William H. Cornish

On the Chinese remainder theorem of H. Draškovičová

Mathematica Slovaca, Vol. 27 (1977), No. 3, 213--220

Persistent URL: http://dml.cz/dmlcz/131801

Math. Slovaca 27, 1977, No. 3, 213—220

ON THE CHINESE REMAINDER THEOREM OF H. DRAŠKOVIČOVÁ

WILLIAM H. CORNISH

Introduction. As a by-product of her investigation into strongly permutable sets of congruences, H. Draškovičová [4; Corollary 3.4] proved that an upper subsemilattice of the lattice of congruences of a universal algebra satisfied the Chinese remainder theorem if and only if the sublattice generated by the subsemilattice is not only distributive but also consists of permuting congruences. The purpose of this note is twofold. Firstly, we give a proof of Draškovičová's theorem that is based on B. Jónsson's theorem which gives an elegant necessary and sufficient condition for the sublattice generated by a subset of a modular lattice to be distributive. Since R. Balbes [1] has given an elementary proof of Jónsson's theorem, this means that Draškovičová's theorem is independent of the Axiom of Choice. Secondly, we consider some naturally arising instances of this Chinese remainder theorem. Our main examples involve congruence-kernels in a distributive pseudocomplemented lattice and the so-called distributive ideals of a general lattice. We precede these examples with a brief discussion of ideals (congruence-kernels) in a universal algebra.

1. The Chinese Remainder Theorem.

The notation and terminology of this article is based on that of [5].

A set $\mathcal{P}$ of congruences on a universal algebra $\mathfrak{A}$ is said to *satisfy the Chinese remainder theorem (is finitely strongly permutable,* in the terminology of [4]) if, for any $\theta_1, \theta_2, \ldots, \theta_n \in \mathcal{P}$ and any $a_1, a_2, \ldots, a_n \in A$, $a_i \equiv a_j(\theta_i \vee \theta_j)$ for any $i, j = 1, 2, \ldots, n$ implies that there exists $a \in A$, such that $a \equiv a_j(\theta_j)$ for each $j = 1, 2, \ldots, n$.

In [5; Chapter 5, Exer. 68] G. Grätzer generalized the Chinese remainder theorem of Zariski and Samuel [12; Theorem 18, p. 280] to universal algebras. The proof is via finite induction and can be extended slightly to establish the following lemma which has already been observed by Draškovičová [4; Corollary 3.3].

Lemma 1.1. *Let $\mathscr{L}$ be a sublattice of the lattice of congruences of a universal algebra $\mathfrak{A}$. Then, $\mathscr{L}$ satisfies the Chinese remainder theorem if and only if $\mathscr{L}$ is distributive and consists of permuting congruences.*

The next lemma is due to B. Jónsson [9]; a proof is also presented in the book [11; Theorem 34, p. 93]. Recently an elementary proof was given by R. Balbes [1].

Lemma 1.2. *Let H be a non-empty subset of a modular lattice L. The sublattice of L, which is generated by H, is distributive if and only if, for any $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ in H*

$$\bigwedge_{i=1}^{n} x_i \wedge \bigvee_{j=1}^{m} y_j = \bigvee_{j=1}^{m} \left(\bigwedge_{i=1}^{n} x_i \wedge y_j \right).$$

We now come to Draškovičová's theorem which followed as a consequence of her detailed examination of permutability notions. Moreover, Examples 3.1 and 4.1 of [4] show that in a certain sense the result is the best that can be hoped for.

Theorem 1.3. *Let $\mathscr{P}$ be an upper subsemilattice of the lattice $\mathscr{C}(\mathfrak{A})$ of a universal algebra $\mathfrak{A}$. Then, $\mathscr{P}$ satisfies the Chinese remainder theorem if and only if the sublattice $L(\mathscr{P})$ of $\mathscr{C}(\mathfrak{A})$, generated by $\mathscr{P}$, is distributive and consists of permuting congruences.*

Proof. Because of 1.1 it is sufficient to verify the necessity of the conditions on $L(\mathscr{P})$. To do this we first prove the following:

(1.1) *If $\Psi_1, \Psi_2, \ldots, \Psi_n$; $\Phi_1, \Phi_2, \ldots, \Phi_m \in \mathscr{P}$ then*

$$\bigcap_{i=1}^{n} \bigcap_{j=1}^{m} (\Psi_i \vee \Phi_j) = \bigcap_{i=1}^{n} \Psi_i \circ \bigcap_{j=1}^{m} \Phi_j.$$

Suppose that x and y are elements of A which are congruent with respect to the left-hand side of (1.1). Let $\theta_1 = \Psi_1, \ldots, \theta_n = \Psi_n, \theta_{n+1} = \Phi_1, \ldots, \theta_{n+m} = \Phi_m$ and $a_1 = \ldots = a_n = x$, while $a_{n+1} = \ldots = a_{n+m} = y$. Then, $a_i \equiv a_j(\theta_i \vee \theta_j)$ for any $i, j = 1, 2, \ldots, n+m$. As $\mathscr{P}$ satisfies the Chinese remainder theorem, there exists $a \in A$ such that $a \equiv a_j(\theta_j)$ for all $j = 1, 2, \ldots, n+m$. It follows that x is congruent to y with respect to the product on the right-hand side of (1.1). It is now clear that 1.1 holds.

Since $\mathscr{P}$ is an upper subsemilattice of $\mathscr{C}(\mathfrak{A})$, (1.1) immediately implies

(1.2) $$L(\mathscr{P}) = \left\{ \bigcap_{i=1}^{n} \theta_i : \theta_1, \theta_2, \ldots, \theta_n \in \mathscr{P};\ n \geqslant 1 \right\}$$

and $L(\mathscr{P})$ consists of permuting congruences.

Hence,

(1.3) *$L(\mathscr{P})$ is a modular lattice.*

We now establish

(1.4) *For any* $\Psi_1, \Psi_2, \ldots, \Psi_n \in \mathcal{P}$ *and* $\Psi, \Phi \in \mathcal{P}$,

$$(\Psi_1 \cap \Psi_2 \cap \ldots \cap \Psi_n) \cap (\Psi \vee \Phi) = (\Psi_1 \cap \ldots \cap \Psi_n \cap \Psi) \circ (\Psi_1 \cap \ldots \cap \Psi_n \cap \Phi).$$

Suppose that x and y are elements of A which are congruent with respect to the left-hand side of (1.4). Let $\theta_1 = \Psi_1, \ldots, \theta_n = \Psi_n, \theta_{n+1} = \Psi, \theta_{n+2} = \Phi$ and $a_1 = \ldots = a_n = a_{n+1} = x$ while $a_{n+2} = y$. Then, $a_i \equiv a_j(\theta_i \vee \theta_j)$ for each $i, j = 1, \ldots, n+1$, $a_{n+1} \equiv a_{n+2}\ (\theta_{n+1} \vee \theta_{n+2})$, and $a_i \equiv a_{n+2}(\theta_i \vee \theta_{n+2})$ for all $i = 1, \ldots, n$, since $x \equiv y(\theta_i)$ for each $i = 1, 2, \ldots, n$. That is, $a_i \equiv a_j(\theta_i \vee \theta_j)$ for all $i, j = 1, \ldots, n+2$. Hence, there exists $a \in A$ such that $a \equiv a_j(\theta_j)$ for $j = 1, \ldots, n+2$. Then, $a \equiv x(\Psi_i)$ for $i = 1, \ldots, n$, $a \equiv x(\Psi)$ and $a \equiv y(\Phi)$. But $x \equiv y(\Psi_i)$ for $i = 1, \ldots, n$ and so $a \equiv y(\Psi_i)$ for $i = 1, \ldots, n$. Hence, x is congruent to y under the product on the right-hand side of (1.4), and it follows that (1.4) holds.

Since $\mathcal{P}$ is an upper semilattice, induction on (1.4) yields

(1.5) *For any* $\Psi_1, \ldots, \Psi_n, \Phi_1, \ldots, \Phi_m \in \mathcal{P}$

$$\bigcap_{i=1}^{n} \Psi_i \cap \bigvee_{j=1}^{m} \Phi_j = \bigvee_{j=1}^{m} \left(\bigcap_{i=1}^{n} \Psi_i \cap \Phi_j \right).$$

The theorem now follows from (1.1), (1.2), (1.3), (1.5) and Lemma 1.2.

