P, E\}$.

**Proof.**

(i) \iff (ii). It follows from Proposition 4.7 while $E^\perp_{\perp P} = P$.

(i) \implies (iii). Let $I \in \text{Pol}_P(E)$ and $P$ be prime. Since $P = I^\perp_{\perp P} \cap I_{\perp P}^\perp_{\perp P}$, we have $P = I^\perp_{\perp P}$ or $P = I$, i.e., $I = E$ or $I = P$.

(iii) \implies (i). Assume that $a \in E \setminus P$ and $P \subset a^\perp_{\perp P}$. Since $a^\perp_{\perp P} \in \text{Pol}_P(E)$, we have $a^\perp_{\perp P} = E$, i.e., $a \in a^\perp_{\perp P} = E^\perp_{\perp P} = P$, a contradiction. \(\square\)

5. **C-Carriers of pseudo-effect algebras and C-regularity**

Let $a$ be an element of a pseudo-effect algebra $E$ and let $C$ be an ideal of $E$. The **C-carrier** of $a$, $a^{\land(C)}$, is the set

$$a^{\land(C)} = \{ b \in E : b^\perp_{c^\perp\perp c} = a^\perp_{c^\perp\perp c} \}.$$

In particular, if $C = \{0\}$, we call $a^{\land} := a^{\land(\{0\})}$ the **carrier** of $a$.

The following basic properties of C-carriers can be easily proved.

**Proposition 5.1.** Let $E$ be a pseudo-effect algebra and let $a \in E$ and $C \in \mathcal{I}(E)$. Then

(i) $a^{\land(C)} = C$ for any $a \in C$. In particular, $0^{\land} = \{0\}$.  

(ii) $a \in a^{\land(C)} \subseteq a^\perp_{c^\perp\perp c}$, $a^\perp_{c^\perp\perp c} = (a^{\land(C)})^\perp_{c^\perp\perp c}$.

Let $E$ satisfy (RDP$^0$).

(iii) If $b_1, b_2 \in a^{\land(C)}$ and $b_1 + b_2 \in E$, then $b_1 + b_2 \in a^{\land(C)}$.

(iv) If $a \in E \setminus C$, then $C \cap a^{\land(C)} = \emptyset$.  

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We say that a pseudo-effect algebra $E$ is $C$-regular if $C$ is a normal ideal of $E$, and $a \perp C$ is normal for any $a \in E$.

**Proposition 5.2.** Let $E$ be a pseudo-effect algebra with $(\text{RDP}_0)$ and let $C$ be an ideal of $E$. Then $E$ is $C$-regular if and only if $a + x \in E$ and $y + a \in E$ imply $a^{\wedge(C)} = (x / (a + x))^{\wedge(C)} = ((y + a) \setminus y)^{\wedge(C)}$.

**Proof.** Let $E$ be $C$ regular, and let $z \in a \perp C$. Then $a \in z \perp C$ and the normality of $z \perp C$ yields $x / (a + x), (y + a) \setminus y \in z \perp C$, i.e., $z \in (x / (a + x)) \perp C$ and $z \in ((y + a) \setminus y) \perp C$. Conversely, if $z \in ((y + a) \setminus y) \perp C$, then $z \in (x / (a + x)) \perp C$, i.e., $a \in z \perp C$, $z \in a \perp C$, and similarly $z \in ((y + a) \setminus a) \perp C$ implies $z \in a \perp C$.

Assume now $a^{\wedge(C)} = (x / (a + x))^{\wedge(C)} = ((y + a) \setminus y)^{\wedge(C)}$. Let $x_0 \in a \perp C$ and let $y_0 / (x_0 + y_0) \in E$. Then $a \in x \perp C = (y_0 / (x_0 + y_0))^\perp C$. Hence, $y_0 / (x_0 + y_0) \in a \perp C$, and similarly we can prove $(y_0' + x_0) \setminus y_0' \in a \perp C$ for some $y_0' \in E$ for which $y_0' + x_0$ is defined in $E$. □

Let $C$ be an ideal of a pseudo-effect algebra $E$. Let us set

$$K_C(E) := \{a^{\wedge(C)} : a \in E\},$$

and define a partial order $\leq$ on $K_C(E)$ as follows: $a^{\wedge(C)} \leq b^{\wedge(C)}$ if and only if $b \perp C \subseteq a \perp C$. Then, for all $a, b \in E$ such that $a \leq b$, we have

$$0^{\wedge(C)} \leq a^{\wedge(C)} \leq b^{\wedge(C)} \leq 1^{\wedge(C)}.$$

**Theorem 5.3.** Let $E$ be a pseudo-effect algebra with $(\text{RDP})$.

(i) If $c = a + b$, then $c^{\wedge(C)}$ is the join of $a^{\wedge(C)}$ and $b^{\wedge(C)}$ in the space $K_C(E)$.