2. Ideals and Examples.

Let z be a fixed element of a universal algebra. For θ in $\mathcal{C}(\mathfrak{A})$, define z-ker $\theta = \{a \in A\colon a \equiv z(\theta)\}$ and call it the *z-kernel* of θ. A *z-ideal* or *congruence-z-kernel* is a (non-empty) subset J of A such that $J = z$-ker θ for some congruence θ in $\mathcal{C}(\mathfrak{A})$. In practice the element z is usually identified with a nullary operation in $\mathfrak{A}$. Either for the sake of brevity or because there is no ambiguity the prefix "z-" is generally omitted from the above. Thus, in what follows we will speak of "ideals" and denote the collection of all ideals by $J(\mathfrak{A})$.

Ideals in this general context are discussed in [7; 0.2.39—0.2.52, p. 78—82] and a fundamental result is [7; Theorem 0.2.41, p. 78] which we single out in the next lemma.

Lemma 2.1. *Ordered by set-inclusion, the collection* $J(\mathfrak{A})$ *of ideals is an algebraic closure system on A and hence a complete compactly generated lattice.*

For an ideal J, let $\theta(J)$ be the smallest congruence on $\mathfrak{A}$ which has J as its kernel. Of course, $\theta(J) = \cap\{\theta \in \mathcal{C}(\mathfrak{A})\colon \ker \theta = J\}$ and the map $J \mapsto \theta(J)$ is an order-embedding of $J(\mathfrak{A})$ into $\mathcal{C}(\mathfrak{A})$. In general, only the following property of this map would seem to hold.

Proposition 2.2. *For any collection* $\{J_\lambda\}$ *of ideals,* $\theta(\vee J_\lambda) = \vee \theta(J_\lambda)$. *In particular, if* $\{J_\lambda\}$ *is directed then* $\theta(\vee J_\lambda) = \theta(\cup J_\lambda) = \cup \theta(J_\lambda)$.

Proof. For $J \in J(\mathfrak{A})$, let $\theta[J] = \vee\{\theta(a, b): a, b \in J\}$. Thus, $\theta[J]$ is the smallest congruence on $\mathfrak{A}$ identifying all the elements of J. Hence, $\theta[J] \subseteq \theta(J)$. But this inclusion implies $\ker(\theta[J]) \subseteq J$ and so $\ker(\theta[J]) = J$. Hence, $\theta[J] = \theta(J)$.

For $J, K \in J(\mathfrak{A})$, $J \subseteq K$ certainly implies $\theta[J] \subseteq \theta[K]$. Hence, $J \subseteq K$ if only if $\theta(J) \subseteq \theta(K)$. We are now able to prove the assertion. For if $J_\mu \in \{J_\lambda\}$ then $J_\mu \subseteq \vee J_\lambda$ and so $\vee\, \theta(J_\lambda) \subseteq \theta(\vee J_\lambda)$, while $J_\mu \subseteq \ker(\vee\, \theta(J_\lambda))$ so that $\vee J_\lambda \subseteq \ker(\vee\, \theta(J_\lambda))$. Hence, $\theta(\vee J_\lambda) \subseteq \theta(\ker(\vee\, \theta(J_\lambda)) \subseteq \vee\, \theta(J_\lambda)$. Hence, $\theta(\vee J_\lambda) = \vee\, \theta(J_\lambda)$. The rest follows from the algebraic nature of the closure systems involved.

Of course, we have no right to expect that $\theta(J \cap K) = \theta(J) \cap \theta(K)$ for any two ideals J and K. Thus, in general, $\{\theta(J): J \in J(\mathfrak{A})\}$ is only an upper subsemilattice of $\mathscr{C}(\mathfrak{A})$. Of course, it does happen when $\mathfrak{A}$ is (z-) weakly regular, i.e. when the map $J \mapsto \theta(J)$ is a bijection and so induces a lattice isomorphism of $J(\mathfrak{A})$ onto $\mathscr{C}(\mathfrak{A})$, whose inverse is the map $\theta \mapsto \ker \theta$ of $\mathscr{C}(\mathfrak{A})$ onto $J(\mathfrak{A})$. This is the case for groups, rings, Boolean algebras etc. However, there is another case which is not without interest.

Theorem 2.3. *Let $\mathscr{P}$ be a distributive sublattice of $J(\mathfrak{A})$. Assume also that $\mathscr{P}$ has the same smallest element and largest element as $J(\mathfrak{A})$ and $\mathscr{P}$ is closed under the formation of arbitrary suprema. Let $P(\mathscr{P})$ be the set of meet-irreducibles of $\mathscr{P}$. Then the following two conditions are equivalent.*

(i) *For any $J, K \in \mathscr{P}$, $\theta(J \cap K) = \theta(J) \cap \theta(K)$.*

(ii) *For any $J \in \mathscr{P}$, $\theta(J) = \cap\{\theta(P): J \subseteq P, P \in P(\mathscr{P})\}$.*

Proof. (i)$\Rightarrow$(ii). Clearly, $\theta(J) \subseteq \cap\{\theta(P): J \subseteq P, P \in P(\mathscr{P})\}$ for any $J \in \mathscr{P}$. We prove the opposite inequality by a contrapositive argument. That is, suppose that $a, b \in A$ and $a \not\equiv b(\theta(J))$. Consider the set $\mathscr{K} = \{K \in \mathscr{P}: J \subseteq K, a \not\equiv b(\theta(K))\}$, partially ordered by set-inclusion. Let $\mathscr{C}$ be a chain in $\mathscr{K}$. By the hypotheses on $\mathscr{P}$, $\cup\{C \in \mathscr{P}: C \in \mathscr{C}\} \in \mathscr{P}$. From Proposition 2.2, this ideal is actually in $\mathscr{K}$. Thus, Zorn's lemma implies that $\mathscr{K}$ has maximal elements. Let Q be such an element. It remains to prove that Q is meet-irreducible in $\mathscr{P}$. Firstly, notice that since $a \not\equiv b(\theta(Q))$, Q is not A, the largest element of $\mathscr{P}$. Let $A, B \in \mathscr{P}$ and assume that $A \not\subseteq Q$ and $B \not\subseteq Q$. Then, $Q \vee A$ and $Q \vee B$ both properly contain Q. Hence, $a \equiv b(\theta(Q \vee A) \cap \theta(Q \vee B))$. By *(i)* and the distributivity of $\mathscr{P}$, it follows that $a \equiv b(\theta(Q \vee (A \cap B)))$. Hence, $A \cap B \not\subseteq Q$. The distributivity of $\mathscr{P}$ ensures that Q is meet-irreducible.

(ii)$\Rightarrow$(i) is easy due to the distributivity of $\mathscr{P}$. Indeed, $\theta(J \cap K) \subseteq \theta(J) \cap \theta(K)$ holds for any $J, K \in J(\mathfrak{A})$. Now, let $P \in \mathscr{P}$ be meet-irreducible and suppose $J \cap K \subseteq P$ for given $J, K \in \mathscr{P}$. Then, $J \subseteq P$ or $K \subseteq P$. That is, $\theta(J) \subseteq \theta(P)$ or $\theta(K) \subseteq \theta(P)$. In either case $\theta(J) \cap \theta(K) \subseteq \theta(P)$. Thus, *(ii)* implies $\theta(J) \cap \theta(K) \subseteq \theta(J \cap K)$.

Taking $\mathscr{P} = J(\mathfrak{A})$, $z = 0$, and $\mathfrak{A}$ as a distributive lattice with 0, considered as an algebra $(A; \vee, \wedge, 0)$ of type $\langle 2, 2, 0 \rangle$, Theorem 2.3 can be applied to obtain a new proof of [3; Lemma 1.2 (ii)]. It can even be applied to the lattice $\mathcal{P}$ of *standard ideals* of a general lattice with 0. This is because of [6; Theorem 3, p. 33]; for the sake of brevity, we will omit the details and implications of this application. It can also be used to give new interpretations of the pathological behaviour of semirings as noted in [2] and an explanation of some of the conditions, in particular (x), of [3; Theorem 3.2], wherein Stone lattices are characterized by the behaviour of their congruence kernels in the variety of distributive pseudocomplemented algebras. Let us look a little more closely at these two instances, particularly in relation to Theorem 1.3.