(ii) $a^{\wedge(C)} \lor b^{\wedge(C)}$ is defined in $K_C(E)$ for all $a, b \in E$. Moreover, there exists an element $d \in E$ such that $d \geq a, b$ and $d^{\wedge(C)} = a^{\wedge(C)} \lor b^{\wedge(C)}$. For an element $e \in E$, we have $e^{\wedge(C)} = a^{\wedge(C)} \lor b^{\wedge(C)}$ if and only if $e \perp C = a \perp C \cap b \perp C$.

(iii) If $a \lor b$ is defined in $E$, then $(a \lor b)^{\wedge(C)} = a^{\wedge(C)} \lor b^{\wedge(C)}$. If $a \land b$ is defined in $E$, then $(a \land b)^{\wedge(C)} = a^{\wedge(C)} \land b^{\wedge(C)}$.

(iv) If $d \perp C = (a \perp C \cup b \perp C) \perp C \perp C$, then $d^{\wedge(C)} = a^{\wedge(C)} \land b^{\wedge(C)}$.

(v) Let $a^{\wedge(C)} \leq b^{\wedge(C)}$. Then, for any $a_1 \in a^{\wedge(C)}$ there exists $b_1 \in b^{\wedge(C)}$ such that $a_1 \leq b_1$.

(vi) If $a^{\wedge(C)} \land b^{\wedge(C)}$ is defined in $K_C(E)$, then so is $(a^{\wedge(C)} \lor c^{\wedge(C)}) \land (b^{\wedge(C)} \lor c^{\wedge(C)})$, and it is equal to $(a^{\wedge(C)} \land b^{\wedge(C)}) \lor c^{\wedge(C)}$, and if also $a^{\wedge(C)} \land d^{\wedge(C)}$ exists in $K_C(E)$, then so does $a^{\wedge(C)} \land (b^{\wedge(C)} \lor d^{\wedge(C)})$ and it is equal to $(a^{\wedge(C)} \land b^{\wedge(C)}) \lor (a^{\wedge(C)} \land d^{\wedge(C)})$.

(vii) If $K_C(E)$ is finite, then it is a Boolean algebra.
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Proof.

(i) Let $c = a + b$. According to (viii) of Proposition 4.1, we have $c_{\perp} = a_{\perp} \cap b_{\perp}$, which proves easily $c^{\land}(C) = a^{\land}(C) \lor b^{\land}(C)$.

(ii) Let $a$ and $b$ be arbitrary elements of $E$. (RDP) implies that there are three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$ and $a_1 + b_1 + c = b_1 + a_1 + c \in E$. Let $d := a_1 + b = b_1 + a$. Then $d_{\perp} = a_{\perp}^{\land} \cap b_{\perp}^{\land} = b_{\perp}^{\land} \cap a_{\perp}^{\land}$, i.e., $d^{\land}(C) \leq a^{\land}(C), b^{\land}(C) = a_{\perp}^{\land} \cap b_{\perp}^{\land}$. Assume $y^{\land}(C) \geq a^{\land}(C), b^{\land}(C)$. Hence, $y_{\perp}^{\land} \subseteq a_{\perp}^{\land} \cap b_{\perp}^{\land} = d^{\land}(C)$, i.e., $d^{\land}(C) \leq y^{\land}(C)$.

The rest is evident.

(iii) Assume $a \lor b \in E$. Then $a, b \leq a \lor b \leq d$, where $d$ is the element from (ii). This gives $a^{\land}(C), b^{\land}(C) \leq (a \lor b)^{\land}(C) \leq d^{\land}(C) = a^{\land}(C) \lor b^{\land}(C)$.

Assume now $a \land b \in E$. Hence, $(a \land b)^{\land}(C) \leq a^{\land}(C), b^{\land}(C)$. Suppose $x^{\land}(C) \leq a^{\land}(C), b^{\land}(C)$. Since $I_{0}(a \land b) = I_{0}(a) \cap I_{0}(b)$, according to Proposition 4.4, we have $(a \land b)^{\perp} = (a_{\perp}^{\land} \cup b_{\perp}^{\land})^{\perp} \subseteq x_{\perp}^{\land}$. This gives $(a \land b)^{\land}(C) \geq x^{\land}(C)$.

(iv) Suppose $d_{\perp} = (a_{\perp}^{\land} \cup b_{\perp}^{\land})^{\perp}$. Then $d_{\perp} \supseteq a_{\perp}^{\land}, b_{\perp}^{\land}$, i.e., $d^{\land}(C) \leq a^{\land}(C), b^{\land}(C)$. Assume $x^{\land}(C) \leq a^{\land}(C), b^{\land}(C)$. Then $x_{\perp}^{\land} \supseteq a_{\perp}^{\land} \cup b_{\perp}^{\land}$, i.e., $x_{\perp}^{\land} \supseteq (a_{\perp}^{\land} \cup b_{\perp}^{\land})^{\perp} = d_{\perp}^{\land}$, which gives $x^{\land}(C) \leq d^{\land}(C)$, and $d^{\land}(C) = a^{\land}(C) \land b^{\land}(C)$.

(v) By (ii), there exists $b_1 \geq a, b$ such that $b_1^{\land}(C) = a_1^{\land}(C) \lor b^{\land}(C) = a^{\land}(C) \lor b^{\land}(C)$, which gives $b_1 \in b^{\land}(C)$.

(vi) Put $x^{\land}(C) = a^{\land}(C) \land b^{\land}(C)$. Then obviously $x^{\land}(C) \lor c^{\land}(C) \leq a^{\land}(C) \lor c^{\land}(C)$ and $x^{\land}(C) \lor c^{\land}(C) \leq b^{\land}(C) \lor c^{\land}(C)$. Assume that $u^{\land}(C) \leq a^{\land}(C) \lor c^{\land}(C)$ and $u^{\land}(C) \leq b^{\land}(C) \lor c^{\land}(C)$ but it is not less than $x^{\land}(C) \lor c^{\land}(C)$. By (v) and (ii), there is a $u^{\land}(C)$ such that

\[ x^{\land}(C) \lor c^{\land}(C) < u^{\land}(C) \tag{*} \]

(we change $u^{\land}(C)$ to $u^{\land}(C) \lor x^{\land}(C) \lor c^{\land}(C)$ if necessary). As in the proof of (ii), we have $x_1 \leq x, a_1 \leq a$ and $b_1 \leq b$ such that $(x_1 + c)^{\land}(C) = x^{\land}(C) \lor c^{\land}(C) = u^{\land}(C) \leq (a_1 + c)^{\land}(C) = a^{\land}(C) \lor c^{\land}(C)$ and $u^{\land}(C) \leq (b_1 + c)^{\land}(C) = b^{\land}(C) \lor c^{\land}(C)$. By (iv), we can assume that they satisfy also $x_1 + c < u < a_1 + c, u < b_1 + c$. Since $x_1^{\land}(C) \leq (u \land c)^{\land}(C)$, we have $x_1^{\land}(C) < (u \land c)^{\land}(C)$, otherwise the equality $x_1^{\land}(C) = (u \land c)^{\land}(C)$ would imply, by (i), $(x_1 + c)^{\land}(C) = x^{\land}(C) \lor c^{\land}(C) = x_1^{\land}(C) \lor c^{\land}(C) = (u \land c)^{\land}(C) \lor c^{\land}(C) = u^{\land}(C)$ against (*). Since $u \land c \leq a_1, b_1$, i.e., $u \land c \leq a, b$, we have $(u \land c)^{\land}(C) \leq a^{\land}(C) \land b^{\land}(C)$, which contradicts the choice of $u^{\land}(C)$.

For the second equality. Let $a_1^{\land}(C) = a^{\land}(C) \land b^{\land}(C)$ and $a_2^{\land}(C) = a^{\land}(C) \land d^{\land}(C)$. Then $a_1^{\land}(C) \lor a_2^{\land}(C) \leq a^{\land}(C)$ and $a_1^{\land}(C) \lor a_2^{\land}(C) \leq b^{\land}(C) \lor d^{\land}(C)$. Assume $x^{\land}(C) \leq a^{\land}(C), b^{\land}(C) \lor d^{\land}(C)$. Then $x_{\perp}^{\land} \supseteq a_{\perp}^{\land} \lor (b_{\perp}^{\land} \cap d_{\perp}^{\land})$, which gives by Theorem 4.6, $x_{\perp}^{\land} \supseteq a_{\perp}^{\land} \lor (b_{\perp}^{\land} \lor d_{\perp}^{\land}) = (a_{\perp}^{\land} \lor C b_{\perp}^{\land}) \lor (a_{\perp}^{\land} \lor C d_{\perp}^{\land}) = a_{\perp}^{\land} \lor a_{\perp}^{\land}$. Then $x^{\land}(C) \leq a_1^{\land}(C) \lor a_2^{\land}(C)$. 