By a semiring, we mean an algebra $\mathfrak{A} = \langle A; +, \cdot, 0\rangle$ of type (2, 2, 0) such that $(A; +, 0)$ is a commutative semigroup with 0 as its zero, $(A, \cdot)$ is a semigroup and the multiplication "$\cdot$" distributes over addition from either side and $a \cdot 0 = 0$. Thus, we consider semirings as generalizations of both rings and distributive lattices. Also, we take $z = 0$. It is easily checked that a non-empty subset J of A is an ideal if and only if

(2.1) $a, b \in J$ imply $a + b \in J$,

(2.2) $a \in J$, $b \in A$ imply ab, $ba \in J$, and

(2.3) $a + b \in J$ and $a \in J$ imply $b \in J$

hold simultaneously. Indeed, to verify this we also need to verify that $\theta(J)$ is given by

(2.4) $x \equiv y(\theta(J))(x, y \in A)$ if and only if $x + a = y + b$ for some suitable $a, b \in J$.

In general we have no right to assume the permutability of the $\theta(J)$ nor to assume the intersection formula: $\theta(J \cap K) = \theta(J) \cap \theta(K)$. In [2], a 5-element semiring in which both operations $+$ and $\cdot$ are commutative and idempotent is given. The intersection formula failed but all the congruences on $\mathfrak{A}$ are in the sublattice generated by the subsemilattice $\{\theta(J): J \in J(\mathfrak{A})\}$ and what is more $\mathscr{C}(\mathfrak{A})$ is distributive and consists of permuting congruences. Thus, we get a particular case of Theorem 1.3, where the fact that we are dealing only with an upper semilattice of $\mathscr{C}(\mathfrak{A})$ was of prime importance to [2].

Now consider a distributive pseudocomplemented lattice considered as an algebra $\mathfrak{A} = (A; \vee, \wedge, {}^*, 0, 1)$ of type $\langle 2, 2, 1, 0, 0\rangle$. Take $z = 0$. Then, the ideals of $\mathfrak{A}$ are examined in detail in [3]. We will use the notation and terminology of [3] freely, except that we will speak of ideals in place of congruence-kernels of *-congruences. There the intersection formulae of Theorem 2.3 failed badly, but after all that was the interest in the paper, c.f. [3; Theorem 2.5 and Theorem 3.2, (x), (xi) and (xii)]. However, for $J \in J(\mathfrak{A})$ and $0 \leqslant n \leqslant \omega$, each of the congruences $\Sigma_n(J)$ (see [3; Theorems 1.5, 1.6, 1.7 and 2.5]) is determined by the minimal congruence identifying some filter on A. And the minimal congruences $\Psi(F)$ associated with filters F of a distributive lattice form a distributive sublattice of $\mathscr{C}(\mathfrak{A})$ which consists of permuting congruences c.f. the "dual" of Theorem 2.3 and

[3; Lemma 1.2, Lemma 1.3] (N.B. The statement of [3; Lemma 1.3] should be ended by the clause "if they are congruence relations"). Thus, we can summarize and apply Theorem 1.3 to obtain

Theorem 2.4. *Let* $\mathfrak{A}=(A; \vee, \wedge, {}^*, 0, 1)$ *be a distributive pseudocomplemented lattice considered as an algebra of type* $\langle 2, 2, 1, 0, 0\rangle$. *With the notation of* [3], $0\leqslant n\leqslant\omega$ *and* $\mathscr{P}_n=(\Sigma_n(J): J\in J(\mathfrak{A}))$, $\mathscr{P}_n$ *is an upper subsemilattice of* $\mathscr{C}(\mathfrak{A})$ *satisfying the Chinese remainder theorem and consequently generates a distributive sublattice of permuting congruences in* $\mathscr{C}(\mathfrak{A})$. *When* $n=0$, $\mathscr{P}_n$ *is actually a sublattice of* $\mathscr{C}(\mathfrak{A})$, *while for* $1\leqslant n\leqslant\omega$, $\mathscr{P}_n$ *is a sublattice if and only if* $\mathfrak{A}$ *is a Stone lattice.*

This is an easy consequence of our previous remarks and [3; Theorems 2.10 and 3.2 (xii)].