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(vii) Since $K_C(E)$ is finite, for any two elements $a, b \in E$, there is only a finite number of elements $c^{\wedge(C)}$ of $K_C(E)$ such that $c^{\wedge(C)} \leq a^{\wedge(C)}, b^{\wedge(C)}$. Hence, the element $\bigvee c^{\wedge(C)}$ is the infimum of $a^{\wedge(C)}$ and $b^{\wedge(C)}$.

By (vi), $K_C(E)$ is distributive.

Let $a_1^{\wedge(C)}, \ldots, a_n^{\wedge(C)}$ be the atoms of $K_C(E)$. Let $b^{\wedge(C)} \in K_C(E)$ and let $a_1^{\wedge(C)}, \ldots, a_k^{\wedge(C)}$ be the atoms which are less than $b^{\wedge(C)}$. Then $b^{\wedge(C)} = \bigvee_{i=1}^{k+1} a_i^{\wedge(C)}$, and the element $c^{\wedge(C)} := \bigvee_{i=k+1}^{n} a_i^{\wedge(C)}$ is the complement of $b^{\wedge(C)}$. Indeed, $b^{\wedge(C)} \wedge c^{\wedge(C)} = \bigvee_{i=k+1}^{n} (b^{\wedge(C)} \wedge a_i^{\wedge(C)}) = 0^{\wedge(C)}$, and $b^{\wedge(C)} \vee c^{\wedge(C)} = \bigvee_{i=1}^{n} a_i^{\wedge(C)} = 1^{\wedge(C)}$.

**Proposition 5.4.** Let $E$ be a pseudo-effect algebra with (RDP) and let $C$ be an ideal of $E$. The mapping $\phi : E \to K_C(E)$ defined by $\phi(a) = a^{\wedge(C)}$, $a \in E$, is an order-preserving mapping of $E$ onto $K_C(E)$ preserving all existing finite suprema and infima which exist in $E$, and $\{a \in E : \phi(a) = 0^{\wedge(C)}\} = C$.

**Proof.** It follows from Theorem 5.3. \qed

6. Representable pseudo-effect algebras

Let $\{E_i\}_{i \in I}$ be an indexed system of pseudo-effect algebras. The Cartesian product $\prod_{i \in I} E_i$ can be organized into a pseudo-effect algebra with the partial addition defined by coordinates. Each $E_i$ has the property (RDP) ((RDP$_1$), (RDP$_2$)) if and only if $\prod_{i \in I} E_i$ has this property.

We say that a pseudo-effect algebra $E$ is a **subdirect product** of pseudo-effect algebras $\{E_i\}_{i \in I}$ if there is an injective homomorphism of pseudo-effect algebras $f : E \to \prod_{i \in I} E_i$ such that $f(a) \leq f(b)$ if and only if $a \leq b$ ($a, b \in E$), and for every $j \in I$, $\pi_j \circ f$ is a surjective homomorphism from $E$ onto $E_j$, where $\pi_j$ is the $j$th projection of $\prod_{i \in I} E_i$ onto $E_j$.

We say that a po-group $G$ is a **subdirect product** of a system $\{G_i\}_{i \in I}$ of po-groups if there exists an injective group homomorphism $f : G \to \prod_{i \in I} G_i$ such that $f(a) \leq f(b)$ if and only if $a \leq b$ ($a, b \in G$), and for every $j \in I$, $\pi_j \circ f$ is a surjective homomorphism from $G$ onto $G_j$, where $\pi_j$ is the $j$th projection of $\prod_{i \in I} G_i$ onto $G_j$.

We recall that a poset $(E; \leq)$ is an **antilattice** if only comparable elements of $E$ have an infimum or a supremum. If $E$ is a pseudo-effect algebra, then
$E$ is an antilattice if and only if $a \land b = 0$ implies $a = 0$ or $b = 0$, while $(a \land (a \land b)) \land (b \land (a \land b)) = 0$, see [Dvu3].

We say that a pseudo-effect algebra $E$ is representable if $E$ is a subdirect product of antilattice pseudo-effect algebras such that all finite suprema and infima which exist in $E$ are preserved in the subdirect product.

In the paper [Dvu], we have proved that the system of all representable pseudo-effect algebras forms a variety. Not all pseudo MV-algebras are representable, but every effect algebra with (RDP) is representable, as it was proved in [Rav] and [Dvu2].

**Theorem 6.1.** Every effect algebra $E$ with (RDP) is a subdirect product of antilattice effect algebras with (RDP), and all existing meets and joins in $E$ are preserved in the subdirect product.

**Proposition 6.2.** Let a pseudo-effect algebra $E$ with (RDP) be representable. Then every polar $A^\perp$ is a normal ideal.

**Proof.** Let $E$ be a subdirect product of a system $\{E_i\}_{i \in I}$ of antilattice pseudo-effect algebras. Assume $x \in A^\perp$ and let $x + y$ be defined in $E$. We show that $y / (x + y) \in A^\perp$. Let $z < y / (x + y)$ and $z < a$ for any $a \in A$. Write $z = (z_i)_{i \in I}$, $y = (y_i)_{i \in I}$, $x = (x_i)_{i \in I}$ and $a = (a_i)_{i \in I}$, where $z_i, y_i, x_i, a_i \in E_i$, $i \in I$. Then $z_i < y_i / (x_i + y_i)$ and $z_i < a_i$ for any $i \in I$. Since $a_i \land x_i = 0$ for each $i \in I$, if $a_i = 0$, then $z_i = 0$, if $a_i > 0$, then $x_i = 0$, which yields $z_i < y_i / (0 + y_i) = 0$. Hence $z = 0$, which proves $(y / (x + y)) \land a = 0$ for any $a \in A$.

In a similar way, if $x \in A^\perp$ and $u + x \in E$, then $(u + x) \setminus u \in A^\perp$.

We recall that every polar is normal in $E$ if and only if $a^\perp$ is normal for every $a \in E$. In addition, in [Gel0], it is proved that a pseudo MV-algebra is representable if and only if every polar is normal, while $A^\perp = \left( \bigcup_{a \in A} \{a\} \right)^\perp = \bigcap_{a \in A} a^\perp$.

**7. Regular pseudo-effect algebras and Lorenzen’s theorem**

We say that a pseudo-effect algebra $E$ is regular if $a^\perp$ is a normal ideal for any $a \in E$. This is equivalent with the statement $A^\perp$ is a normal ideal for any $\emptyset \neq A \subseteq E$. We recall that if a regular $E$ satisfies (RDP$_0$), then for any $a \in E$, we have $N_0(a)^\perp = a^\perp = I_0(a)^\perp$, where $N_0(a)$ is the normal ideal of $E$ generated by $a$. Indeed, we have $I_0(a) \subseteq N_0(a) \subseteq a^\perp$. Hence, $a^\perp \subseteq N_0(a)^\perp \subseteq a^\perp$.

We say that a pseudo-effect algebra $E$ is finitely irreducible if, for any two ideals $I$ and $J$ of $E$ with $I \cap J = \{0\}$, we have $I = \{0\}$ or $J = \{0\}$. 37
We recall that according to [DvVe1], if \( a \) and \( b \) are two elements of a pseudo-effect algebra \( E \) with \((\text{RDP}_0)\), then \( a \wedge b = 0 \) implies \( a + b, b + a, a \vee b \) are defined in \( E \), and
\[
a + b = a \vee b = b + a.
\]

**PROPOSITION 7.1.** Any antilattice pseudo-effect algebra with \((\text{RDP}_0)\) is finitely irreducible and regular.

**Proof.** If a pseudo-effect algebra \( E \) with \((\text{RDP}_0)\) is not finitely irreducible, then there exist two non-zero ideals \( I \) and \( J \) such that \( I \cap J = \{0\} \). Hence, if \( a \in I \) and \( b \in J \) are non-zero elements, then \( a \wedge b = 0 \), whence \( E \) cannot be an antilattice.

Assume \( x \in a^\perp \) and let \( x + y \) be defined in \( E \). We show that \( y / (x + y) \in a^\perp \). Let \( z \leq y / (x + y) \) and \( z \leq a \) for any \( a \in A \). Since \( a \wedge x = 0 \), then if \( a = 0 \), then \( z = 0 \), if \( a > 0 \), then \( x = 0 \), which yields \( z \leq y / (0 + y) = 0 \). Hence \( z = 0 \), which proves \( (y / (x + y)) \wedge a = 0 \).