Finally, let us turn to the so-called distributive ideals which can be defined as those ideals J of a lattice which satisfy either of the equivalent conditions of the following lemma. The details are omitted; for a comparison with standard ideals, see [6].

Lemma 2.5. *For a (lattice)-ideal* J *of a lattice* $\mathfrak{A}=(A; \vee, \wedge)$, *the following conditions are equivalent.*

(i) $J\vee(H\cap K)=(J\vee H)\cap(J\vee K)$ *for any (lattice)-ideals* H *and* K.

(ii) *The relation* $\theta(J)$ *defined by* $x\equiv y(\theta(J))(x, y\in A)$ *if and only if* $x\vee j=y\vee j$ *for some* $j\in J$ *is a congruence relation on* $\mathfrak{A}$.

(Note: $\theta(J)$ can be defined equivalently by $x\equiv y(\theta(J))(x, y\in A)$ if and only if $(x]\vee J=(y]\vee J$).

Of course, here we have not insisted on the existence of $0\in A$ (i.e. a distinguished element z) and $\theta(J)$ of Lemma 2.5 is the smallest congruence on $\mathfrak{A}$ having the distributive ideal J as a class. The collection $D(\mathfrak{A})$ of distributive ideals of $\mathfrak{A}$ is a complete upper subsemilattice of the lattice of ideals of lattice $\mathfrak{A}$ and $\mathscr{P}(D(\mathfrak{A}))=\{\theta(J): J\in D(\mathfrak{A})\}$ is a complete upper subsemilattice of $\mathscr{C}(\mathscr{A})$ (the proof is similar to that of Proposition 2.2).

Theorem 2.6. *Let* $\mathfrak{A}=(A; \vee, \wedge)$ *be a lattice. Then the upper subsemilattice* $\mathscr{P}(D(\mathfrak{A}))$ *of* $\mathscr{C}(\mathfrak{A})$ *and consisting of the minimal congruences associated with the set* $D(\mathfrak{A})$ *of distributive ideals satisfies the Chinese remainder theorem.*

Proof. Let $J_1, J_2, \ldots, J_n\in D(\mathfrak{A})$ and $x_1, x_2, \ldots, x_n\in A$ be such that $x_i\equiv x_j(\theta(J_i)\vee\theta(J_j)=\theta(J_i\vee J_j))$ for each $i, j=1, 2, \ldots, n$. It follows that there exist $k_i\in J_i$ and $k_j\in J_j$ such that $x_i\vee k_i\vee k_j=x_j\vee k_i\vee k_j$ for each $i, j=1, \ldots, n$. Hence, we certainly have $(x_i\vee k_i]\vee J_j=(x_j\vee k_j]\vee J_j$ for each i, $j=1, \ldots, n$. Let $x=(x_1\vee k_1)\wedge\ldots\wedge(x_n\vee k_n)$. Then,

$$\begin{aligned}(x]\vee J_j&=((x_1\vee k_1]\cap\ldots\cap(x_n\vee k_n])\vee J_j\\&=((x_1\vee k_1]\vee J_j)\cap\ldots\cap((x_j\vee k_j]\vee J_j)\cap\ldots\cap((x_n\vee k_n]\vee J_j)\end{aligned}$$

$= ((x_i \vee k_1] \vee J_j) \cap \ldots \cap ((x_j] \vee J_j) \cap \ldots \cap ((x_i \vee k_n] \vee J_j)$
$= (x_j] \vee J_j,$

using the distributivity of J_j. That is, $x \equiv x_j(\theta(J_j))$ for $j = 1, \ldots, n$.