In a similar way, if \( x \in a^\perp \) and \( u + x \in E \), then \((u + x) \setminus u \in a^\perp \), which proves \( E \) is regular. \( \square \)

**PROPOSITION 7.2.** Any regular finitely irreducible pseudo-effect algebra \( E \) with \((\text{RDP}_0)\) is an antilattice.

**Proof.** Assume that there are \( a, b \in E \setminus \{0\} \) with \( a \wedge b = 0 \). Then \( a \in b^\perp \) and \( b \in a^\perp \). In view of (7.1), \( 0 \neq a + b = a \vee b \in E \), so that \( a^\perp \cap b^\perp = (a + b)^{\perp \perp} \).

While \( (a + b)^{\perp \perp} \cap (a + b)^{\perp} = \{0\} \) and \( a + b \in (a + b)^{\perp \perp} \), the irreducibility implies \( (a + b)^{\perp} = \{0\} \), i.e., \( a^\perp \cap b^\perp = \{0\} \), which gives \( b \in a^\perp = \{0\} \) or \( a \in b^\perp = \{0\} \), i.e., \( b = 0 \) or \( a = 0 \), a contradiction. \( \square \)

**PROPOSITION 7.3.** Let \( E \) be a pseudo-effect algebra with \((\text{RDP})\) and let \( P \) be a proper normal ideal of \( E \).

(i) If \( I \) is an ideal of \( E \), so is \( I/P \) in \( E/P \). Moreover, if \( I \) is a proper ideal of \( E \) containing \( P \), then \( I/P \) is a proper ideal of \( E/P \).

(ii) If \( M \) is an ideal of \( E/P \), then
\[
\kappa(M) := \{x \in E : x/P \in M\}
\]
is an ideal of \( E \), and \( \kappa(M)/P = M \). If \( M \) is a proper ideal of \( E \) so is \( \kappa(M) \) in \( E \).

(iii) \( N(E/P) = \{N/P : N \in N(E) \text{ and } P \subseteq N\} \).

(iv) If \( P \) is an \( o \)-ideal of a directed po-group \( G \) with \((\text{RDP}_1)\) and if \( M \) is an \( o \)-ideal of \( G/P \), then \( \kappa(M) := \{x \in G : x/P \in M\} \) is an \( o \)-ideal of \( G \), and \( \kappa(M)/P = M \). In addition, \( \mathcal{O}(G/P) = \{N/P : N \in \mathcal{O}(G) \text{ and } P \subseteq N\} \).
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Proof.

(i) $0/P \in I/P$. Let $x/P \leq y/P$, where $y \in I$. There exists $x_1 \in [x]_P$ such that $x_1 \leq y$, which gives $x_1 \in I$, and $x_1/P = x/P \leq y/P$. Assume $x/P + y/P$ is defined in $E/P$ for some $x, y \in I$. There are $x_1 \in [x]_P, y_1 \in [y]_P$ and $e, f, u, v \in P$ such that $x_1 \setminus e = x \setminus f \in I, y_1 \setminus u = y \setminus v \in I, x_1 + y_1 \in E$. Then $x/P + y/P = x_1/P + y_1/P = (x_1 + y_1)/P = ((x \setminus f) + e + (y \setminus v) + u)/P = ((x \setminus f) + (y \setminus v))/P$ and $(x \setminus f) + (y \setminus v) \in I$.

Let now $I \supseteq P$ and $1/P = x/P$, where $x \in I$. There are $e, f \in P$ such that $1 \setminus e = x \setminus f$, i.e., $x \setminus 1 = f / e \in P \subseteq I$, which gives a contradiction.

(ii) We have $\kappa(M) \supseteq P$. If $x \leq y \in \kappa(M)$, then $x/P \leq y/P \in M$, so that $x \in \kappa(M)$. Let now $x, y \in \kappa(M)$ and $x + y \in E$. Then $(x + y)/P = x/P + y/P \in M$, i.e., $x + y \in \kappa(M)$.

Finally, assume $M$ is a proper ideal of $E/P$. Then $1/P \notin M$, hence, $1 \notin \kappa(M)$.

(iii) It follows from (ii).

(iv) It follows the same steps as (iii).

PROPOSITION 7.4.

(1) Let $I$ and $J$ be two normal ideals of a pseudo-effect algebra $E$ with $(RDP_1)$ such that $I \cap J = \{0\}$. Then $E$ is a subdirect product of $E/I$ and $E/J$ with the embedding $f : E \to E/I \times E/J$ defined $f(a) = (a/I, a/J), \ a \in E$.

(2) Let $I$ and $J$ be two $o$-ideals of a directed po-group $G$ with $(RDP_1)$ such that $I \cap J = \{0\}$. Then $G$ is a subdirect product of $G/I$ and $G/J$ with the embedding $f : G \to G/I \times G/J$ defined $f(a) = (a/I, a/J), \ a \in G$.

Proof.

(1) The mapping $f : E \to E/I \times E/J$ given by $f(a) = (a/I, a/J), \ a \in E$, is a homomorphism of pseudo-effect algebras. If $f(a) = f(b)$, then there are $e, f_1 \in I$ and $u_1, v \in J$ such that $a \setminus e = b \setminus f_1$ and $a \setminus u_1 = b \setminus v$. If we now take the addition and subtraction in the corresponding unital interpolation group $(G, u)$ such that $E = \Gamma(G, u)$, then $a - b = e - f_1 \in \phi(I)$ and $a - b = u_1 - f_1 \in \phi(J)$, i.e., $a - b = 0$, and $f$ is an injective homomorphism.

Assume $f(x) \leq f(y)$ for some $x, y \in E$, i.e., $x/I \leq y/I$ and $x/J \leq y/J$. There are two elements $a \in I$ and $b \in J$ with $a, b \leq x$ such that $x \setminus a \leq y$ and $x \setminus b \leq y$. Since $a \wedge b = 0$, then $x = x \setminus (a \wedge b) = (x \setminus a) \vee (x \setminus b)$ (while all existing meets in $E$ are preserved in the corresponding representation group $(G, u)$), which gives $x \leq y$.

Hence, $E$ is a subdirect product of $E/I$ and $E/J$, as claimed.

(2) The second statement follows the same ideas as the first one.
PROPOSITION 7.5. Let $E$ be a pseudo-effect algebra with (RDP$_1$). The following statements are equivalent:

(i) $E$ is finitely irreducible.

(ii) If $E$ is a subdirect product of $E_1$ and $E_2$, and if $f$ is an injective homomorphism from $E$ into $E_1 \times E_2$ such that $f(x) \leq f(y)$ whenever $x \leq y$, and $\pi_1 \circ f$ and $\pi_2 \circ f$ being surjective, then $\text{Ker}(\pi_1 \circ f) = \{0\}$ or $\text{Ker}(\pi_2 \circ f) = \{0\}$.

Proof. 
\(\neg(i) \implies \neg(ii)\). Suppose $E$ is not finitely irreducible, i.e., there are two normal non-zero ideals $A$ and $B$ of $E$ such that $A \cap B = \{0\}$. By Proposition 7.4, $E$ is a subdirect product of $E/A$ and $E/B$ with the embedding $f(a) = (a/A, a/B)$, $a \in E$. Hence, for the mappings $f_A: a \mapsto a/A$ and $f_B: a \mapsto a/B$, we have $\text{Ker}(f_A) = A \neq \{0\}$ and $\text{Ker}(f_B) = B \neq \{0\}$, so that $E$ does not satisfy (ii).

\(\neg(ii) \implies \neg(i)\). Suppose $E$ is a subdirect product of $E_1$ and $E_2$ and let $f: E \to E_1 \times E_2$ be an injective homomorphism with $f(x) \leq f(y)$ if and only if $x \leq y$ such that, for every $A_i = \{a \in E : \pi_i \circ f(a) = 0\} \neq \{0\}$, $i = 1, 2$. Then $A_1$ and $A_2$ are normal non-zero ideals of $E$. Assume $x \in A_1 \cap A_2$, then $f(x) = (0, 0)$, and the injectivity of $f$ gives $x = 0$, which proves $A_1 \cap A_2 = \{0\}$. Hence, $E$ is not finitely irreducible. 

\section*{Theorem 7.6.}
Every pseudo-effect algebra $E$ with (RDP$_1$) is a subdirect product of finitely irreducible pseudo-effect algebras with (RDP$_1$) preserving all finite joins and meets from $E$.

Proof. Without loss of generality, we can assume that $E = \Gamma(G, u)$, where $(G, u)$ is a unital po-group with (RDP$_1$). Let $g \in G$, $g \not\leq 0$, and set $U(g) := \{h \in G : h \geq g\}$. We denote by $A(g)$ a proper normal ideal of $E$ which is maximal among normal proper ideals $A$ of $E$ with respect to the property $U(g) \cap A = \emptyset$. Since $0 \not\in U(g)$, $A(g)$ exists due to the Zorn lemma. Moreover, $igcap A(g) = \{0\}$.