In general, the intersection $J \cap K$ of two distributive ideals J and K may even turn out to be distributive and yet $\theta(J \cap K) \neq \theta(J) \cap \theta(K)$. This is illustrated by the five element non-modular lattice $\{0, a, b, c, 1 : 0 < a < 1; 0 < c < b < 1\}$, wherein $(a] \cap (b] = (0]$, $(a]$, $(b]$ and $(0]$ are all distributive ideals, yet $b \neq c$ and $b \equiv c(\theta((a]) \cap \theta((b]))$. On the other hand Theorem 2.6 and Theorem 1.3 yield positive results.

Corollary 2.7. *Let $J_1, \ldots, J_n, K_1, \ldots, K_m$ be $n + m$ distributive ideals in a lattice $\mathfrak{A}$. Then, the congruences $\bigcap_{r=1}^{n} \theta(J_r)$ and $\bigcap_{s=1}^{m} \theta(J_s)$ are permutable.*

Corollary 2.8. *Let $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$ be lattices with 0. Then, $\mathfrak{A}$ is isomorphic to the direct product of $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$ if and only if there exist distributive ideals $J_1, J_2, \ldots, J_n$ in $\mathfrak{A}$ such that*

(i) *the J_i are pairwise comaximal i.e. $r \neq s$ implies $J_r \vee J_s = A$,*

(ii) $\bigcap_{i=1}^{n} \theta(J_i) = \omega$, *and*

(iii) *for each $i = 1, \ldots, n$, $\mathfrak{A}_i \cong \mathfrak{A}/\theta(J_i)$.*

Proof. Corollary 2.8 follows from Corollary 2.7 using a familiar argument, see, for example [10; Corollary 1.6, p. 67].

We note that Corollary 2.7 simplifies for standard ideals ([6; Theorem 3, p. 33]) and takes into account many known results, see [8; Theorem 2 and its corollaries]. The ideals J_i of Corollary 2.8 are easily seen to be standard when $\mathfrak{A} = \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_n$ ($J_i = \{f \in A : f(i) = 0\}$ and $\theta(J_i)$ is given by $f \equiv g(f, g \in A)$ if and only if $f(i) = g(i)$ or equivalently either $f \vee k = g \vee k$ or $f \vee g = (f \wedge g) \vee k$ for some $k \in J_i$) and hence, an alternative form of Corollary 2.8 can be stated, wherein the ideals J_i are assumed to be standard and (ii) is replaced by (ii)' $\bigcap_{i=1}^{n} J_i = (0]$.

REFERENCES

[1] BALBES, R.: A note on distributive sublattices of a modular lattice. Fund. Math. 65, 1969, 219—222.

[2] CORNISH, W. H.: Subdirectly irreducible semirings and semigroups without nonzero nilpotents. Canad. Math. Bull. 16, 1973, 45—47.

[3] CORNISH, W. H.: Congruences on distributive pseudocomplemented lattices. Bull. Austral. Math. Soc. 8, 1973, 161—179.

[4] DRAŠKOVIČOVÁ, H.: Permutability, distributivity of equivalence relations and direct products. Mat. Čas. 23, 1973, 69—87.

[5] GRÄTZER, G.: Universal algebra. Princeton 1968.
[6] GRÄTZER, G., SCHMIDT, E. T.: Standard ideals in lattices. Acta Math. Acad. Sci. Hungar. 12, 1961, 17—86.
[7] HENKIN, L., MONK, J. D., TARSKI, A.: Cylindric algebras. Part I. Amsterdam 1971.
[8] IQBALUNISSA: Permutable congruences in a lattice. Illinois J. Math. 10, 1966, 235—239.
[9] JÓNSSON, B.: Distributive sublattices of a modular lattice. Proc. Amer. Math. Soc. 6, 1955, 682—688.
[10] PIERCE, R. S.: Introduction to the theory of abstract algebras. New York 1968.
[11] SZÁSZ, G.: Introduction to lattice theory. New York 1963.
[12] ZARISKI, O., SAMUEL, P.: Commutative algebra. Vol. 1. Princeton 1958.

Received January 13, 1975

Department of Mathematics
Dalhousie University
Halifax, N.S., Canada

Permanent address:

School of Mathematical Science
The Flinders University of South Australia
Bedford Park, 5042, S.A.
Australia