We assert that $E$ is a subdirect product of $\{E/A(g)\}_g$. Let $f(a) := \{a/A(g)\}_g \leq \{b/A(g)\}_g =: f(b)$, $a, b \in E$. Then $(a - b)/\phi(A(g)) \leq 0$ for any $g \not\leq 0$. Set $g_0 = a - b$. If $g_0 \not\leq 0$, there is an element $e \in A(g_0)$ such that $a - b \leq e$, which implies $e \in U(g_0) \cap A(g_0)$, which is absurd.

Therefore, $E$ is a subdirect product of $\{E/A(g)\}_g$, moreover, the embedding $a \mapsto f(a)$ $(a \in E)$ preserves all existing finite joins and meets from $E$.

To prove the finite irreducibility of $E/A(g)$, assume that $I$ and $J$ are normal ideals of $E/A(g)$ such that $I \cap J = \{0\}$. By Proposition 7.3, the sets $\kappa(I) = \{a \in E : a/A(g) \in I\}$ and $\kappa(J) = \{b \in E : b/A(g) \in J\}$ are normal ideals of $E$ containing $A(g)$ such that $\kappa(I)/A(g) = I$ and $\kappa(J)/A(g) = J$. Since
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$I = \{0\}$ if and only if $\kappa(I) = A(g)$, assume $\kappa(I) \supset A(g)$ and $\kappa(J) \supset A(g)$. The maximality of $A(g)$ implies there are $a \in \kappa(I) \cap U(g)$ and $b \in \kappa(J) \cap U(g)$. Hence, $0, g \leq a, b$. (RIP) holding in $G$ entails there exists an element $c \in G$ such that $0, g \leq c \leq a, b$. Then $c \in E$, $c \in U(g)$, $c \notin A(g)$, and $c \in \kappa(I) \cap \kappa(J)$, i.e., $0 \neq c/A(g) \in I$ and $c/A(g) \in J$, which is a contradiction. Hence, $I = \{0\}$ or $J = \{0\}$.

**Theorem 7.7.** Let $E$ be a pseudo-effect algebra with $(RDP_1)$. If $E$ is representable, then $E$ is regular.

If $E$ is $C$-regular for any normal ideal $C$ of $E$, then $E$ is representable.

If $E$ is a pseudo-effect algebra with $(RDP_2)$, then $E$ is representable if and only if $E$ is regular.

**Proof.** The first statement follows from Proposition 6.2.

Suppose now that $E = \Gamma(G, u)$ for some unital po-group $(G, u)$ with $(RDP_1)$. For any element $g \in G$, $g \notin 0$, let $A(g)$ be a normal ideal of $E$ having the same sense as that in the proof of Theorem 7.6. If $E$ is $C$-regular for any normal ideal $C$ of $E$, then $A(g)$ is prime. Indeed, set $C = A(g)$, and let $A(g) = I \cap J$, where $I, J \in I(E)$. Then $A(g) = A(g) \perp c \perp C = I \perp c \perp C \cap J \perp c \perp C$ by Proposition 4.4. Since $I \perp c \perp C$ and $J \perp c \perp C$ are normal ideals of $E$, we have $A(g) = I \perp c \perp C = I$ or $A(g) = J \perp c \perp C = J$. Applying the proof of Theorem 7.6, we have that $E$ is a subdirect product of $\{E/A(g)\}_g$, and the embedding $a \mapsto f(a)$ ($a \in E$) preserves all existing finite joins and meets from $E$.

Finally, let $E$ satisfy $(RDP_2)$. Then $E$ is a lattice. Assume $a/A(g) \land b/A(g) = 0$. Hence, if $a \land b = 0$, then $a \in b^\perp \subseteq A(g)$ or $b \in b^\perp \subseteq A(g)$, i.e., $a/A(g) = 0$ or $b/A(g) = 0$. If $a \land b \notin A(g)$, then $(a \land (a \land b)) \land (b \land (a \land b)) = 0$, which gives again $a/A(g) = 0$ or $b/A(g) = 0$. Consequently, $A(g)$ is prime, which yields that $E$ is a subdirect product of $\{E/A(g)\}_g$.

We note that we do not know whether the condition $E$ is $C$-regular for any normal ideal $C$ of $E$ can be replaced by the condition $E$ is regular in order to be $E$ representable.

**Acknowledgment**

The author is indebted to the referee for his valuable remarks, which improved the readability of the paper.
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Received May 19, 2003